# Solutions to Exercise Sheet 10

1. We first compute the partial derivatives:

$$\frac{\partial f}{\partial x} = 5x^4 e^y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^5 e^y.$$

Thus

$$f(1.01, 0.02) \approx f(1,0) + 0.01 \cdot \frac{\partial f}{\partial x}(1,0) + 0.02 \cdot \frac{\partial f}{\partial y}(1,0) = 1 + 0.05 + 0.02 = 1.07.$$

(The exact value is 1.07224 to 5 decimal places.)

2. (a) We compute

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 y^6 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 6x^3 y^5.$$

(b) Here we compute

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}.$$

(c) In this case, we have three partial derivatives, namely

$$\frac{\partial f}{\partial x}(x,y,z) = yze^{xy} - \sin(x-z), \quad \frac{\partial f}{\partial y} = xze^{xy}, \quad \text{and} \quad \frac{\partial f}{\partial z} = e^{xy} + \sin(x-z).$$

3. We first compute the partial derivatives

$$\frac{\partial z}{\partial t} = x^y, \quad \frac{\partial z}{\partial x} = tyx^{y-1}, \quad \text{and} \quad \frac{\partial z}{\partial y} = tx^y \ln x.$$

Moreover,

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = -\frac{1}{t^2}.$$

Hence

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= x^y + tyx^{y-1} \cdot 2t + tx^y \ln x \cdot \left(-\frac{1}{t^2}\right) \\
&= x^{y-1}\left(x + 2t^2 y - \frac{x \ln x}{t}\right) \\
&= t^{2/t-2}\left(t^2 + 2t - 2t\ln t\right) \\
&= t^{2/t-1}(t + 2 - 2\ln t).
\end{aligned}$$

4. We first calculate the first order partial derivatives:

$$\frac{\partial f}{\partial x}(x, y, z) = 2x\cos(yz), \quad \frac{\partial f}{\partial y}(x, y, z) = -x^2 z\sin(yz),$$

and

$$\frac{\partial f}{\partial z}(x, y, z) = -x^2 y\sin(yz).$$

Differentiating again, we obtain

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, y, z) &= 2\cos(yz), & \frac{\partial^2 f}{\partial x \partial y}(x, y, z) &= -2xz\sin(yz), \\ \frac{\partial^2 f}{\partial x \partial z}(x, y, z) &= -2xy\sin(yz), & \frac{\partial^2 f}{\partial y^2}(x, y, z) &= -x^2 z^2\cos(yz), \\ \frac{\partial^2 f}{\partial y \partial z}(x, y, z) &= -x^2\sin(yz) - x^2 yz\cos(yz), & \frac{\partial^2 f}{\partial z^2}(x, y, z) &= -x^2 y^2\cos(yz). \end{aligned}$$

This is already sufficient to answer the question, because the symmetry of the second order derivatives now gives

$$\frac{\partial^2 f}{\partial y \partial x}(x, y, z) = -2xz\sin(yz), \quad \frac{\partial^2 f}{\partial z \partial x}(x, y, z) = -2xy\sin(yz),$$

and

$$\frac{\partial^2 f}{\partial z \partial y}(x, y, z) = -x^2\sin(yz) - x^2 yz\cos(yz)$$

as well.

RM, 28/11/2017