MA10192 Mathematics I

Lecture Notes

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Contents

Chapter 1

Functions and Equations

1.1 What are functions and equations?

An *equation* is a statement saying that two things (in this course typically numbers or quantities) are equal. For example, the equation

$$2 + 3 = 5$$

expresses the fact that the sum of 2 and 3 is equal to 5. We often consider equations involving an unknown quantity or number, say x. For example, consider the equation

$$x^2 = 2.$$

To *solve* an equation of this type means to determine all values of x such that the equation is true.

Example 1.1.1. Consider the equation

$$x^2 - 4 = 0.$$

It has two solutions, namely $x = 2$ and $x = -2$.

A *function* is a rule by which one quantity depends unambiguously on another. We can think of a function as a process that takes a certain input and turns it into a unique output. (The word 'unambiguously' in the previous definition means that every input generates one single output.)

Example 1.1.2. The formula

$$y = x^2 + 2x - \frac{1}{2}$$

gives rise to a function by which the quantity y depends on x. The input $x = 2$, say, leads to $y = 2^2 + 2 \cdot 2 - \frac{1}{2} = \frac{15}{2}$, and every other value for x will give a single value for y.

We will mostly study functions that are given by formulas such as in the example, because these are particularly useful.

We may use symbols to represent a given function, just as we use x to represent numbers in the above examples. If our function is denoted by f, then $f(x)$ stands for the output arising from the input x. If we wish to assign a symbol to this quantity as well, say y, then we can write $y = f(x)$. Here x is called the *independent variable* and y is called the *dependent variable* of the function.

Example 1.1.3. Consider the function determined by the formula

$$y = x^2 + 2x - \frac{1}{2}.$$

If we represent this function by the symbol f, then we may also write

$$f(x) = x^2 + 2x - \frac{1}{2}.$$

Then we have, for example, $f(2) = \frac{15}{2}$.

A function f can be represented geometrically in terms of the set of all points in the plane with coordinates (x, y) such that

$$y = f(x).$$

This is called the *graph* of the function f.

Example 1.1.4. Fig. 1.1.1 represents the graph of the function from Example 1.1.3.

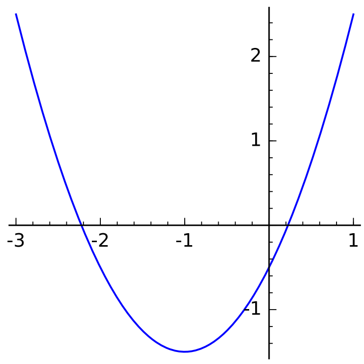

Figure 1.1.1: The graph of the function f with $f(x) = x^2 + 2x - \frac{1}{2}$

A *zero* of the function f is a solution of the equation $f(x) = 0$.

1.2 Special functions

1.2.1 Polynomials

Among the simplest functions are the *linear* ones, given by a formula such as

$$y = ax + b.$$

Here a and b are *constants* (i.e., fixed numbers), whereas x and y are the independent variable and the dependent variable, respectively. The graph of a linear function is a straight line (hence the name) with slope a and y-intercept b. (An example is shown in Fig. 1.2.1.)

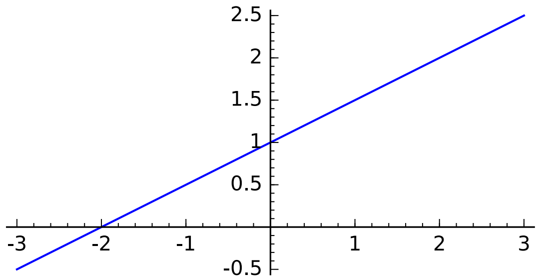

Figure 1.2.1: The line given by $y = \frac{1}{2}x + 1$

If we look for the zeros of such a function, then we consider the equation

$$ax + b = 0,$$

which we may attempt to solve for x. This is possible if $a \neq 0$ and gives the unique solution

$$x = -\frac{b}{a}.$$

If $a = 0$, then it can happen that the equation has no solutions (which is the case if $b \neq 0$) or that every number is a solution (if $b = 0$).

We now consider the next more complicated thing, a *quadratic function*. For given constants a, b, c, consider the formula

$$y = ax^2 + bx + c,$$

giving rise to a function. If $a = 0$, then we can ignore the first term on the right-hand side and we have a linear function. If $a \neq 0$ (which we assume henceforth), then we have something fundamentally different here. The graph of such a function is a parabola. If we look for the zeros, we obtain the *quadratic equation*

$$ax^2 + bx + c = 0.$$

If we want to solve the equation, it is convenient to divide by a on both sides first (which we can do as $a \neq 0$):

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Then we 'complete the square', i.e., we reformulate this as follows:

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

Thus

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Assuming that $b^2 - 4ac \geq 0$, we can take the square root on both sides:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Now subtract $b/2a$ on both sides to obtain the well-known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So for $b^2 - 4ac > 0$, we have two solutions, and for $b^2 - 4ac = 0$, we have one, which is $-\frac{b}{2a}$. If $b^2 - 4ac < 0$, then the equation has no solutions.[1]

Another method to solve a quadratic equation is *factorisation*. Assuming that we have an equation of the form

$$x^2 + bx + c = 0,$$

(which is the case after the first step for the above method), we try to find two numbers α, β such that

$$x^2 + bx + c = (x - \alpha)(x - \beta).$$

This is true if $\alpha + \beta = -b$ and $\alpha\beta = c$. It is not always possible to find α and β that satisfy both conditions, but if they do, then we have the solutions $x = \alpha$ and $x = \beta$ of the quadratic equation.

A *cubic function* arises from a formula of the form

$$y = ax^3 + bx^2 + cx + d$$

and a *quartic function* from a formula of the form

$$y = ax^4 + bx^3 + cx^2 + dx + e,$$

[1] This is the case if you work with *real* numbers as we do here. In MA10193 Mathematics 2, you will see *complex* numbers, and if you work with these, then you have two solutions in this case as well.

where a, b, c, d, e are constants and in both cases it is assumed that $a \neq 0$. There are still methods to solve the corresponding equations, but they are rather more complicated.

There is no reason to stop at fourth powers of x, of course. Any function built up from powers of x in this way is called a *polynomial*. For example, the function given by

$$p(x) = x^{22} - x^{16} + 8x^5 - 9$$

is a polynomial. The highest power in this expression is called the *degree* of the polynomial. So in this example, the degree of p is 22.

1.2.2 Exponential functions

An *exponential function* is a function f that can be represented in the form

$$f(x) = a^x$$

for some constant $a > 0$. Here a is called the *base* and x is called the *exponent*.

If $a > 0$, then a^x exists for any number x and is always positive (i.e., $a^x > 0$). A central property of exponentials is expressed by the formula

$$a^{x+y} = a^x a^y, \tag{1.2.1}$$

which holds true for all numbers x and y. In terms of the function f, this can be written as

$$f(x+y) = f(x)f(y).$$

This formula has a number of important consequences. For example, it follows that

$$f(0) = f(0+0) = f(0)f(0).$$

Since $f(0) = a^0 > 0$, we can divide by $f(0)$ on both sides of the equation and we obtain $f(0) = 1$. That is, we have

$$a^0 = 1,$$

regardless of the value of a. Also, since $x + (-x) = 0$ for any number x, we conclude that

$$1 = f(0) = f(x)f(-x).$$

Dividing by $f(x)$ on both sides, we obtain $f(-x) = 1/f(x)$. That is, we always have

$$a^{-x} = \frac{1}{a^x}.$$

Another formula to remember is

$$(a^x)^y = a^{xy}, \tag{1.2.2}$$

which is true for any pair of numbers x and y.

Example 1.2.1. What is $(2^{14})^{1/7}$?
Solution. We have $(2^{14})^{1/7} = 2^{14/7} = 2^2 = 4$.

The shape of the graph of an exponential function depends on a. If $a > 1$, then the values of a^x increase as x increases. When x decreases towards $-\infty$, then the values of a^x approach 0 (without ever reaching it). As x gets large, the values of a^x grow rapidly. This is known as *exponential growth* (see Fig. 1.2.2). If $0 < a < 1$, then a^x decreases as x increases.

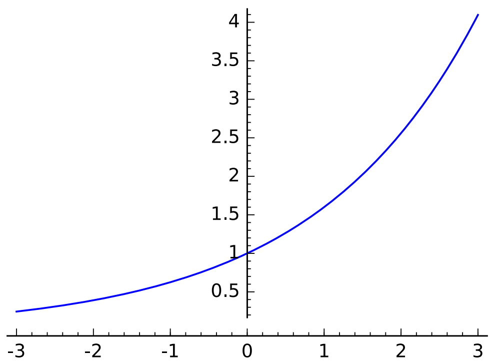

Figure 1.2.2: The graph of a function growing exponentially

When x approaches $-\infty$, then a^x grows rapidly, and when x gets large, it approaches 0. This is *exponential decay* (see Fig. 1.2.3).

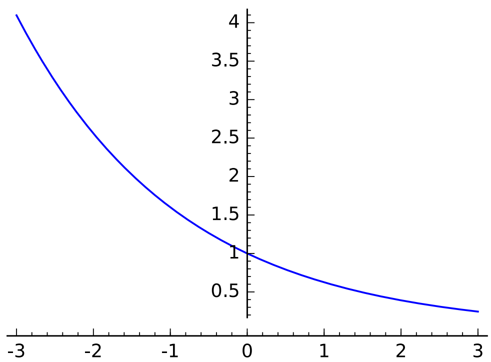

Figure 1.2.3: The graph of a function decaying exponentially

If $a = 1$, then $a^x = 1$ for every x. So in this case, the graph of the exponential function is a horizontal line.

There is a specific base such that the exponential function has particularly nice properties. (We will see this later when we differentiate functions.) This number is denoted e and is approximately (but not precisely) 2.71828.

1.2.3 Logarithms

Suppose that we fix a number $y > 0$ and we want to solve the equation

$$y = a^x. \tag{1.2.3}$$

Geometrically, this means finding the intersection points of the graph of the exponential function with a horizontal line at height y. If $a \neq 1$, then in view of the above observations, it is clear that there exists a unique solution. In other words, given $y > 0$, there exists a unique number x satisfying the equation $y = a^x$. This number is called the *logarithm* of y to the base a. We use the notation

$$x = \log_a y \tag{1.2.4}$$

for this concept.

Since the number x from (1.2.4) solves equation (1.2.3), we have

$$y = a^{\log_a y}$$

for every $y > 0$. Moreover, for any x, if we substitute a^x for y in (1.2.3), then the equation is clearly satisfied. Therefore, we have

$$\log_a a^x = x.$$

So we can think of $\log_a$ as the function that does the reverse of the exponential function with the same base. We say that they are *inverse functions*.

Example 1.2.2. Find $\log_5 125$.
Solution. We calculate that $5^3 = 125$. So $\log_5 125 = 3$.

When we work with the base e, then we write ln for the corresponding logarithm. That is, we abbreviate $\ln x = \log_e x$. (But in some books you may find different conventions.)

Now recall formula (1.2.1). If we choose two numbers x and y and set $u = a^x$, $v = a^y$, and $w = a^{x+y}$, then we can write the equation as $uv = w$. But we also have $x = \log_a u$, $y = \log_a v$, and $x + y = \log_a w = \log_a(uv)$. So

$$\log_a(uv) = \log_a u + \log_a v.$$

This formula is true for all positive numbers $u, v > 0$. (Recall the the logarithm only exists for positive numbers.)

How does equation (1.2.2) translate to the language of logarithms? Given x and y, set $b = a^x$ and $z = b^y$. Then $y = \log_b z$ and $x = \log_a b$. Moreover, equation (1.2.2) says that $z = a^{xy}$. So $\log_a z = xy$. It follows that

$$\log_a z = \log_a b \cdot \log_b z.$$

This formula is particularly useful when you want to convert logarithms for one base into logarithms for another, because it means that

$$\log_b z = \frac{\log_a z}{\log_a b}$$

for all $z > 0$.

Another consequence of (1.2.2) is the following. Suppose that $a > 0$ and $b > 0$. Then for any number x, we have

$$a^{x \log_a b} = \left(a^{\log_a b}\right)^x = b^x.$$

Therefore,

$$\log_a b^x = x \log_a b.$$

This fact is useful when we want to solve equations where the unknown is in the exponent.

Example 1.2.3. Solve the equation $e^{7x+2} = 13$ for x.
Solution. The equation implies that $7x + 2 = \ln 13$. This is now easy to solve for x and we obtain

$$x = \frac{1}{7}(\ln 13 - 2).$$

Example 1.2.4. Solve the equation $\log_2(x^2 - 4) = 4$ for x.
Solution. The equation implies that $x^2 - 4 = 2^4 = 16$. Hence $x^2 = 20$ and so $x = \pm\sqrt{20} = \pm 2\sqrt{5}$.

Example 1.2.5. Solve the equation $2^{x^2} = 3^x$ for x.
Solution. Apply ln on both sides:

$$x^2 \ln 2 = x \ln 3.$$

This is equivalent to $x^2 \ln 2 - x \ln 3 = 0$, and we can factorise $x(x \ln 2 - \ln 3) = 0$. We have either $x = 0$ or $x \ln 2 - \ln 3 = 0$, and in the second case, we obtain

$$x = \frac{\ln 3}{\ln 2} = \log_2 3.$$

(Remark: we could have used logarithms for any base here and would have obtained the same result.)

1.2.4 Trigonometric functions

The following definition of the *sine* and *cosine* may seem a bit odd at first, but bear with me.

Imagine a circle of radius 1 in the plane, the centre of which has the coordinates $(0, 0)$. (Henceforth we will call this the *unit circle*.) So the point with coordinates $(1, 0)$ is on this circle. Suppose that $t \geq 0$ is a given number. Starting at the point $(0, 1)$, travel anticlockwise along the circle and stop when you have covered the distance t. You will have reached a certain point on the circle with a certain x-coordinate and a certain y-coordinate, both of which are unambiguously determined by t. Define $\cos t$ to be the x-coordinate and $\sin t$ to be the y-coordinate of that point (see Fig. 1.2.4). For $t < 0$, use the same method, but travelling clockwise for the distance $-t$.

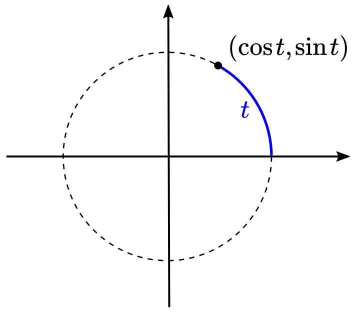

Figure 1.2.4: Definition of the trigonometric functions

This procedure gives rise to two functions sin and cos. They can be defined geometrically, involving angles and right triangles, but the above method has some advantages. (Most importantly, there is no question what measure of angles to use. Our definition is equivalent to the geometric definition when angles are measured in radians.)

