Solutions to Exercise Sheet 2

1. Expanding, we obtain

$$\cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}\left((e^x)^2 + 2e^x e^{-x} + (e^{-x})^2\right).$$

By the exponential laws, $(e^x)^2 = e^{2x}$, $e^x e^{-x} = 1$, and $(e^{-x})^2 = e^{-2x}$. Hence

$$\cosh^2 x = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right).$$

Similarly, we compute

$$\sinh^2 x = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right).$$

So

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}(2+2) = 1.$$

2. The basic solution is $x = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. The other solution in the period $[-\pi, \pi]$ is $x = -\frac{\pi}{4}$, as $\cos(-x) = \cos x$. Hence the general solution is $x = \pm\frac{\pi}{4} + 2k\pi$, for any integer k.

3. Dividing both sides by 11, we obtain $\sin(2x) = \frac{3}{11}$. One solution corresponds to $2x = \arcsin\frac{3}{11}$ and the other solution in the same period to $2x = \pi - \arcsin\frac{3}{11}$. So all solutions are given by

$$2x = 2k\pi + \arcsin\frac{3}{11} \quad \text{and} \quad 2x = (2k+1)\pi - \arcsin\frac{3}{11}$$

for an integer k. To solve for x, we divide by 2 and obtain

$$x = k\pi + \frac{1}{2}\arcsin\frac{3}{11} \quad \text{and} \quad x = \frac{2k+1}{2}\pi - \frac{1}{2}\arcsin\frac{3}{11},$$

where k stands for any integer.

4. (a) We find that $(x, y) = (r\cos\theta, r\sin\theta) = \left(4\cos\frac{\pi}{4}, 4\sin\frac{\pi}{4}\right) = \left(2\sqrt{2}, 2\sqrt{2}\right)$.

   (b) We find that $r = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$. The point $(-2, 3)$ has $x < 0$, hence the angle is $\theta = \pi + \arctan(-\frac{3}{2}) = \pi - \arctan\frac{3}{2}$.

5. We write the left-hand-side in harmonic form, that is, we want to find r and θ with

$$2\sin x - 5\cos x = r\cos(x+\theta) = r\cos\theta\cos x - r\sin\theta\sin x.$$

Hence, we want to solve

$$r\cos\theta = -5,$$
$$r\sin\theta = -2.$$

To this end, we need to find the polar coordinates of the point $(-5,-2)$. We find that $r = \sqrt{5^2+2^2} = \sqrt{29}$ and, since we have a point in the third quadrant, $\theta = \pi + \arctan\frac{2}{5}$.

Our equation now becomes $r\cos(x+\theta) = 1$. Dividing by r and substituting the above values, we obtain

$$\cos\left(x+\pi+\arctan\frac{2}{5}\right) = \frac{1}{\sqrt{29}}.$$

Hence

$$x+\pi+\arctan\frac{2}{5} = \pm\arccos\frac{1}{\sqrt{29}} + 2k\pi$$

for an integer k. We now solve this for x:

$$\begin{aligned} x &= -\pi - \arctan\frac{2}{5} \pm \arccos\frac{1}{\sqrt{29}} + 2k\pi \\ &= (2k-1)\pi - \arctan\frac{2}{5} \pm \arccos\frac{1}{\sqrt{29}}, \end{aligned}$$

where k is an arbitrary integer.

6. Pythagoras' theorem tells us that $\sin^2 x = 1-\cos^2 x$. Substituting this into our equation, we obtain

$$1-\cos^2 x+\cos x+1 = 0.$$

So

$$\cos^2 x - \cos x - 2 = 0.$$

Now let $u = \cos x$, then we end up with the quadratic equation

$$u^2 - u - 2 = 0.$$

Factorisation gives $(u-2)(u+1) = 0$, which has the solutions $u = 2$ and $u = -1$. But we note that $u = 2$ is *not* a solution for $u = \cos x$. Hence our solutions are given by $\cos x = -1$, which gives $x = \pi + 2\pi k = (2k+1)\pi$, where k is an arbitrary integer.

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