Solutions to Exercise Sheet 11

1. (a) We calculate

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 3x + 3y^2.$$

If the first of these vanishes, then $y = -x^2$. If both vanish, then

$$0 = x + y^2 = x + x^4 = x(1 + x^3).$$

The right hand side has zeros at $x = 0$ and $x = -1$. These conditions give rise to $y = 0$ and $y = -1$, respectively. So we need to examine the points $(0, 0)$ and $(-1, -1)$.

In order to distinguish local minima from local maxima and other points, we consider the quantity

$$\Delta = \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 = 36xy - 9.$$

At $(0, 0)$, this gives $\Delta = -9 < 0$, so we have neither a local minimum nor a local maximum here.

At $(-1, -1)$, we find that $\Delta = 27 > 0$. Therefore, we consider

$$E = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 6x + 6y.$$

At $(-1, -1)$, this gives $E = -12 < 0$, thus we have a local maximum here.

(b) Here we calculate

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 + 4xy - 2x \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 2x^2 + 6y.$$

If the second of these vanishes, then $y = -\frac{1}{3}x^2$. If both vanish, then

$$0 = 4x^3 + 4xy - 2x = 4x^3 - \frac{4}{3}x^3 - 2x = x\left(\frac{8}{3}x^2 - 2\right).$$

This means that either $x = 0$ or $x = \pm\frac{\sqrt{3}}{2}$. If $x = 0$, then $y = 0$. If $x = \pm\frac{\sqrt{3}}{2}$, then $y = -\frac{1}{4}$ (regardless of the sign of x). Thus we need to examine the points $(0, 0)$, $(\frac{\sqrt{3}}{2}, -\frac{1}{4})$, and $(-\frac{\sqrt{3}}{2}, -\frac{1}{4})$.

We further compute

$$\begin{aligned}\Delta = \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 &= 6(12x^2 + 4y - 2) - 16x^2 \\ &= 56x^2 + 24y^2 - 12.\end{aligned}$$

At $(0,0)$, we find that $\Delta = -12 < 0$, so this is neither a local minimum nor a local maximum. At $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{4})$, we obtain $\Delta = \frac{63}{2} > 0$. Here we also compute

$$E = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 12x^2 + 4y - 2 + 6 = 12x^2 + 4y + 4.$$

At $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{4})$, this gives $E = 12 > 0$. Hence we have local minima at both of these points.

2. Rewriting the equation in the form

$$(3x^2e^y - y) + (x^3e^y - x)\frac{dy}{dx} = 0$$

and setting $M = 3x^2e^y - y$ and $N = x^3e^y - x$, we obtain the equation $M + Ny' = 0$. This is an exact equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which is true in this case, as

$$\frac{\partial}{\partial y}(3x^2e^y - y) = 3x^2e^y - 1 = \frac{\partial}{\partial x}(x^3e^y - x).$$

In order to find h, we first solve $\frac{\partial h}{\partial x} = M$, which gives $h(x,y) = x^3e^y - xy + g(y)$ for some function g. In order to determine g, we solve $\frac{\partial h}{\partial y} = N$, which reduces to $g'(y) = 0$ and hence $g(y) = C$.

Thus the implicit solution is given by $x^3e^y - xy = C$.

RM, 28/11/2017