We can immediately see that these functions have the following properties.

Periodicity If we travel the distance $t+2\pi$ instead of t, we just add a full rotation of the circle and end up at the same point. Therefore,

$$\cos(t + 2\pi) = \cos t \quad \text{and} \quad \sin(t + 2\pi) = \sin t.$$

Symmetry Replacing t by $-t$ corresponds to a reflection on the x-axis. Therefore,

$$\cos(-t) = \cos t \quad \text{and} \quad \sin(-t) = -\sin t.$$

Pythagoras' theorem The final point will be on the unit circle; therefore it has distance 1 from the point $(0, 0)$. This means that

$$\cos^2 t + \sin^2 t = 1. \tag{1.2.5}$$

Less obvious, but nevertheless important and true, are the following addition formulas:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t \quad \text{and} \quad \sin(s + t) = \cos s \sin t + \sin s \cos t.$$

Once we have the sine and cosine, we can also define a number of other trigonometric functions:

$$\tan t = \frac{\sin t}{\cos t}, \quad \cot t = \frac{\cos t}{\sin t}, \quad \sec t = \frac{1}{\cos t}, \quad \csc t = \frac{1}{\sin t}.$$

These are not defined for all values of t, because we always have to avoid a vanishing denominator. (The functions cos and sin, in contrast, are defined for all values.) Most of these we will rarely use explicitly, because we can, after all, write them as a fraction involving the sine and cosine.

1.2.5 Inverse trigonometric functions

Given a fixed number y, suppose that we want to solve the equation

$$\sin x = y$$

for x. Geometrically, this amounts to the following question: how far do we need to travel along the unit circle in order to reach a point with a given y-coordinate? If there is a solution at all, there are always several answers (in fact infinitely many), because we can always decide to add a full turn of the circle and we will get back to the same point.

Example 1.2.6. Solve the equation

$$\sin t = \frac{1}{2}.$$

Solution. Consider the intersection of the unit circle with the horizontal line $y = \frac{1}{2}$ (see Fig. 1.2.5). There are two intersection points. In order to

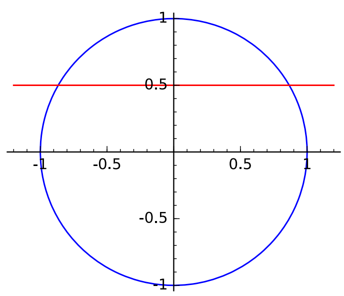

Figure 1.2.5: The unit circle and the line with $y = \frac{1}{2}$

reach the first, we travel the distance $\pi/6$. So $t = \pi/6$ is one solution. But $t = 5\pi/6$ is another solution, corresponding to the second intersection point. Moreover, we can add arbitrary multiples (positive or negative) of 2π. So every number of the form $t = \pi/6 + 2\pi n$ or $t = 5\pi/6 + 2\pi n$, for any integer n, is a solution.

Example 1.2.7. Solve the equation

$$\cos t = \frac{1}{2}.$$

Solution. Now we consider the intersection of the unit circle with the vertical line $x = \frac{1}{2}$ (see Fig. 1.2.6). Again there are two intersection points, and both $t = \pi/3$ and $t = -\pi/3$ are solutions. Again we can add arbitrary multiples of 2π. So every number of the form $t = 2\pi n \pm \pi/3$, for any integer n, is a solution.

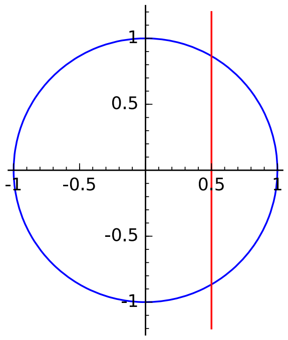

Figure 1.2.6: The unit circle and the line with $x = \frac{1}{2}$

Because an equation such as

$$\cos x = y \tag{1.2.6}$$

does not have a unique solution, there exists no single function that describes all solutions. If we want a function that is the inverse of sin or cos in the same sense as $\log_a$ is the inverse of the exponential function with base a, then we have to choose a specific solution. To this end, we use the following observations. For every given y with $-1 \le y \le 1$, equation (1.2.6) has exactly one solution x with $0 \le x \le \pi$. This solution is called the *arccosine* of y and denoted by $\arccos y$. (Sometimes the notation $\cos^{-1} y$ is used instead, but we will not use it here.)

Similarly, for any given y with $-1 \le y \le 1$, the equation

$$\sin x = y \tag{1.2.7}$$

has exactly one solution x with $-\pi/2 \le x \le \pi/2$. This is called the *arcsine* of y and denoted by $\arcsin y$.

Example 1.2.8. Solve the equation

$$\cos x = \frac{5}{7}.$$

Solution. By definition, the number $\arccos\frac{5}{7}$ is one solution. But it is not the only one. By the symmetry and the periodicity, any number of the form $2\pi n \pm \arccos\frac{5}{7}$, for any integer n, is a solution.

Note that for $y > 1$ or $y < -1$, equations (1.2.6) and (1.2.7) have no solutions. So there is no arccosine or arcsine of such a number.

Now consider the function tan. For any number y, the equation

$$\tan x = y$$

has infinitely many solutions, but exactly one of them satisfies $-\pi/2 < x < \pi/2$. This solution is called the *arctangent* of y and denoted $\arctan y$.

1.2.6 Hyperbolic functions

The *hyperbolic sine*, denoted sinh, and the *hyperbolic cosine*, denoted cosh, are the functions defined by the formulas

$$\sinh x = \frac{1}{2}(e^x - e^{-x}),$$
$$\cosh x = \frac{1}{2}(e^x + e^{-x}).$$

The reason for these names is that the behaviour of these functions resembles that of the sine and cosine in some respects. One example is the identity

$$\cosh^2 x - \sinh^2 x = 1,$$

which should be compared with (1.2.5). The identity can easily be checked by inserting the above expressions. We will see other similarities later.

In addition to these, we can now define other hyperbolic functions analogously to the trigonometric functions:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

1.3 Applications of trigonometric functions

1.3.1 Polar coordinates

Instead of representing a point P in the plane by its Cartesian coordinates (x, y), it is sometimes convenient to use polar coordinates. In order to find the polar coordinates, assume that P is not the origin O (the point with Cartesian coordinates $(0, 0)$). Draw a circle centred at O through P. Denote the radius of this circle by r. (This is the distance between O and P.) Moreover, denote the length of the arc between the points $(0, r)$ and P, travelling anticlockwise, by s (see Fig. 1.3.1). The numbers r and $t = s/r$ determine the position of P uniquely, and (r, t) are the polar coordinates

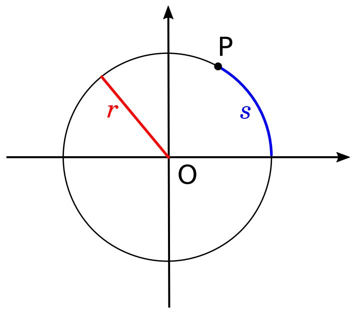

Figure 1.3.1: Construction of polar coordinates (r, t) of the point P

of P. We usually insist that t satisfies $0 \leq t < 2\pi$, and then the polar coordinates are determined uniquely by P as well, unless $P = O$. Given the polar coordinates (r, t), it is easy to determine the Cartesian coordinates of the same point:

$$x = r\cos t, \tag{1.3.1}$$

$$y = r\sin t. \tag{1.3.2}$$

The converse is a bit more complicated. Given the Cartesian coordinates (x, y), we have $r = \sqrt{x^2 + y^2}$ by Pythagoras' theorem. In order to find t, we want to solve the system of equations (1.3.1), (1.3.2) for t. We first eliminate r by taking a quotient:

$$\frac{y}{x} = \frac{\sin t}{\cos t} = \tan t$$

(provided that $x \neq 0$). So in order to solve for t, we take the arctangent,

$$t = \arctan\frac{y}{x},$$

provided that we expect $-\pi/2 < t < \pi/2$. This is the case if P is in the first quadrant, i.e., if $x > 0$ and $y \geq 0$. Otherwise, however, this gives the wrong answer! (That's because solutions of this equation are not unique and we've made a choice when defining the arctangent.) The true answer is

$$t = \begin{cases} \arctan\frac{y}{x} & \text{if } x > 0 \text{ and } y \geq 0, \\ \arctan\frac{y}{x} + \pi & \text{if } x < 0, \\ \arctan\frac{y}{x} + 2\pi & \text{if } x > 0 \text{ and } y < 0, \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ \frac{3\pi}{2} & \text{if } x = 0 \text{ and } y < 0. \end{cases}$$

This covers all cases except $x = y = 0$, where we do not have polar coordinates. In practice, rather than remembering all of these cases, it is usually more convenient to draw a picture and find out what range to expect for t.

Example 1.3.1. Find the Cartesian coordinates of the point with polar coordinates $(3, \pi/3)$.
Solution. This is

$$x = 3\cos\frac{\pi}{3} = \frac{3}{2},$$
$$y = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}.$$

Example 1.3.2. Find the polar coordinates of the point with Cartesian coordinates $(3, -4)$.
Solution. We have $r = \sqrt{9+16} = 5$. The point is in the fourth quadrant, so we expect that $3\pi/2 < t < 2\pi$. As we have

$$-\pi/2 < \arctan\frac{-4}{3} < 0,$$

we should add 2π. So

$$t = \arctan\frac{-4}{3} + 2\pi = 2\pi - \arctan\frac{4}{3}.$$

1.3.2 Solving trigonometric equations

Suppose that you want to solve the equation

$$3\cos t + 2\sin t = 2.$$

Should we use the arcsine or the arccosine here, or something else?

More generally, suppose that A, B, C are three given numbers and we want to solve

$$A\cos t + B\sin t = C. \tag{1.3.3}$$

The idea is to use the addition formulas for the sine and cosine. Try to find r and θ such that

$$A\cos t + B\sin t = r\cos(t+\theta).$$

By the addition formula, this works if

$$A = r\cos\theta \quad \text{and} \quad B = -r\sin\theta.$$

But this means that (r, θ) are the polar coordinates of the point with Euclidean coordinates $(A, -B)$. We know how to solve for r and θ by the previous section, even if it can be a bit complicated. Having done that, we have the equation

$$r\cos(t+\theta) = C.$$

This is called the *harmonic form* of equation (1.3.3).

We can now solve it. Dividing by r, we obtain $\cos(t+\theta) = C/r$. Hence $t + \theta = \pm\arccos(C/r) + 2\pi n$ for some integer n, and

$$t = \pm\arccos(C/r) + 2\pi n - \theta.$$

Example 1.3.3. Solve the equation

$$3\cos t + 4\sin t = 4.$$

Solution. We first need to find the polar coordinates of the point $(3, -4)$, which we have done in Example 1.3.2: we have $r = 5$ and $\theta = \pi - \arctan\frac{4}{3}$. Thus the equation becomes

$$\cos\left(t + \pi - \arctan\frac{4}{3}\right) = \frac{4}{5}.$$

So the solutions are of the form

$$t = \pm\arccos\frac{4}{5} + \arctan\frac{4}{3} - \pi + 2\pi n$$

for any integer n.

The above equation is still fairly simple. In general, when you need to solve an equation involving trigonometric functions, you need to keep the trigonometric identities from section 1.2.4 in mind, because they often allow you to simplify the equation. Also remember that you can in general expect many solutions.

Example 1.3.4. Solve the equation

$$\cos^2 x - 3\cos x + 2 = 0.$$

Solution. Substitute $u = \cos x$ to obtain the equation

$$u^2 - 3u + 2 = 0.$$

This equation we can solve, using the factorisation $u^2 - 3u + 2 = (u-2)(u-1)$. So $u = 2$ or $u = 1$. There is no number x such that $\cos x = 2$, so we can rule out one of the solutions. This leaves $u = 1$ and therefore $\cos x = 1$. The solutions of this equation are of the form $x = 2\pi n$ for an integer n.

Example 1.3.5. Solve the equation

$$3 - \sin^2 x - 3\cos x = 0.$$

Solution. Using the identity $\cos^2 x + \sin^2 x = 1$, we can reformulate the equation as follows:

$$\cos^2 x - 3\cos x + 2 = 0.$$

This is the equation from the previous example and we obtain the same solutions.

Example 1.3.6. Solve the equation

$$\cos(2x) - 2\sin x = 1.$$

Solution. We first use the addition formula and the formula from Pythagoras' theorem to rewrite

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

Hence the equation becomes

$$1 - 2\sin^2 x - 2\sin x = 1,$$

which is equivalent to

$$\sin^2 x + \sin x = 0.$$

The substitution $u = \sin x$ yields $0 = u^2 + u = u(u+1)$. So $u = 0$ or $u = -1$. If $u = 0$, then $\sin x = 0$, so $x = n\pi$ for an integer n. If $u = -1$, then $\sin x = -1$, so $x = m\pi - \frac{\pi}{2}$ for an integer m. All of these are solutions of the equation.

1.4 Limits

Sometimes a quantity approaches a certain value without necessarily ever reaching it. For example, consider a radioactive material with half-life T. That means after time T, the quantity of the material is reduced by half through radioactive decay. After time $2T$, there is a quarter of the original quantity left, after $3T$ an eighth, and so on. We may prescribe any acceptable level, and as long as it is positive and we are willing to wait long enough, the remaining quantity will eventually be below. In particular, it approaches 0 as the time tends to ∞.

More generally, consider a function f of the independent variable x. If there exists a number L such that $f(x)$ approaches L as x tends to ∞, in the sense that the distance between $f(x)$ and L will eventually remain smaller than any prescribed positive error level,[2] then we say that L is the *limit* of $f(x)$ as $x \to \infty$. In symbols, we express this as follows:

$$L = \lim_{x\to\infty} f(x).$$

Sometimes, we also write $f(x) \to L$ as $x \to \infty$.

Example 1.4.1. If $f(x) = 1 + \frac{1}{x}$, then

$$\lim_{x\to\infty} f(x) = 1,$$

since the quantity $\frac{1}{x}$ can be made arbitrarily small in absolute value by increasing x sufficiently. Fig. 1.4.1 illustrates this.

[2] More formally, the condition is the following. For every given number $\epsilon > 0$ there exists a corresponding number $R > 0$ such that for all $x > R$, the inequality $|f(x) - L| < \epsilon$ is satisfied.

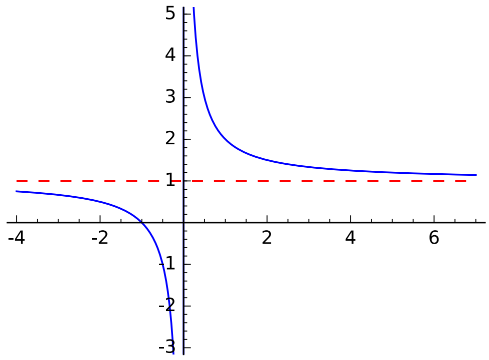

Figure 1.4.1: The graph of the function f with $f(x) = 1 + \frac{1}{x}$

We have limits not just for $x \to \infty$. The expression

$$\lim_{x \to -\infty} f(x),$$

if it exists, is defined similarly, considering values of x that are large in magnitude but negative. Furthermore, we can define limits when x approaches a finite number.

Suppose that a and L are numbers such that $f(x)$ approaches L as x approaches a, in the sense that the distance between $f(x)$ and L shrinks below any prescribed positive error level as soon as x is sufficiently close to a.[3] Then we say that L is the limit of $f(x)$ as $x \to a$ and we write

$$L = \lim_{x \to a} f(x).$$

Example 1.4.2. Consider the function f given by $f(x) = \sqrt{|x|}$ (see Fig. 1.4.2). Since the distance between $f(x)$ and 0 is always $\sqrt{|x|}$, which becomes arbitrarily small when $|x|$ is small enough, we have

$$\lim_{x \to 0} f(x) = 0.$$

Note that when we determine $\lim_{x \to a} f(x)$, then the value $f(a)$ is irrelevant. It need not even be defined.

Sometimes we take *one-sided limits*. If $f(x)$ approaches L as x approaches a from above, meaning that $x \to a$ but at the same time $x > a$, then we write

$$L = \lim_{x \to a^+} f(x).$$

[3] For any given $\epsilon > 0$ there exists $h > 0$ such that for all x with $0 < |x - a| < h$, the inequality $|f(x) - L| < \epsilon$ is satisfied.

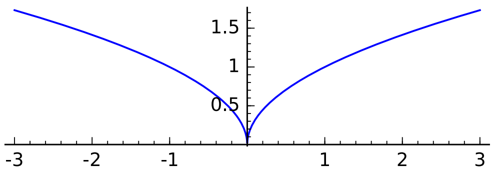

Figure 1.4.2: The graph of the function f with $f(x) = \sqrt{|x|}$

If $f(x)$ approaches L as x approaches a from below (i.e., $x \to a$ while $x < a$), then we write

$$L = \lim_{x \to a^-} f(x).$$

Example 1.4.3. Suppose that $f(x) = \frac{x}{|x|}$. Then we have $f(x) = 1$ for all $x > 0$ and $f(x) = -1$ for all $x < 0$. Therefore,

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1.$$

In addition to these limits, we also consider the situation where $f(x)$ grows beyond all bounds when $x \to a$, where a may be a finite number or $a = \infty$ or $a = -\infty$. If this is the case then we write

$$\lim_{x \to a} f(x) = \infty.$$

If $f(x)$ decreases below any bound as $x \to a$, then

$$\lim_{x \to a} f(x) = -\infty.$$

These are not limits in the strictest sense, but often we can treat these situations the same way.

Example 1.4.4. The following is easy to see by inspecting Fig. 1.4.3:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

If we need to find the limit of a function that is composed of simpler functions, then it is typically sufficient to find the limits of the constituent parts. Suppose that we have two functions f and g, and suppose that a is a finite number or $a = \pm\infty$. Then the following statements hold true.

- $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$;
- $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$;
- $\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$;

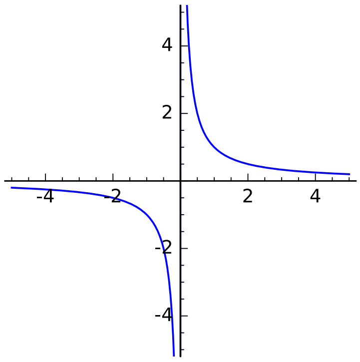

Figure 1.4.3: The graph of the function f with $f(x) = \frac{1}{x}$

- if $\lim_{x\to a} g(x) \neq 0$, then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

This applies when we have actual limits, but in most cases also for ∞ and $-\infty$, provided that we interpret the resulting expressions correctly. For example, we use the convention that $L + \infty = \infty$ and $\frac{L}{\infty} = 0$ for any finite number L. In some cases, however, we have to be more careful. There is no natural way to interpret expressions such as $\infty - \infty$ or $\frac{\infty}{\infty}$, and if they appear, then these rules do not help. (But in the second case, another rule, which we will discuss in Sect. 2.4.2, often helps.)

The same rules apply to one-sided limits.

Example 1.4.5. We know that $\lim_{x\to\infty} \frac{1}{x} = 0$. Therefore,

$$\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{1}{x} = 0$$

Example 1.4.6. Find

$$\lim_{x\to\infty} \frac{x^2 - x + 3}{2x^2 - 7}.$$

Solution. We have

$$\lim_{x\to\infty} \frac{x^2 - x + 3}{2x^2 - 7} = \lim_{x\to\infty} \frac{1 - \frac{1}{x} + \frac{3}{x^2}}{2 - \frac{7}{x^2}} = \frac{1 - \lim_{x\to\infty} \frac{1}{x} + 3 \lim \frac{1}{x^2}}{2 - 7 \lim_{x\to\infty} \frac{1}{x^2}} = \frac{1}{2}.$$

There is also a rule for limits of the form $\lim_{x\to a} f(g(x))$, where f and g are two given functions, but this requires some conditions on f that we need to discuss first.

A function is called *continuous* if a small change of the independent variable will result only in a small change of the dependent variable. Geometrically this means that the graph of the function has no gaps. Nearly all of the functions discussed so far are continuous.

Now suppose that f is a continuous function and g is any function such that $\lim_{x\to a} g(x)$ exists, where again, a is a finite number or $\pm\infty$. Then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Again, the rule applies to one-sided limits as well.

Example 1.4.7. For any base $b > 0$, the exponential function given by $y = b^x$ is continuous. Hence

$$\lim_{x\to\infty} b^{1/x} = b^{\lim_{x\to\infty} 1/x} = b^0 = 1.$$

1.5 The bisection method

So far we have only seen examples of equations that we were able to solve exactly. This is only because the examples were chosen to demonstrate certain solution methods. For even moderately complicated equations, it is rare that we can solve them exactly and we normally have to make do with a numerical approximation. But often we can say with certainty that a suitably constructed number is in fact a good approximation of an actual solution, and even give a bound for the error.

The following is a simple scheme to find approximate solutions and error bounds. There are more efficient methods (in terms of required computation time), but what makes this one useful, especially for theoretical purposes, is that it gives you a high degree of certainty. The method works for all continuous functions.

The basis of the bisection method is the *intermediate value theorem*, which states the following. Suppose that f is a continuous function and y a fixed number. Furthermore, suppose that a, b are two numbers with $a < b$, such that $f(a)$ and $f(b)$ are on different sides of y. That is, either

- $f(a) < y$ and $f(b) > y$, or
- $f(a) > y$ and $f(b) < y$.

Then there exists a number c between a and b (that is, with $a < c < b$) such that $f(c) = y$.

This is a useful statement because it tells us that in certain circumstances, there is definitely a solution of the equation $f(x) = y$ in the interval (a, b). The bisection method now works as follows.

Step 1 Find two numbers a_0 and b_0 with $a_0 < b_0$ such that either $f(a_0) < y < f(b_0)$ or $f(a_0) > y > f(b_0)$. (So a solution of the equation $f(x) = y$ is guaranteed between a_0 and b_0.)

Step 2 Define the number $c_0 = \frac{1}{2}(a_0 + b_0)$.

- If $f(c_0) = y$, then we have found an exact solution.
- If $f(a_0) < y < f(c_0)$ or $f(a_0) > y > f(c_0)$, then define $a_1 = a_0$ and $b_1 = c_0$.
- If $f(c_0) < y < f(b_0)$ or $f(c_0) > y > f(b_0)$, then define $a_1 = c_0$ and $b_1 = b_0$.

Step 3 If $b_1 - a_1$ is smaller than a tolerable error bound, use any number in the interval (a_1, b_1) as an approximate solution. Otherwise, go back to Step 2 and repeat with a_1 and b_1 instead of a_0 and b_0.

Example 1.5.1. Find an approximate solution of the equation

$$\ln x = 2$$

with error smaller than $\frac{1}{4}$.
Solution. We compute $\ln 7 = 1.945910149$ and $\ln 8 = 2.079441542$. So there is a solutions somewhere between 7 and 8. Next we compute $\ln 7.5 = 2.014903021$. So the solution must be in the interval $(7, 7.5)$. As $\ln 7.25 = 1.981001469$, it's in fact in the interval $(7.25, 7.5)$. We may use as an approximate solution:

$$x \approx 7.375.$$

Chapter 2

Differentiation

2.1 Fundamentals

2.1.1 Definition

Consider the graph of a function f. For a given number x, we have a point with coordinates $(x, f(x))$ on the graph. If this graph is a sufficiently nice curve, then there is a unique tangent line at this point. The slope of the tangent line is called the *derivative* of f at x and denoted by $f'(x)$. (Different notation for the same thing will be discussed later.)

Since a tangent line can be a tricky thing to work with, we also consider secant lines. These are lines passing through two points of the graph (see Fig. 2.1.1). More specifically, suppose that h is a number different from 0.

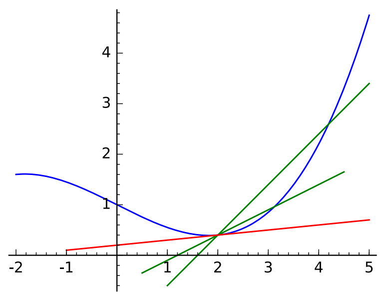

Figure 2.1.1: Two secant lines (green) and a tangent line (red)

Then $(x + h, f(x + h))$ is another point on the graph, and the two points determine a unique secant line, the slope of which is

$$\frac{f(x+h) - f(x)}{h}.$$

This is called a *difference quotient*. Assuming that f is nice enough to possess a derivative at x, this expression will approach $f'(x)$ when h tends to 0. That is,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 2.1.1. Consider the function defined by $f(x) = x^2$. We have $f(x+h) = x^2 + 2hx + h^2$, so

$$\frac{f(x+h) - f(x)}{h} = 2x + h.$$

When h approaches 0, the last term is small, so the difference quotient will tend to $2x$. So $f'(x) = 2x$.

Not every function has a derivative at every point. For example, consider the function with $f(x) = |x|$. The graph has a corner at $(0, 0)$. The secant lines through $(0, 0)$ and $(h, |h|)$ have slope 1 if $h > 0$ and slope -1 if $h < 0$. There is no number that is approached by both of these, and there is no tangent line to the graph at $(0, 0)$. In other words, this function does not have a derivative at 0.

2.1.2 Notation

Several conventions for the notation of the derivative are in use. Among the most common ways to write the object defined above are:

$$f', \quad \dot{f}, \quad \text{and} \quad \frac{df}{dx}.$$

These have different historical origins and have been introduced by Lagrange, Newton, and Leibniz, respectively. The first one is the most common in theoretical treatments of the derivative. The notation $\dot{f}$ is common in science, especially if a function of time is studied. Leibniz's notation is convenient for certain computations. It is no coincidence that it resembles a quotient; in fact, it reminds us that the derivative is computed through approximations by difference quotients as in section 2.1.1. Nevertheless, when using this notation, it is worth keeping in mind that the quotient structure of the expression $\frac{df}{dx}$ is merely symbolic. It does not represent an actual quotient, but rather a limit of difference quotients.

Sometimes the derivative of a function is differentiated again. The result is the *second derivative*, which is denoted by f'', $\ddot{f}$, or $\frac{d^2f}{dx^2}$. Similarly, the third derivative is the derivative of the second derivative and denoted f''' or $\dddot{f}$ or $\frac{d^3f}{dx^3}$. For even higher derivatives, the first two types of notation become too cumbersome and we write $f^{(n)}$ or $\frac{d^nf}{dx^n}$ for the n-th derivative.

2.1.3 Derivatives of a few simple functions

As we have seen in Sect. 2.1.1, we can compute a derivative by considering difference quotients. We don't need to do this for all functions, however, because once we know the derivatives of certain functions, we can figure out the derivatives of other functions composed from these by simple rules.

First we consider the simplest functions of all. For a fixed number a, let f be the constant function with $f(x) = a$ for all x. Then for $h \neq 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{a-a}{h} = 0. \tag{2.1.1}$$

So $f'(x) = 0$. In other words, a constant function has the derivative 0.

We have already seen the derivative of the function with $f(x) = x^2$. But more fundamental is the function f given by $f(x) = x$. For $h \neq 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{x+h-x}{h} = \frac{h}{h} = 1.$$

Hence for this function, we have $f'(x) = 1$. Every polynomial can be composed from this function and constants. We will see later how this helps to determine the derivative of any polynomial.

Now consider exponential functions. Let $a > 0$ and suppose that the function f is defined by $f(x) = a^x$. Then for $h \neq 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{a^{x+h} - a^x}{h} = \frac{a^x a^h - a^x}{h} = a^x \frac{a^h - 1}{h} = a^x \frac{f(h) - 1}{h}.$$

This does not immediately tell us what the derivative is, but it does give us some information about it. Note that the expression

$$\frac{f(h) - 1}{h}$$

is the difference quotient of f at 0. So while the left-hand side will approach $f'(x)$ when h approaches 0, the right-hand side will approach $a^x f'(0)$. That is, we have $f'(x) = a^x f'(0)$. Here $f'(0)$ is just a constant, albeit for the moment an unknown one. Writing for convenience $C = f'(0)$, we have

$$f'(x) = Cf(x).$$

In other words, the derivative of an exponential function coincides with the function up to a constant. For the base e, that constant is 1. So

$$\frac{d}{dx} e^x = e^x.$$

Now consider the trigonometric function sin. For $h \neq 0$, we use the addition formula to compute

$$\begin{aligned} \frac{\sin(x+h) - \sin x}{h} &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}. \end{aligned}$$

It can be shown that $\frac{\cos h - 1}{h}$ approaches 0 and $\frac{\sin h}{h}$ approaches 1 as h tends to 0. Therefore, we have

$$\frac{d}{dx}\sin x = \cos x.$$

Similarly, we compute

$$\begin{aligned}\frac{\cos(x+h) - \cos x}{h} &= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}.\end{aligned}$$

Hence

$$\frac{d}{dx}\cos x = -\sin x.$$

Other trigonometric functions can be composed from the sine and the cosine, and we will find their derivatives later.

2.2 Differentiation rules

In this section, we will see how to differentiate a an expression composed from simpler functions.

2.2.1 The sum rule

Our first rule, the *sum rule*, is very simple: the derivative of a sum is the sum of the derivatives of the summands. In other words, if f and g are two functions and a third function h is given by $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$. The rule may also be expressed by the formula

$$(f+g)'(x) = f'(x) + g'(x)$$

or

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}.$$

Example 2.2.1. Find the derivative of the function f with $f(x) = x^2 + x$.
Solution. We know that $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}x = 1$, so

$$\frac{d}{dx}(x^2 + x) = 2x + 1.$$

2.2.2 The product rule

The *product rule*, as the name suggests, is about products of functions. (It is *not* as simple as the sum rule, and the false analogy with the latter is a source of many mistakes.) For two given functions f and g, the rule is

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

or, in different notation,

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Why do we have this expression? Let's look at the difference quotients. For $h \neq 0$, we have

$$\begin{aligned} &\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &\quad = \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &\quad = \frac{f(x+h) - f(x)}{h}g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}. \end{aligned}$$

Keeping in mind that the difference quotients will tend to the corresponding derivatives, and observing that $g(x+h)$ will approach $g(x)$ when we let h approach 0, we conclude that the above formula holds true.

As an application, we can now compute the value of $\frac{d}{dx}x^2$ from $\frac{d}{dx}x$, rather than deriving it from first principles. As $x^2 = x \cdot x$, we have

$$\frac{d}{dx}x^2 = 1 \cdot x + x \cdot 1 = 2x.$$

Similarly, we have

$$\frac{d}{dx}x^3 = \frac{d}{dx}(x \cdot x^2) = 1 \cdot x^2 + x \cdot 2x = 3x^2.$$

More generally, for any positive integer n, we obtain

$$\frac{d}{dx}x^n = nx^{n-1}.$$

(The formula is actually true not just for positive integers, but it's easiest to see in this case.)

Example 2.2.2. Differentiate $x^2 \sin x$.
Solution. By the product rule,

$$\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x.$$

2.2.3 The quotient rule

For the derivative of a quotient, we have the formula

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2};$$

that is,

$$\frac{d}{dx}(f/g) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}.$$

Example 2.2.3. Find the derivative of the function tan.
Solution. We know that $\tan x = \frac{\sin x}{\cos x}$. Hence

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

By Pythagoras' theorem, we have $\cos^2 x + \sin^2 x = 1$. Hence we have the alternative representation

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}.$$

Example 2.2.4. Differentiate $\frac{e^x}{x}$.
Solution. By the quotient rule,

$$\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{xe^x - e^x}{x^2} = \frac{x-1}{x^2}e^x.$$

2.2.4 The chain rule

This rule is about the composition of functions in the sense of using the output of one function as the input of another. Suppose that we have two functions f and g. Then we can define another function h with $h(x) = f(g(x))$. Then we have

$$h'(x) = f'(g(x))g'(x).$$

This can be expressed very conveniently with Leibniz's notation. For this purpose, we write $u = g(x)$ and $y = f(u) = h(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Written like this, the formula looks obvious. (But remember that these are not actual fractions. This may look like the reduction of a fraction, but it's really something more complicated.)

The chain rule can be extended to chains of more than two functions. For example, in the case of three functions, we have

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}.$$

This is a convenient form to write it in, but it suppresses some information. In particular, it's not immediately clear in which way these quantities depend on each other, and this can lead to some confusion. To make everything more explicit: we have three functions here, say f, g, and h, and the composition i with $i(x) = f(g(h(x)))$. We write $u = h(x)$, $v = g(u) = g(h(x))$, and $y = f(v) = f(g(h(x)))$. The formula can also be written as follows:

$$i'(x) = f'(g(h(x)))g'(h(x))h'(x).$$

Example 2.2.5. Given a constant c, differentiate e^{cx}.
Solution. This is the composition of the exponential function (with base e) and the function f with $f(x) = cx$, the derivative of which is $f'(x) = c$. So

$$\frac{d}{dx}e^{cx} = \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x) = ce^{cx}.$$

(In Leibniz notation, write $u = cx$ and $y = e^u$. Then

$$\frac{d}{dx}e^{cx} = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot c = ce^{cx}.)$$

Using this example, we may now also give a more explicit expression for the derivatives of exponential functions. Let $a > 0$ and consider a^x. Recall that

$$a^x = e^{x \ln a}.$$

Thus setting $c = \ln a$ in Example 2.2.5, we find

$$\frac{d}{dx}a^x = e^{x \ln a} \ln a = a^x \ln a.$$

Example 2.2.6. Differentiate $\cos^2 x$.
Solution. Set $u = \cos x$ and $y = u^2$. Then

$$\frac{d}{dx}\cos^2 x = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u(-\sin x) = -2x \sin x \cos x.$$

Example 2.2.7. Differentiate $\cos(x^2)$.
Solution. Set $u = x^2$ and $y = \cos u$. Then

$$\frac{d}{dx}\cos(x^2) = \frac{dy}{du}\frac{du}{dx} = -\sin u \cdot 2x = -2x \sin(x^2).$$

Example 2.2.8. Differentiate $e^{\sin^3 x}$.
Solution. Writing $u = \sin x$, $v = u^3$, and $y = e^v$, we find

$$\frac{d}{dx}e^{\sin^3 x} = \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = e^v \cdot 3u^2 \cdot \cos x = 3e^{\sin^3 x} \sin^2 x \cos x.$$

2.3 Further differentiation techniques

In this section we discuss how to find the slope of a curve that is not necessarily given by a simple expression as in the previous examples or where an application of the rules would be too laborious.

2.3.1 Logarithmic differentiation

Suppose that we want to differentiate a product of many functions. Then we may use the product rule several times, but this can become complicated. Often the following method is easier.

Let f be the product of the functions $f_1, f_2, \ldots, f_n$. That is,

$$f(x) = f_1(x)f_2(x)\cdots f_n(x).$$

Apply the natural logarithm to both sides of this equation. Then by the rules for the logarithm,

$$\ln f(x) = \ln f_1(x) + \ln f_2(x) + \cdots + \ln f_n(x).$$

Now we can use the sum rule rather than the product rule. Differentiating with the help of the chain rule, we obtain

$$\frac{f'(x)}{f(x)} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}.$$

Therefore,

$$f'(x) = f_1(x)\cdots f_n(x)\left(\frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}\right).$$

Example 2.3.1. Differentiate the expression $y = \sqrt{x}e^x \cos x \cosh x$.
Solution. Using logarithmic differentiation, we find

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{x}e^x \cos x \cosh x \left(\frac{1}{2\sqrt{x}\cdot\sqrt{x}} + \frac{e^x}{e^x} + \frac{-\sin x}{\cos x} + \frac{\sinh x}{\cosh x}\right) \\ &= \sqrt{x}e^x \cos x \cosh x \left(\frac{1}{2x} + 1 - \tan x + \tanh x\right).\end{aligned}$$

2.3.2 Derivatives of inverse functions

We have seen some functions (namely logarithms and inverse trigonometric functions) that we obtain by solving an equation of the form $y = f(x)$ for x. The resulting function is called the *inverse* of f and denoted by f^{-1}. If we know the derivatives of f, then we can easily compute the derivatives of the inverse as well by the following formula, which is most easily represented in Leibniz's notation. Write $y = f(x)$, so that $x = f^{-1}(y)$. Then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

Written differently, that is

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Example 2.3.2. Differentiate $\sqrt{y}$.
Solution. The root is the inverse of the function $y = x^2$. So

$$\frac{d}{dy}\sqrt{y} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$

Note that we can write $\sqrt{y} = y^{1/2}$, and then the result from this example is consistent with the formula

$$\frac{d}{dy}y^n = ny^{n-1},$$

which we have seen earlier, although n is no longer an integer here.

We can use the above formula to find derivatives of logarithms and inverse trigonometric functions. For $a > 0$, we write $y = a^x$, so that $x = \log_a y$. Then $\frac{dy}{dx} = a^x \ln a$, as we have seen earlier. Hence

$$\frac{d}{dy}\log_a y = \frac{1}{a^x \ln a} = \frac{1}{y \ln a}.$$

For base e, we have the particular case

$$\frac{d}{dy}\ln y = \frac{1}{y}.$$

Now suppose that $y = \sin x$ and $x = \arcsin y$. Then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x} = \frac{1}{\sqrt{\cos^2 x}} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

That is,

$$\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}}.$$

Similarly,

$$\frac{d}{dy}\arccos y = -\frac{1}{\sqrt{1-y^2}}.$$

Finally, if we have $y = \tan x$ and $x = \arctan y$, then $\frac{dy}{dx} = 1+\tan^2 x = 1+y^2$. Hence

$$\frac{d}{dy}\arctan y = \frac{1}{1+y^2}.$$

2.3.3 Parametric differentiation

Sometimes a curve in the plane is given not as the graph of a function, but rather through a *parametrisation*. This is a pair of functions x and y describing a motion along the curve as follows: if we think of the independent variable t as time, then we reach the point with coordinates $(x(t), y(t))$ at the time t. We have already seen an example for this: when introducing the trigonometric functions, we considered the functions given by $x(t) = \cos t$ and $y(t) = \sin t$. Because we think of t as time here, we denote the derivatives by $\dot{x}$ and $\dot{y}$.

A curve like this does not necessarily correspond to the graph of a function. But if it does, say if y is a function of x, then

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}},$$

provided that these derivatives exist and $\dot{x}$ does not vanish. (Writing the right-hand side in Leibniz notation may again help you memorise this.)

Example 2.3.3. Find the slope at the point $(1, 0)$ of the curve parametrised by $x(t) = \cos t + t$ and $y(t) = t^4 + t$.
Solution. We have $\dot{x}(t) = -\sin t + 1$ and $\dot{y}(t) = 4t^3 + 1$. The point $(1, 0)$ corresponds to a time t with $x(t) = 1$ and $y(t) = 0$; that is,

$$\begin{aligned} \cos t + t &= 1, \\ t^4 + t &= 0. \end{aligned}$$

The second equation gives $t(t^3 + 1) = 0$. We have two solutions: $t = 0$ and $t = -1$. But the second one does not satisfy the first equation. So we need to consider the time $t = 0$ here. Now $\dot{x}(0) = 1$ and $\dot{y}(0) = 1$. Therefore, the slope is

$$\frac{dy}{dx}(1) = \frac{\dot{y}(0)}{\dot{x}(0)} = 1.$$

2.3.4 Implicit differentiation

Sometimes a curve is given only implicitly through an equation in x and y. For example, the equation $x^2 + y^2 = 1$ describes the unit circle. While in this case, we can solve for y and then differentiate (namely, $y = \pm\sqrt{1 - x^2}$ and $\frac{dy}{dx} = \pm\frac{-x}{\sqrt{1-x^2}}$) or we can parametrise as in the previous section, in general it may be difficult to do either. Nevertheless, we can still find the slopes of the curve by the following process, demonstrated with the example $x^2 + y^2 = 1$.

We first differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}1.$$

Using the chain rule for the terms involving y (in this case only one term), we obtain

$$2x + 2y\frac{dy}{dx} = 0.$$

Now we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

(You can check easily that the result agrees with the calculations at the beginning of this section.)

Example 2.3.4. Consider the curve implicitly given by the equation

$$ye^x + xe^y = 1.$$

The points $(1, 0)$ and $(0, 1)$ lie on this curve. Find the slopes at both points.
Solution. We differentiate both sides of the equation:

$$\frac{dy}{dx}e^x + ye^x + e^y + xe^y\frac{dy}{dx} = 0.$$

We can simplify as follows:

$$\frac{dy}{dx}(e^x + xe^y) + ye^x + e^y = 0.$$

Now we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{ye^x + e^y}{e^x + xe^y}.$$

If we insert $x = 1$ and $y = 0$, we obtain $\frac{dy}{dx} = \frac{1}{e+1}$. For $x = 0$ and $y = 1$, we obtain $\frac{dy}{dx} = e + 1$.

2.4 Applications

2.4.1 Maxima and minima

Suppose that you are given a function and want to understand its behaviour. One of the most notable features of a function are its local maxima and minima, so it helps to find them.

Given a function f and a number c, we say that f has a *local maximum* at c if $f(x) \leq f(c)$ for all x in the immediate neighbourhood of c. We say that f has a *local minimum* at c if $f(x) \geq f(c)$ for all x in the immediate neighbourhood of c.

If f has a local minimum or a local maximum at c and has a derivative at c, then it must necessarily satisfy $f'(c) = 0$. Therefore, if we look for local maxima and local minima, then we may differentiate and solve the equation

$$f'(x) = 0$$

for x. This, together with the set of points where the derivative does *not* exist, gives a set of candidates for local maxima and minima. But there may be solutions of the equation that are neither maxima nor minima.

Example 2.4.1. Consider the function given by $f(x) = (1-x^2)^2$. We have $f'(x) = -4x(1-x^2) = -4x(1-x)(1+x)$. The solutions of the equation $f'(x) = 0$ are $x = -1$, $x = 0$, and $x = 1$. Since $f(x) \geq 0$ for all x (as it is a square), while $f(-1) = f(1) = 0$, we have local minima at -1 and 1. We will see later that we have a local maximum at 0.

Example 2.4.2. Consider the function given by $f(x) = x^3$. Its derivative is $f'(x) = 3x^2$, which vanishes at 0 and nowhere else. We have $f(0) = 0$, but this function has positive and negative values in the immediate neighbourhood of 0. So here we have no local minima or maxima.

Example 2.4.3. Consider the function given by $f(x) = |x|$. The derivative is $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$. At 0, the derivative does not exist. The function satisfies $f(0) = 0$ and $f(x) \geq 0$ for all x. Therefore, it has a local minimum at 0.

Now that we have these candidates, how do we distinguish local maxima and local minima from each other and from all the 'false' candidates? This is not always easy, but in many cases, it helps to differentiate again.

Suppose that we have a point c where both the first and the second derivatives exist. If $f'(c) = 0$ and $f''(c) < 0$, then it is guaranteed that c is a local maximum of f. If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum. However, if $f'(c) = 0$ and $f''(c) = 0$, then we are none the wiser.

Example 2.4.4. Consider the function f with $f(x) = (1-x^2)^2$ again. We have $f''(x) = 12x^2 - 4$. As $f'(0) = 0$ and $f''(0) = -4 < 0$, it has a local maximum at 0. (We have already seen in example 2.4.1 that f has local minima at ± 1, but if necessary, we could verify it with this method as well.)

Example 2.4.5. Find the local maxima and minima of the function given by $f(x) = \sin x$.

Solution. We have $f'(x) = \cos x$ and $f''(x) = -\sin x$. The equation $f'(x) = 0$ (that is, $\cos x = 0$) has solutions of the form $\frac{\pi}{2} + n\pi$ for all integers n. Now we test the sign of the second derivative

$$f''\left(\frac{\pi}{2} + n\pi\right) = -\sin\left(\frac{\pi}{2} + n\pi\right)$$

We note that $-\sin(\frac{\pi}{2}) = -1$ and $-\sin(-\frac{\pi}{2}) = 1$. Therefore, we have a local maximum at $\frac{\pi}{2}$ and a local minimum at $-\frac{\pi}{2}$. By the periodicity, we also have local maxima at $\frac{\pi}{2} + 2\pi n$ and local minima at $-\frac{\pi}{2} + 2\pi n$ for all integers n.

2.4.2 L'Hopital's rule

Finding limits can be tricky when we have fractions that give the formal limits $\frac{\infty}{\infty}$ or $\frac{0}{0}$. Differentiation can help here. Suppose that you want to find a limit of the form

$$\lim_{x\to c} \frac{f(x)}{g(x)},$$

where f and g are two functions and c may be a finite number or $c = \infty$ or $c = -\infty$. If either

- $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ or
- $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$,

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Example 2.4.6. We have

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example 2.4.7. Find

$$\lim_{x\to\infty} \frac{x}{\ln x}.$$

Solution. We are in a situation where we can apply l'Hopital's rule. So

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Example 2.4.8. Find

$$\lim_{x\to\pi} \frac{\cos x - 1}{(x-\pi)^2}.$$

Solution. Again we can apply l'Hopital's rule, as $\cos x - 1 \to 0$ and $(x-\pi)^2 \to 0$ as $x \to \pi$. We obtain

$$\lim_{x\to\pi} \frac{\cos x - 1}{(x-\pi)^2} = \lim_{x\to\pi} \frac{-\sin x}{2(x-\pi)}.$$

It is still not obvious what the limit is, but we can use the rule again:

$$\lim_{x\to\pi} \frac{-\sin x}{2(x-\pi)} = \lim_{x\to\pi} \frac{-\cos x}{2} = \frac{1}{2}.$$

Hence

$$\lim_{x\to\pi} \frac{\cos x - 1}{(x-\pi)^2} = \frac{1}{2}.$$

2.4.3 Newton's method

The following is another method to solve an equation approximately. It is more efficient than the bisection method that we have seen in Sect. 1.5, but it has a disadvantage, too: unlike the bisection method, it comes with no guarantees. It works well most of the time, but sometimes it doesn't, and it is not always easy to predict its behaviour.

The basic idea is as follows. Suppose that you want to solve the equation $f(x) = 0$, where f is a given function. This means finding a point where the graph of f intersects the x-axis. We first take an initial guess for the solution, say x_0. Then we approximate the graph of f by the tangent line at the corresponding point, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Unless $f'(x_0) = 0$ (in which case this tangent line is parallel to the x-axis), there exists a unique intersection with the x-axis. We can compute where it occurs, namely, at

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We expect that this is a better approximation of the solution than x_0. We keep repeating the same steps until we are satisfied that we have a good approximation. That is, we define recursively

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and stop after a suitable number of steps.

Example 2.4.9. Find an approximate solution of the equation

$$\ln x = 2.$$

Solution. Here we use the function with $f(x) = \ln x - 2$, which satisfies $f'(x) = \frac{1}{x}$. So the above formula becomes

$$x_{n+1} = x_n - x_n(\ln x_n - 2) = x_n(3 - \ln x_n).$$

Beginning with $x_0 = 8$, we compute

$$\begin{aligned} x_1 &\approx 7.364467667, \\ x_2 &\approx 7.389015142, \\ x_3 &\approx 7.389056099. \end{aligned}$$

The result has not changed up to the fourth decimal place in the last step, so we can be reasonably confident that this is a good approximation to that precision (but it's actually much better).

For comparison, with the bisection method, beginning with an interval of length 1, it would have taken 10 steps to reach this precision. But we would have been certain that our approximation is really that good, whereas here, it's an educated guess.

2.5 Taylor polynomials

Consider a function f and a number a. What we have used for Newton's method is the fact that the graph of f can be approximated by its tangent line near $(a, f(a))$. In other words, the function f is approximated by a linear function near a, which is given by the formula

$$y = f(a) + f'(a)(x - a).$$

This is the unique linear function with the value $f(a)$ at a and slope $f'(a)$.

We can hope to find even better approximations of f by using higher order polynomials. This is indeed possible, and in order to find these polynomials, we have to make sure that as many derivatives as possible agree with the derivatives of f at a.

Given a positive integer n, consider the formula

$$T_{n,a}(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{2 \cdot 3 \cdots n}(x - a)^n,$$

or in shorthand notation,

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

(Here $k! = 1 \cdot 2 \cdot 3 \cdots k$, with the convention that $0! = 1$.) The function $T_{n,a}$ defined thus is called the *Taylor polynomial* or *Taylor expansion* for f of order n at a. It has the property that

$$T_{n,a}^{(k)}(a) = f^{(k)}(a)$$

for $k = 0, \ldots, n$ and is the polynomial of order n providing the best possible approximation of f near a.

In the special case $a = 0$, the Taylor polynomial is also called *Maclaurin polynomial*.

Example 2.5.1. For the function ln, we have

$$\ln'(x) = \frac{1}{x}, \quad \ln''(x) = -\frac{1}{x^2}, \quad \ln'''(x) = \frac{2}{x^3}, \quad \ln^{(4)}(x) = -\frac{6}{x^4}.$$

Therefore, the fourth order Taylor polynomial for ln at 1 is given by

$$T_{4,1}(x) = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4}.$$

As already mentioned, the Taylor polynomial approximates the function f near a; that is,

$$T_{n,a}(x) \approx f(x)$$

when $x \approx a$. We can also say something about the quality of this approximation. Suppose that

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

represents the error of the approximation and define

$$h_{n,a}(x) = \frac{R_{n,a}(x)}{(x-a)^n}.$$

Then *Taylor's theorem* states the following. If f has derivatives up to order n, then

$$\lim_{x \to a} h_{n,a}(x) = 0.$$

That is, if x is sufficiently close to a, then the error term is even smaller than any multiple of $(x-a)^n$, which is very small itself, especially if n is chosen large. In most cases, the error behaves in fact like $(x-a)^{n+1}$.

Instead of computing the terms of the Taylor polynomials up to a certain order, we can just continue indefinitely and write down an infinite sum, called the *Taylor series* or *infinite Taylor expansion* of f at a:

$$\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

or, in shorthand notation,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

But we first have to think about whether this makes any sense. How can we sum infinitely many terms? In fact, it does not always make sense, but in many cases, we can interpret this series as a limit:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

If this is the case, then we may hope that the approximation for Taylor polynomials turns into equality for Taylor series and we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

But caution: this is not true for all functions, even if they have derivatives of all orders, and not necessarily for all values of x. It is not easy to find out if a given function satisfies the identity, but it is known, for example, for the following functions, giving rise to the following identities (all of them for $a = 0$):

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots && \text{for all } x,\\
\cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots && \text{for all } x,\\
\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots && \text{for all } x,\\
\frac{1}{1+x} &= \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \cdots && \text{for } -1 < x < 1,\\
\ln(1+x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots && \text{for } -1 < x < 1.
\end{aligned}$$

If you know that a function is represented in this way, then you can calculate with Taylor series term by term as you do for polynomials.

Example 2.5.2. Using the above series for $\sin x$, we obtain

$$x \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+1)!} = x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \cdots$$

and this hold for all x again. Moreover,

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots$$

for all $x \neq 0$ (because the left-hand side is meaningless for $x = 0$).

Example 2.5.3. We have

$$\begin{aligned}
\frac{e^x}{1+x} &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - x + x^2 - x^3 + x^4 - \cdots\right)\\
&= 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \cdots
\end{aligned}$$

for $-1 < x < 1$.

Example 2.5.4. Using the expansion for $\frac{1}{1+x}$ and replacing x by x^2, we obtain

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots,$$

which holds for $-1 < x < 1$ as well (because then $0 \le x^2 < 1$).

Example 2.5.5. The derivative of $\ln(1+x)$ is $\frac{1}{1+x}$. This fact is reflected in the Taylor expansions: every term in the expansion of $\frac{1}{1+x}$ is the derivative of the corresponding term for $\ln(1+x)$.

Example 2.5.6. If an unknown function f has the Taylor expansion

$$f(x) = \sum_{k=0}^{\infty} kx^k$$

for $-1 < x < 1$, what is its derivative?
Solution. Without further information, we will be able to give an answer only in terms of the Taylor series again, which is

$$f'(x) = \sum_{k=0}^{\infty} k^2 x^{k-1} = \sum_{\ell=0}^{\infty} (\ell+1)^2 x^{\ell}.$$

This is again true for $-1 < x < 1$. (In the last step of the computation we have substituted $\ell = k - 1$ in order to obtain a nicer expression.)

2.6 Numerical differentiation

With the rules that we've seen so far, we can differentiate any function that is composed of the standard functions for which we already know the derivative (and we also have some methods for implicitly given functions). In practice, however, the functions may not be in the required form or you may even have to deal with functions that you don't know exactly. (For example, you may only know the values at certain data points.) For this reason, you may sometimes have to make do with numerical approximations.

As the derivative is defined as a limit, there is an obvious way to obtain approximations. We have

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{2.6.1}$$

and the approximation gets better the smaller h is chosen. But there is a better approximation, and we can see why using Taylor series.

We assume here that the function f is represented by its Taylor series near x. (This is not true for all functions, but we can still use similar arguments with finite Taylor polynomials otherwise.) Hence

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \cdots . \tag{2.6.2}$$

This implies that

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(x)}{2}h + \frac{f'''(x)}{6}h^2 + \cdots .$$

So when we approximate $f'(x)$ by $\frac{f(x+h)-f(x)}{h}$, the error is

$$\frac{f''(x)}{2}h + \frac{f'''(x)}{6}h^2 + \cdots .$$

When h is small, the first term is dominant and the error is of the order h.

Evaluating the Taylor series at $x-h$, we also obtain

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \cdots . \tag{2.6.3}$$

Using (2.6.2) and (2.6.3) simultaneously, we obtain

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{f'''(x)}{6}h^2 + \cdots .$$

So when we approximate $f'(x)$ by $\frac{f(x+h)-f(x-h)}{2h}$, the error is

$$\frac{f'''(x)}{6}h^2 + \cdots ,$$

which is of order h^2.

Example 2.6.1. The following table gives the values of the above approximations for $f(x) = x^3$ at 1. Note that $f'(1) = 3$.

h	$\frac{1}{h}(f(1+h)-f(1))$	$\frac{1}{2h}(f(x+h)-f(x-h))$
1	7	4
0.1	3.31	3.01
0.01	3.0301	3.0001

Chapter 3

Integration

3.1 Two approaches to integration

3.1.1 Antidifferentiation and the indefinite integral

Consider the following question: for a given function f, is there another function F such that $F' = f$? In other words, is there a way to reverse differentiation? For reasonably nice functions, the answer is typically yes.

Example 3.1.1. Is there a function F such that $F'(x) = x^5 + 6$?
Solution. It is easy to check that $F(x) = \frac{x^6}{6} + 6x$ provides a solution.

If F is a function that satisfies $F' = f$, then we say that F is an *antiderivative* of f (also called *primitive* of f). In practice, the difficulty is usually not in deciding whether an antiderivative exists, but rather to find it. We will see some methods for this later. First, we discuss a few general facts.

Note that antiderivatives are not unique. In Example 3.1.1, we could have given the answer $\frac{x^6}{6} + 6x + 10$, and you can write down many more similar answers. In fact, whenever F is an antiderivative of a function f and C is a constant, then $F + C$ is another antiderivative of f. We can verify this by differentiation: if $F' = f$, then

$$\frac{d}{dx}(F(x) + C) = F'(x) = f(x)$$

for all x by the summation rule and (2.1.1). Moreover, it is also true that any two antiderivatives of a function differ by a constant. Therefore, by combining these two facts, we obtain the following statement.

If F is an antiderivative of a given function f, then *all* antiderivatives of f are of the form $F + C$, where C is a constant.

The *indefinite integral* of f is a symbolic representation of a generic antiderivative. When we write

$$\int f(x)\, dx,$$

we mean an unspecified antiderivative of f, with the understanding that adding various constants will produce *all* antiderivatives. When evaluating an indefinite integral, we usually use the symbol C to represent this unspecified constant. For example,

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

Example 3.1.2. Evaluate

$$\int (x^5 + 6)\,dx.$$

Solution. This is

$$\int (x^5 + 6)\,dx = \frac{x^6}{6} + 6x + C.$$

3.1.2 Area and the definite integral

Consider the graph of a function f, describing a curve in the plane. Given an interval (a, b) (meaning that a and b are two numbers with $a < b$), we also consider the vertical lines given by $x = a$ and $x = b$ and the x-axis, given by $y = 0$. These four components generally form the boundary of a region in the plane (see Fig. 3.1.1). We now study the area of regions like this.

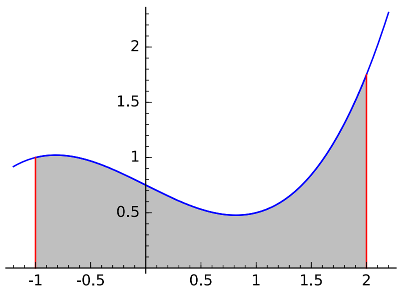

Figure 3.1.1: The region under the curve $y = f(x)$ between $x = -1$ and $x = 2$

It is convenient here to modify the usual notion of area a bit and consider *signed area* instead: if a region is above the x-axis, we regard its area as positive, but if it is below the x-axis, we regard the area as negative. (If a region has parts on either side, we split it in two and subtract the area of the lower part from the area of the upper part.)

Example 3.1.3. The curve given by $y = \sqrt{1 - x^2}$, together with the lines given by $x = 0$ and $y = 0$ (and, for completeness, $x = 1$, although this plays no role here), bounds a quarter disk of radius 1 above the x-axis. The area is $\frac{\pi}{4}$.

On the other hand, if we consider the curve given by $y = -\sqrt{1-x^2}$, then we have a quarter disk below the x-axis. According to our convention, the area of this region is $-\frac{\pi}{4}$.

We use the following notation for this concept. The *definite integral*

$$\int_a^b f(x)\,dx$$

is the signed area (in the above sense) of the region bounded by the curve described $y = f(x)$ and the lines given by $x = a$, $x = b$, and $y = 0$. The numbers a and b are called the (lower and upper) *limits* of the integral. (The symbol x here is a dummy variable and may be replaced by any other symbol.)

Why do we use notation and terminology so similar to the indefinite integral discussed in Sect. 3.1.1? It turns out that computing area is closely related to antidifferentiation. In order to see why this is the case, fix the lower limit a, but replace the upper limit by a variable t. Then we may define a function F by the formula

$$F(t) = \int_a^t f(x)\,dx.$$

Now let h be a number with $h \neq 0$ and compare the value $F(t)$ with $F(t+h)$. The difference $F(t+h)-F(t)$ corresponds to the area of the region bounded by the graph of f, the x-axis, and the lines given by $x = t$ and $x = t+h$. Assuming that h is small, this is a narrow strip, and assuming that f is continuous (and therefore does not vary much between t and $t+h$), the area is approximately $hf(t)$. Hence

$$\frac{F(t+h)-F(t)}{h} \approx f(t).$$

In fact, taking limits, we obtain

$$F'(t) = \lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = f(t).$$

In other words, we have found an antiderivative for f.

The following statement is known as the *first fundamental theorem of calculus*. Suppose that f is a continuous function and a is a fixed number. Let the function F be defined through

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$ for all values of x.

Now remember that any other antiderivative differs from F by a constant. So if we consider an arbitrary antiderivative of f, say G, then there exists a constant C such that $G = F + C$. It is clear that $F(a) = 0$ (as this is the area of a strip of width 0), so $G(a) = C$. On the other hand,

$$\int_a^b f(x)\,dx = F(b) = G(b) - C = G(b) - G(a).$$

This is the *second fundamental theorem of calculus*. If f is a continuous function and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Because expressions like this appear a lot when we work with definite integrals, we use the following shorthand notation:

$$[G(x)]_a^b = G(b) - G(a).$$

If it may be unclear what the relevant variable is, we may instead write

$$[G(x)]_{x=a}^b.$$

The condition that f be continuous can be relaxed, although it takes a sophisticated theory to determine a more appropriate condition. In practice, the condition can almost always be ignored. It is important, however, that the condition $F'(x) = f(x)$ is satisfied for all x in the interval $[a, b]$, unless you add other conditions instead.

Example 3.1.4. Consider the functions f and F with $f(x) = \frac{x}{|x|}$ and $F(x) = |x|$. Then $F'(x) = f(x)$ at any x except when $x = 0$, and we still have

$$\int_{-1}^1 f(x)\,dx = 0 = F(1) - F(-1).$$

However, the function given by $G(x) = |x| + \frac{x}{|x|}$ also satisfies $G'(x) = f(x)$ everywhere except when $x = 0$, and we have $G(1) - G(0) = 2$. So we cannot replace F by G in the above formula.

3.2 Integration techniques

Since we can think of integration as the reverse of differentiation, it is no surprise that the differentiation rules in Sect. 2.2 give rise to integration rules. However, with integration, it is often far less obvious which rule to apply, and so this often takes some experience.

3.2.1 The substitution rule

The *substitution rule* is the counterpart of the chain rule for differentiation. Recall that the chain rule says that for two functions F and g, we have

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

Thus if we write $f = F'$, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C. \tag{3.2.1}$$

We can write the same rule in more convenient notation: write $u = g(x)$, so that $\frac{du}{dx} = g'(x)$. Then we may substitute $du = g'(x)\,dx$ and thus

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

(As before, expressions such as dx or du should be thought of as convenient symbols rather than objects in their own right. In this context, they can actually be given precise meaning, but this theory is outside of the scope of this course.)

While it appears that you need a very special integrand for this, in practice many functions can be written in this form.

Example 3.2.1. Evaluate

$$\int xe^{x^2}\,dx.$$

Solution. Write $u = x^2$, so that $du = 2x\,dx$. Then

$$\int xe^{x^2}\,dx = \frac{1}{2}\int 2xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Example 3.2.2. Evaluate

$$\int \cos(7x)\,dx.$$

Solution. The substitution $u = 7x$ gives $du = 7dx$ and

$$\int \cos(7x)\,dx = \frac{1}{7}\int \cos u\,du = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x) + C.$$

Example 3.2.3. Evaluate

$$\int \frac{dx}{(2-x)^2}.$$

Solution. The substitution $u = 2 - x$ gives $du = -dx$ and

$$\int \frac{dx}{(2-x)^2} = -\int \frac{du}{u^2} = \frac{1}{u} + C = \frac{1}{2-x} + C.$$

Instead of writing $u = g(x)$, we may turn the substitution around and write $x = h(u)$ for some function h. This amounts to the previous method if we choose $h = g^{-1}$. Then by Sect. 2.3.2, we have $h'(u) = \frac{1}{g'(x)}$ and thus we still have $dx = h'(u)\,du$, as expected.

Example 3.2.4. Evaluate

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

Solution. Use the substitution $x = \sin u$, so that $dx = \cos u\,du$. Then

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,du = \int du = u + C = \arcsin x + C.$$

When we want to evaluate a define integral, then one method is to first evaluate the indefinite integral (perhaps using the substitution rule) and then insert the limits. But when we use the substitution rule, then there is a more direct method. Using the functions F and g again and setting $f = F'$, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a))$$

by (3.2.1). That is,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Thus after the substitution $u = g(x)$, there is no need to go back to the variable x, provided that we substitute the limits as well.

Example 3.2.5. Evaluate

$$\int_0^1 \sqrt{x+1}\,dx.$$

Solution. With the substitution $u = x+1$, giving rise to $du = dx$, we obtain

$$\int_0^1 \sqrt{x+1}\,dx = \int_1^2 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_{u=1}^2 = \frac{4\sqrt{2}-2}{3}.$$

Example 3.2.6. Evaluate

$$\int_0^4 \frac{x}{x^2+1}\,dx.$$

Solution. The substitution $u = x^2+1$ gives $du = 2x\,dx$ and

$$\int_0^4 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{17} \frac{du}{u} = \frac{1}{2}[\ln u]_1^{17} = \frac{1}{2}(\ln 17 - \ln 1) = \frac{1}{2}\ln 17.$$

It can happen that after the substitution, the lower limit is larger than the upper limit. But this is no problem. We may just ignore this issue or use the convention

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example 3.2.7. Evaluate

$$\int_{-\sqrt{\pi/2}}^0 x\cos(x^2)\,dx.$$

Solution. The substitution $u = x^2$ gives

$$\int_{-\sqrt{\pi/2}}^0 x\cos(x^2)\,dx = \frac{1}{2}\int_{\pi/2}^0 \cos u\,du = \frac{1}{2}[\sin u]_{u=\pi/2}^0 = -\frac{1}{2}.$$

There are a few substitutions that are far from obvious but very useful. We will discuss these later.

3.2.2 Integration by parts

This is the counterpart to the product rule for differentiation, which tells us that for two functions f and g, we have

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Hence

$$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)g(x) + C.$$

We will rarely have an integrand of this precise form, but we can rearrange the terms as follows:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This only expresses one integral in terms of the other, but we can hope that the one on the right-hand side is easier to evaluate.

Again it is convenient to use the following notation: write $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Substituting this in the above formula yields

$$\int u\,dv = uv - \int v\,du.$$

Example 3.2.8. Evaluate

$$\int xe^x\,dx.$$

Solution. We apply the integration by parts formula for $f(x) = x$ and $g(x) = e^x$. That is, we write $u = x$ and $v = e^x$. Then $du = dx$ and $dv = e^x\,dx$. So

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Example 3.2.9. Evaluate
$$\int x^2 e^x \, dx.$$

Solution. Setting $u = x^2$ and $v = e^x$, we obtain
$$\int x^2 e^x \, dx = \int u \, dv = uv - \int v \, du = x^2 e^x - 2 \int x e^x \, dx.$$
We already know how to evaluate the last integral by Example 3.2.8. Inserting the expression, we obtain
$$\int x^2 e^x \, dx = x^2 e^x - 2(xe^x - e^x) + C = (x^2 - 2x + 2)e^x + C.$$

Example 3.2.10. Evaluate
$$\int e^x \cos x \, dx$$

Solution. With $u = e^x$ and $v = \sin x$, we have
$$\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \sin x \, dx.$$
In order to evaluate the last integral, we use integration by parts again. Let $\tilde{u} = e^x$ and $\tilde{v} = -\cos x$. Then
$$\int e^x \sin x \, dx = \int \tilde{u} \, d\tilde{v} = \tilde{u}\tilde{v} - \int \tilde{v} \, d\tilde{u} = -e^x \cos x + \int e^x \cos x \, dx.$$
Insert this into the first formula:
$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$
Add $\int e^x \cos x \, dx$ on both sides, then divide by 2. This yields
$$\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

The method is exactly the same for definite integrals. We use the formula
$$\int_a^b f(x)g'(x) \, dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) \, dx$$
or we first find an antiderivative and then insert the limits.

Example 3.2.11. Evaluate
$$\int_4^7 x \ln x \, dx.$$

Solution. Using the functions $f(x) = x^2/2$ and $g(x) = \ln x$, we observe that

$$\begin{aligned}\int_4^7 x \ln x \, dx &= \int_4^7 f'(x)g(x)\, dx = [f(x)g(x)]_4^7 - \int_4^7 f(x)g'(x)\, dx \\ &= \left[\frac{x^2}{2}\ln x\right]_4^7 - \int_4^7 \frac{x^2}{2x}\, dx = \frac{49}{2}\ln 7 - 8\ln 4 - \left[\frac{x^2}{4}\right]_4^7 \\ &= \frac{49}{2}\ln 7 - 8\ln 4 - \frac{49}{4} + 4.\end{aligned}$$

3.2.3 Partial fractions

In this section we study quotients of polynomials, which are called rational functions. The method discussed here is not just for integration. It is a method to decompose a rational function into simpler parts, which helps in particular for integration, but may be useful for other purposes, too.

In order to motivate the following, we first observe what happens when we add some simple rational functions. For example,

$$\frac{1}{x+3} + \frac{8}{x-2} = \frac{x-2+8(x+3)}{(x+3)(x-2)} = \frac{9x+22}{x^2+x-6}.$$

The idea is now to reverse this process. That is, given a rational function, we want to split it up into simpler rational functions. It turns out that this is always possible up to a certain point.

Suppose that we have a rational function, that is, a function of the form

$$\frac{p(x)}{q(x)},$$

where p and q are polynomials.

Step 1: Use polynomial long division This will give two polynomials $p_1(x)$ and $r(x)$ such that

$$\frac{p(x)}{q(x)} = \frac{p_1(x)}{q(x)} + r(x)$$

and the degree of $p_1(x)$ is strictly less than the degree of $q(x)$.

Example 3.2.12. Consider the rational function

$$\frac{x^2-x+1}{x+2}.$$

Polynomial long division gives:

$$
\begin{array}{r|l}
 & x - 3 \\
\hline
x + 2 & x^2 - x + 1 \\
 & -x^2 - 2x \\
\hline
 & -3x + 1 \\
 & 3x + 6 \\
\hline
 & 7
\end{array}
$$

Thus

$$\frac{x^2 - x + 1}{x + 2} = \frac{7}{x + 2} + x - 3.$$

Example 3.2.13. Consider the rational function

$$\frac{x^3 - 5x^2 + 7x - 19}{x^2 - 8}.$$

Polynomial long division gives:

$$
\begin{array}{r|l}
 & x - 5 \\
\hline
x^2 - 8 & x^3 - 5x^2 + 7x - 19 \\
 & -x^3 + 8x \\
\hline
 & -5x^2 + 15x - 19 \\
 & 5x^2 - 40 \\
\hline
 & 15x - 59
\end{array}
$$

Thus

$$\frac{x^3 - 5x^2 + 7x - 19}{x^2 - 8} = \frac{15x - 59}{x^2 - 8} + x - 5.$$

Step 2: decompose the denominator into irreducible factors Here 'irreducible' means that it is not possible to factorise any further. In principle, it is possible to decompose any polynomials into a constant and the following two types of irreducible factors:

- linear factors of the form $x - a$, where a is a zero of $q(x)$; and
- quadratic factors of the form $x^2 + bx + c$ with $b^2 < 4c$. (Note that the condition $b^2 < 4ac$ means that there are no zeros. If a polynomial has any zeros, then it is not irreducible.)

Unfortunately, these are not always easy to find. It helps if we can find the zeros of $q(x)$, although this can be difficult too. In fact, if $q(x)$ has a zero at a point a, then $x - a$ is automatically a factor of $q(x)$.

Example 3.2.14. Consider the polynomial $q(x) = x^2 - 3x + 2$. This has zeros at 1 and 2, and we have $q(x) = (x - 1)(x - 2)$.

Example 3.2.15. Consider the polynomial $q(x) = x^3 + x$. We have a zero at 0, and $q(x) = x(x^2 + 1)$. The quadratic factor has no zeros, so this is an irreducible factorisation.

Example 3.2.16. Consider the polynomial $q(x) = x^4 + 2x^2 + 1$. We note that $q(x) = (x^2 + 1)^2 = (x^2 + 1)(x^2 + 1)$. These factors have no zeros, so they are irreducible.

An irreducible factor in the factorisation of $q(x)$ may appear just once, as all factors in Example 3.2.14 and Example 3.2.15 do, or several times, as in Example 3.2.16. In any case, we combine all factors of the same form and write them as a power. That is, for linear factors, we have $(x - a)^n$ and for quadratic factors we have $(x^2 + bx + c)^n$ for some positive integer n and for certain numbers a, b, and c.

Step 3: find the partial fractions For each factor of $q(x)$ of the form $(x - a)^n$, we may expect partial fractions of the form

$$\frac{A_1}{x-a}, \quad \frac{A_2}{(x-a)^2}, \quad \ldots, \quad \frac{A_n}{(x-a)^n}$$

for certain numbers $A_1, \ldots, A_n$. So if $n = 1$, we only have a partial fraction of the form

$$\frac{A}{x-a}.$$

For any factor of the form $(x^2 + bx + c)^n$, where $x^2 + bx + c$ is an irreducible quadratic polynomial, we may expect partial fractions of the form

$$\frac{B_1 x + C_1}{x^2 + bx + c}, \quad \frac{B_2 x + C_2}{(x^2 + bx + c)^2}, \quad \ldots, \quad \frac{B_n x + C_n}{(x^2 + bx + c)^n}$$

for certain numbers $B_1, \ldots, B_n, C_1, \ldots, C_n$. So if $n = 1$, we only have

$$\frac{Bx + C}{x^2 + bx + c}.$$

It then suffices to determine the coefficients $A_1, \ldots, A_n$ or $B_1, \ldots, B_n$, $C_1, \ldots, C_n$. We demonstrate with some examples how this is done.

Example 3.2.17. Decompose the rational function

$$\frac{7x - 13}{x^2 - 4}$$

into partial fractions.
Solution. The denominator has the zeros ± 2, so $x^2 - 4 = (x - 2)(x + 2)$. We look for A_1, A_2 such that

$$\frac{7x - 13}{x^2 - 4} = \frac{A_1}{x - 2} + \frac{A_2}{x + 2}.$$

Simplifying the right-hand side, we obtain

$$\frac{A_1}{x-2}+\frac{A_2}{x+2}=\frac{A_1(x+2)+A_2(x-2)}{(x-2)(x+2)}=\frac{(A_1+A_2)x+2A_1-2A_2}{x^2-4}.$$

In order to obtain the desired rational function, we need to solve the system of equations

$$\begin{aligned} A_1+A_2&=7\\ 2A_1-2A_2&=-13. \end{aligned}$$

The solution is $A_1=\frac{1}{4}$ and $A_2=\frac{27}{4}$. Hence

$$\frac{7x-13}{x^2-4}=\frac{A_1}{x-2}+\frac{A_2}{x+2}=\frac{1}{4(x-2)}+\frac{27}{4(x+2)}.$$

Example 3.2.18. Decompose

$$\frac{x+6}{x^2+8x+16}$$

into partial fractions.
Solution. We have $x^2+8x+16=(x+4)^2$. Note that we have a repeated factor here. Thus we look for A_1, A_2 such that

$$\frac{x+6}{x^2+8x+16}=\frac{A_1}{x+4}+\frac{A_2}{(x+4)^2}=\frac{A_1(x+4)+A_2}{x^2+8x+16}=\frac{A_1x+4A_1+A_2}{x^2+8x+16}.$$

We have to solve the system

$$\begin{aligned} A_1&=1,\\ 4A_1+A_2&=6. \end{aligned}$$

The solution is $A_1=1$ and $A_2=2$. So

$$\frac{x+6}{x^2+8x+16}=\frac{1}{x+4}+\frac{2}{(x+4)^2}.$$

Example 3.2.19. Decompose

$$\frac{5x^2-6x+3}{x^3+x^2-2}$$

into partial fractions
Solution. Here it is not obvious how to find the irreducible factors of the denominator. However, we can check that 1 is a zero. Hence $x-1$ is a factor. Polynomial long division reveals that $x^3+x^2-2=(x-1)(x^2+2x+2)$.

Since the quadratic polynomial $x^2 + 2x + 2$ has no zeros, it is irreducible as well. Therefore, we look for A, B, C such that

$$\frac{5x^2 - 6x + 3}{x^3 + x^2 - 2} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 2x + 2} = \frac{A(x^2 + 2x + 2) + (Bx + C)(x-1)}{(x-1)(x^2 + 2x + 2)}$$
$$= \frac{(A+B)x^2 + (2A - B + C)x + 2A - C}{x^3 + x^2 - 2}.$$

We need to solve the system

$$\begin{aligned} A + B &= 5, \\ 2A - B + C &= -6, \\ 2A - C &= 3. \end{aligned}$$

The solution is $A = \frac{2}{5}$, $B = \frac{23}{5}$, and $C = -\frac{11}{5}$. So we have found that

$$\frac{5x^2 - 6x + 1}{x^3 + x^2 - 2} = \frac{2}{5(x-1)} + \frac{23x - 11}{5(x^2 + 2x + 2)}.$$

Step 4: find the integrals Assuming that we can find the partial fractions of our rational function, we now have a sum of simpler expressions. We finally need to integrate these.

This is quite easy for an expression of the form

$$\frac{A}{(x-a)^n}.$$

We can use the substitution $u = x - a$ to compute

$$\int \frac{A}{x-a}\,dx = A \int \frac{du}{u} = A \ln|u| + C = A \ln|x-a| + C$$

and

$$\int \frac{A}{(x-a)^n}\,dx = A \int \frac{du}{u^n} = \frac{A}{(1-n)u^{n-1}} + C = \frac{A}{(1-n)(x-a)^{n-1}} + C$$

if $n \geq 2$.

Now we have a closer look at the partial fractions of the form

$$\frac{Bx + C}{x^2 + bx + c},$$

where $b^2 < 4c$. Here the first step is to rewrite the denominator. Recall that we have solved quadratic equations in Sect. 1.2.1 by completing the square. We do the same thing here, finding that

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}.$$

Set $\beta = \frac{b}{2}$ and $\gamma = \sqrt{c - b^2/4}$. Then

$$\frac{Bx + C}{x^2 + bx + c} = \frac{Bx + C}{(x + \beta)^2 + \gamma^2} = \frac{Bx + C}{\gamma^2 \left(\left(\frac{x+\beta}{\gamma} \right)^2 + 1 \right)}.$$

In order to evaluate the integral, use the substitution $u = \frac{x+\beta}{\gamma}$. Then $x = \gamma u - \beta$ and thus

$$\begin{aligned} \int \frac{Bx + C}{x^2 + bx + c} \, dx &= \int \frac{B(\gamma u - \beta) + C}{\gamma(u^2 + 1)} \, du \\ &= B \int \frac{u}{u^2 + 1} \, du + \frac{C - B\beta}{\gamma} \int \frac{du}{u^2 + 1}. \end{aligned}$$

The first of the integrals on the right-hand side we can evaluate with the substitution $v = u^2 + 1$, giving $dv = 2u \, du$ and

$$\int \frac{u}{u^2 + 1} \, du = \frac{1}{2} \int \frac{dv}{v} = \frac{1}{2} \ln v + C = \frac{1}{2} \ln(u^2 + 1) + C.$$

For the second one, we use the fact that $\frac{d}{du} \arctan u = \frac{1}{u^2+1}$. Hence

$$\int \frac{du}{u^2 + 1} = \arctan u + C.$$

Combining everything, we obtain

$$\begin{aligned} \int \frac{Bx + C}{x^2 + bx + c} &= \frac{B}{2} \ln(u^2 + 1) + \frac{C - B\beta}{\gamma} \arctan u + C \\ &= \frac{B}{2} \ln \left(\left(\frac{x + \beta}{\gamma} \right)^2 + 1 \right) + \frac{C - B\beta}{\gamma} \arctan \frac{x + \beta}{\gamma} + C, \end{aligned}$$

where $\beta = \frac{b}{2}$ and $\gamma = \sqrt{c - b^2/4}$.

We may also have partial fractions of the form

$$\frac{Bx + C}{(x^2 + bx + c)^n}$$

for some integer $n \geq 0$. In practice, however, this is rare, and so we discuss this only briefly. The first steps are the same as in the case $n = 1$ above. This will now give rise to integrals of the form

$$\int \frac{u}{(u^2 + 1)^n} \, du \quad \text{and} \quad \int \frac{du}{(u^2 + 1)^n}.$$

The first of these can be evaluated with the substitution $v = u^2 + 1$ again, which yields

$$\int \frac{u}{(u^2 + 1)^n} \, du = \frac{1}{2} \int \frac{dv}{v^n} = \frac{1}{2(1 - n)v^{n-1}} = \frac{1}{2(1 - n)(u^2 + 1)^{n-1}}.$$

For the second integral, we use the substitution $u = \tan v$. Then $u^2 + 1 = \tan^2 v + 1 = \frac{1}{\cos^2 v}$ and $du = \frac{dv}{\cos^2 v}$. Hence

$$\int \frac{du}{(u^2+1)^n} = \int \cos^{2n-2} v \, dv.$$

These integrals can now be evaluated with a repeated integration by parts.

We have already found the partial fractions of some rational functions, so we can now evaluate their integrals.

Example 3.2.20. Evaluate

$$\int \frac{7x-13}{x^2-4}\, dx.$$

Solution. We have seen in Example 3.2.17 that

$$\frac{7x-13}{x^2-4} = \frac{1}{4(x-2)} + \frac{27}{4(x+2)}.$$

Hence

$$\int \frac{7x-13}{x^2-4}\, dx = \frac{1}{4}\ln|x-2| + \frac{27}{4}\ln|x+2| + C.$$

Example 3.2.21. Evaluate

$$\int \frac{x+6}{x^2+8x+16}\, dx.$$

Solution. We have seen in Example 3.2.18 that

$$\frac{x+6}{x^2+8x+16} = \frac{1}{x+4} + \frac{2}{(x+4)^2}.$$

Hence

$$\int \frac{x+6}{x^2+8x+16} = \ln|x+4| - \frac{2}{x+4} + C.$$

Example 3.2.22. Evaluate

$$\int \frac{5x^2-6x+3}{x^3+x^2-2}\, dx.$$

Solution. We have seen in Example 3.2.19 that

$$\frac{5x^2-6x+1}{x^3+x^2-2} = \frac{2}{5(x-1)} + \frac{23x-11}{5(x^2+2x+2)}.$$

Moreover, we have

$$\int \frac{2}{5(x-1)}\, dx = \frac{2}{5}\ln|x-1| + C.$$

In order to evaluate the remaining integral, we write

$$x^2 + 2x + 2 = (x+1)^2 + 1.$$

We use the substitution $u = x + 1$, which gives

$$\begin{aligned}
\int \frac{23x - 11}{5(x^2 + 2x + 2)} &= \int \frac{23(u-1) - 11}{5(u^2+1)}\,du \\
&= \frac{23}{5}\int \frac{u}{u^2+1}\,du - \frac{34}{5}\int \frac{du}{u^2+1} \\
&= \frac{23}{10}\ln(u^2+1) - \frac{34}{5}\arctan u + C \\
&= \frac{23}{10}\ln(x^2+2x+2) - \frac{34}{5}\arctan(x+1) + C.
\end{aligned}$$

3.3 Further integration techniques

3.3.1 Integrating an inverse function

Recall that we have seen in Sect. 2.3.2 how to differentiate the inverse f^{-1} of a function f. We can use this information, combined with integration by parts and the substitution rule, to find antiderivatives of f^{-1} as well.

We demonstrate the method with an example. Consider the integral

$$\int \arcsin x\,dx.$$

Use integration by parts with $u = \arcsin x$ and $v = x$, giving rise to

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

Now substitute $w = 1 - x^2$, so that $dw = -2x\,dx$. Thus

$$\begin{aligned}
\int \arcsin x\,dx &= x\arcsin x + \frac{1}{2}\int \frac{dw}{\sqrt{w}} \\
&= x\arcsin x + \sqrt{w} + C = x\arcsin x + \sqrt{1-x^2} + C.
\end{aligned}$$

In general, if we want to integrate an inverse function f^{-1}, then we can always use integration by parts with $u = f^{-1}(x)$ and $v = x$, followed by a substitution. The resulting formula becomes particularly simple when we write $y = f(x)$ (and therefore $x = f^{-1}(y)$ is the expression that we want to integrate). Then we have

$$\int x\,dy = xy - \int y\,dx.$$

But because this is so terse, using this formula blindly may result in confusion. In practice, it is often better to do the above steps directly.

Example 3.3.1. Evaluate

$$\int \ln x \, dx.$$

Solution. Note that ln is the inverse of the exponential function with base e, so we can use the above method. Set $u = \ln x$ and $v = x$. This gives

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} \, dx = x \ln x - x + C.$$

3.3.2 Trigonometric rational functions

It's easiest to understand what trigonometric rational functions are by looking at some examples such as the following:

$$\frac{\sin^3 x - \cos^2 x + 2}{\cos^3 x - \sin^3 x + 8 \cos x}, \quad \frac{\cos x}{14 \sin^8 x + 5 \cos^7 x + \sin^3 x}, \quad \frac{\cos^4 x + \sin^4 x}{\cos^3 x}.$$

A standard trick to integrate these is the substitution

$$u = \tan \frac{x}{2}.$$

Then

$$\cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2 \, du}{1 + u^2}.$$

It is clear that inserting all of this, we obtain the integral of a rational function, which we can, in principle, evaluate with the method of partial fractions. In practice, however, this can give very complicated expressions. (Imagine using it for the above examples.) But for simple trigonometric rational functions, this works well.

Example 3.3.2. Evaluate

$$\int \sec x \, dx.$$

Solution. Note that sec is a trigonometric rational function, as $\sec x = \frac{1}{\cos x}$. The standard substitution $u = \tan \frac{x}{2}$ gives

$$\int \sec x \, dx = \int \frac{1 + u^2}{1 - u^2} \frac{2}{1 + u^2} \, du = \int \frac{2}{1 - u^2} \, du.$$

The method of partial fractions gives

$$\frac{2}{1 - u^2} = \frac{1}{u + 1} - \frac{1}{u - 1}.$$

Hence

$$\int \frac{2}{1 - u^2} \, du = \int \frac{du}{u + 1} - \int \frac{du}{u - 1} = \ln |u + 1| - \ln |u - 1| + C.$$

We finally obtain

$$\int \sec x\, dx = \ln\left|\tan\frac{x}{2} + 1\right| - \ln\left|\tan\frac{x}{2} - 1\right| + C.$$

(Note that there are other methods to compute this and there are other ways to represent the result.)

3.3.3 Trigonometric substitution

This is a method that is often useful for integrals involving the expression $\sqrt{1-x^2}$. The trick is to substitute $x = \sin u$. Then $dx = \cos u\, du$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \sqrt{\cos^2 u} = \cos u.$$

(Here we have used the assumption that $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.)

Example 3.3.3. Evaluate

$$\int \sqrt{1-x^2}\, dx.$$

Solution. Using the above substitution $x = \sin u$, we obtain

$$\int \sqrt{1-x^2}\, dx = \int \cos^2 u\, du = \frac{1}{2}(u + \sin u \cos u) + C.$$

In order to reverse the substitution, we note that $u = \arcsin x$. Moreover, we have $\sin u = x$ and $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2}$. Thus

$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}(\arcsin x + x\sqrt{1-x^2}) + C.$$

Integrals involving $\sqrt{ax^2+bx+c}$ can often be treated similarly, because we can transform them by completing the square.

Example 3.3.4. Evaluate

$$\int \sqrt{8+2x-x^2}\, dx.$$

Solution. We have

$$8 + 2x - x^2 = 9 - (x-1)^2 = 9\left(1 - \left(\frac{x-1}{3}\right)^2\right),$$

which suggests the substitution $u = \frac{x-1}{3}$. Then

$$\begin{aligned}\int \sqrt{8+2x-x^2}\, dx &= 9\int \sqrt{1-u^2}\, du \\ &= \frac{9}{2}(\arcsin u + u\sqrt{1-u^2}) + C \\ &= \frac{9}{2}\arcsin\frac{x-1}{3} + \frac{1}{2}(x-1)\sqrt{8+2x-x^2} + C.\end{aligned}$$

For integrals involving $\sqrt{1+x^2}$, the substitution $x = \tan u$ is sometimes useful. Then $dx = \frac{du}{\cos^2 u}$ and

$$\sqrt{1+x^2} = \sqrt{1+\tan^2 u} = \sqrt{\frac{1}{\cos^2 u}} = \frac{1}{\cos u}$$

(again assuming that $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$).

Example 3.3.5. Evaluate

$$\int \frac{dx}{\sqrt{1+x^2}}.$$

Solution. With the above substitution, we obtain

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{du}{\cos u} = \int \sec u \, du.$$

We have computed this integral in Example 3.3.2, but it is more convenient to use a different method here. Note that

$$\int \frac{du}{\cos u} = \int \frac{\cos u}{\cos^2 u} du = \int \frac{\cos u}{1-\sin^2 u} du.$$

With $v = \sin u$, we obtain the integral

$$\begin{aligned} \int \frac{dv}{1-v^2} &= \frac{1}{2} \int \frac{dv}{1+v} + \frac{1}{2} \int \frac{dv}{1-v} \\ &= \frac{1}{2} \ln|1+v| - \frac{1}{2} \ln|1-v| + C = \frac{1}{2} \ln \left| \frac{1+v}{1-v} \right| + C. \end{aligned}$$

Since $x = \tan u$ and $v = \sin u$, we have $v = \frac{x}{\sqrt{1+x^2}}$. Hence

$$\begin{aligned} \frac{1+v}{1-v} &= \frac{1+v}{1-v} \frac{1+v}{1+v} = \frac{(1+v)^2}{1-v^2} = \frac{\left(1+\frac{x}{\sqrt{1+x^2}}\right)^2}{1-\frac{x^2}{1+x^2}} \\ &= (1+x^2)\left(1+\frac{x}{\sqrt{1+x^2}}\right)^2 = \left(x+\sqrt{1+x^2}\right)^2. \end{aligned}$$

It follows that

$$\int \frac{dx}{1+x^2} = \frac{1}{2} \ln \left(x+\sqrt{1+x^2}\right)^2 + C = \ln\left(x+\sqrt{1+x^2}\right) + C.$$

3.3.4 Hyperbolic substitution

Recall the functions

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \text{and} \quad \sinh x = \frac{1}{2}(e^x - e^{-x}),$$

which satisfy

$$\cosh^2 x - \sinh^2 x = 1$$

and

$$\frac{d}{dx}\cosh x = \sinh x \quad \text{and} \quad \frac{d}{dx}\sinh x = \cosh x.$$

If we have an integral involving the expression $\sqrt{1+x^2}$, then they can be useful as well. The substitution $x = \sinh u$ then gives

$$\sqrt{1+x^2} = \sqrt{1+\sinh^2 u} = \sqrt{\cosh^2 u} = \cosh u$$

and $dx = \cosh u\, du$.

Example 3.3.6. Evaluate

$$\int \frac{dx}{\sqrt{1+x^2}}.$$

Solution. We have already found the answer in Example 3.3.5, but we now use a different method. With the above substitution, we obtain

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh u}{\cosh u}\, du = \int du = u + C.$$

Now we need to express u in terms of x; that is, we need to find the inverse of the function sinh. To this end, we solve $\sinh u = x$ for u. Recalling the definition of sinh, we obtain

$$\frac{1}{2}(e^u - e^{-u}) = x.$$

If we substitute $v = e^u$, this becomes

$$\frac{v}{2} - \frac{1}{2v} = x.$$

Multiply by $2v$:

$$v^2 - 1 = 2xv.$$

Rearrange:

$$v^2 - 2vx - 1 = 0.$$

This is a quadratic equation in v. The solutions are

$$v = x \pm \sqrt{x^2+1}.$$

But since $v = e^u > 0$, while $x - \sqrt{x^2+1} < 0$, we can only use the positive solution $v = x + \sqrt{x^2+1}$. Hence

$$u = \ln v = \ln\left(x + \sqrt{x^2+1}\right).$$

It follows that

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right) + C.$$

3.4 Numerical integration

Up to now we have calculated our integrals symbolically. But often this is impossible and we have to resort to numerical approximations. As a definite integral, such as

$$\int_a^b f(x)\,dx,$$

is the area of a region in the plane, the basic idea is to approximate this region by simpler shapes. One common way to do this is to divide the interval $[a,b]$ into lots of smaller intervals and thereby divide the region into lots of thin strips. Each of these strips can then be approximated by a rectangle, and we know how to compute the area of a rectangle.

Suppose that we want to divide $[a,b]$ into n intervals of equal length. Then $h = \frac{b-a}{n}$ is this length and the numbers

$$x_k = a + hk, \quad k = 0, \dots, n,$$

are the end points of the intervals. If w_k denotes the height of the k-th rectangle, then its area is hw_k. (This may be negative, as we still consider signed area here.) So the total area of the approximation is

$$h\sum_{k=1}^{n} w_k.$$

The question is now how to choose w_k. There are several common ways to do it.

Left point rule Choose the value of f at the left end point of the corresponding interval, so $w_k = f(x_{k-1})$. Then the total area of the approximation is

$$h\sum_{k=1}^{n} f(x_{k-1}).$$

Right point rule Use the right end point instead. Then $w_k = f(x_k)$ and we obtain

$$h\sum_{k=1}^{n} f(x_k).$$

Mid-point rule Use the mid-point $\frac{x_{k-1}+x_k}{2}$. Then $w_k = f(\frac{x_{k-1}+x_k}{2})$ and we have the expression

$$h\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right).$$

Trapezium rule This rule is motivated by an approximation by trapezia rather than rectangles. It corresponds to $w_k = \frac{f(x_{k-1})+f(x_k)}{2}$ and gives rise to

$$h\sum_{k=1}^{n}\frac{f(x_{k-1})+f(x_k)}{2} = \frac{h}{2}f(x_0) + h\sum_{k=1}^{n-1}f(x_k) + \frac{h}{2}f(x_n).$$

In addition, there is another important method that does not quite fit in this framework. We will see later why it works (and works very well).

Simpson's rule Here we assume that n is even and we use the formula

$$\frac{h}{3}\sum_{k=1}^{n/2}\left(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\right).$$

Example 3.4.1. Suppose that we want to integrate

$$\int_0^1 x^5\,dx$$

numerically. We know that the exact answer is $\frac{1}{6} \approx 0.166667$, and this will allow us to compare the results for different methods.

Here we have $a = 0$ and $b = 1$. We first choose $n = 2$, giving rise to $h = \frac{1}{2}$. Then we have the following approximations

Left point rule $\frac{1}{2}\left(0 + \frac{1}{32}\right) = \frac{1}{64} \approx 0.0156$,

Mid-point rule $\frac{1}{2}\left(\frac{1}{1024} + \frac{243}{1024}\right) = \frac{61}{512} \approx 0.1191$,

Right point rule $\frac{1}{2}\left(\frac{1}{32} + 1\right) = \frac{33}{64} \approx 0.5156$,

Trapezium rule $\frac{1}{2}\left(\frac{0+\frac{1}{32}}{2} + \frac{\frac{1}{32}+1}{2}\right) = \frac{17}{64} \approx 0.2656$,

Simpson's rule $\frac{1}{6}\left(0 + \frac{4}{32} + 1\right) = \frac{3}{16} \approx 0.1875$.

The following table contains the results for various values of n.

n	2	4	8	16
Left point rule	0.0156	0.0674	0.1107	0.1370
Mid-point rule	0.1191	0.1539	0.1634	0.1659
Right point rule	0.5156	0.3174	0.2357	0.1995
Trapezium rule	0.2656	0.1924	0.1732	0.1683
Simpson's rule	0.1875	0.1680	0.1667	0.1667

In order to explain the different performances of these methods, we compare their errors for the integrals over two consecutive intervals in the subdivision of $[a, b]$. More precisely, for a given number c and $h > 0$, we examine the errors in the approximation of

$$I(h) = \int_{c-h}^{c+h} f(x)\, dx$$

when the interval $[c - h, c + h]$ is divided into two subintervals. The total error for the integral $\int_a^b f(x)\, dx$ will then be roughly this error times $n/2$.

We assume that f is represented by its Taylor series near c (otherwise we can use similar arguments with finite Taylor polynomials). Then

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \frac{f'''(c)}{6}(x - c)^3 + \cdots , \quad (3.4.1)$$

and therefore

$$\begin{aligned} I(h) &= \left[f(c)(x - c) + \frac{f'(c)}{2}(x - c)^2 + \frac{f''(c)}{6}(x - c)^3 + \cdots \right]_{c-h}^{c+h} \\ &= 2f(c)h + 2\frac{f''(c)}{6}h^3 + 2\frac{f^{(4)}(c)}{5!}h^5 + \cdots . \end{aligned}$$

Now we compute the results of the approximation methods. The left point rule and the right point rule are similar, and we only treat the latter here. It gives the approximation

$$h(f(c) + f(c + h)).$$

Using (3.4.1) for the second term, we obtain

$$2f(c)h + f'(c)h^2 + \frac{f''(c)}{2}h^3 + \cdots .$$

The error is of order h^2. The mid-point rule gives

$$h(f(c - h/2) + f(c + h/2)) = 2f(c)h + \frac{f''(c)}{4}h^3 + \cdots ,$$

with an error of order h^3. The trapezium rule gives

$$\frac{h}{2}(f(c - h) + 2f(c) + f(c + h)) = 2f(c)h + \frac{f''(c)}{2}h^3 + \cdots ,$$

again with an error of order h^3. Finally, Simpson's rule gives

$$\frac{h}{3}(f(c - h) + 4f(c) + f(c + h)) = 2f(c)h + \frac{f''(c)}{3}h^3 + \frac{f^{(4)}(c)}{36}h^5 + \cdots .$$

This is an error of order h^5.

Once we add up the errors over the whole interval $[a, b]$, we obtain

- order h for the left and right point rules,
- order h^2 for the mid-point and trapezium rules, and
- order h^4 for Simpson's rule.

3.5 Improper integrals

So far we have considered the area of regions bounded by the graph of a function f, the x-axis, and two vertical lines, given by $x = a$ and $x = b$. But sometimes we want to remove one or both of these vertical lines and consider regions stretching to infinity. A similar situation occurs if $f(x)$ tends to ∞ or $-\infty$ as x approaches a or b. It is not always clear what the area of such a region is, but in some cases the notion still makes sense.

We define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

provided that this limit exists. Similarly,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$

provided, again, that the limit exists.

Example 3.5.1. We have

$$\int_0^b e^{-x}\,dx = 1 - e^{-b} \to 1$$

as $b \to \infty$. Hence

$$\int_0^\infty e^{-x}\,dx = 1.$$

Similarly,

$$\int_a^0 e^{x}\,dx = 1 - e^{a} \to 1$$

as $a \to -\infty$. Hence

$$\int_{-\infty}^0 e^{x}\,dx = 1.$$

But

$$\int_{-\infty}^0 e^{-x}\,dx = \infty$$

and

$$\int_0^\infty e^{x}\,dx = \infty.$$

If $\lim_{x\to a^+} f(x) = \pm\infty$, then we define

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx,$$

provided that the limit exists. If $\lim_{x\to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(c)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx,$$

provided that the limit exists.

Example 3.5.2. We have

$$\int_c^1 \frac{dx}{\sqrt{x}} = 2(1 - \sqrt{c}) \to 2$$

as $c \to 0^+$. Hence

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

But

$$\int_0^1 \frac{dx}{x} = \lim_{a \to 0^+} (-\ln a) = \infty.$$

All integrals of this type are called *improper*.

Chapter 4

Differential equations

4.1 Introduction

If an equation involves a function and some of its derivatives, then it is called a *differential equation*. For example, the following is a differential equation:

$$f' = f. \tag{4.1.1}$$

This expresses the fact that a function coincides with its own derivative. Often the equation is written in the form

$$f'(x) = f(x),$$

and then this is usually meant to be true for all x. If we write y for the dependent variable (so that $y = f(x)$), then we may also write

$$\frac{dy}{dx} = y$$

for the same equation.

To *solve* a differential equation means to find the functions that make it a true statement. For example, we know that the exponential function with base e, i.e., the function with $f(x) = e^x$, satisfies (4.1.1). You can also check that for any constant c, the function defined by $f(x) = ce^x$ is another solution. In fact these are all the solutions of this equation.

When we solve a differential equation, we typically obtain infinitely many solutions. For this reason, a differential equation often comes together with additional conditions that are designed to pick a specific solution. For the type of equations that we study here, the most common condition is one of the form

$$f(x_0) = y_0, \tag{4.1.2}$$

where x_0 and y_0 are given numbers. That is, we prescribe the value of the solution at a specific point. Very often in problems of this sort, the independent variable represents time and condition (4.1.2) represents a measurement of the quantity y at an initial time. Therefore, a problem consisting of a differential equation and a condition of this sort is called an *initial value problem*.

For example, equations (4.1.1) and (4.1.2) constitute an initial value problem. Any function of the form $f(x) = ce^x$ is a solution of (4.1.1). We can now use (4.1.2) to determine the value of c. It must satisfy

$$ce^{x_0} = y_0,$$

so $c = y_0 e^{-x_0}$. Therefore, the unique solution of the initial value problem is

$$f(x) = y_0 e^{-x_0} e^x = y_0 e^{x - x_0}.$$

In general, it can be rather difficult to solve a given differential equation, but there are methods for equations of a specific form.

4.2 Separation of variables

A differential equation is called *separable* if it can be written in the form

$$f'(x) = g(x)h(f(x))$$

for some functions g and h. If $y = f(x)$, then we can also write

$$\frac{dy}{dx} = g(x)h(y).$$

These equations are relatively easy to solve. The basic idea is to *separate variables*, meaning that we write all expressions involving the independent variable (including the derivative) on one side and all expressions involving the dependent variable on the other side of the equation. That is,

$$\frac{f'(x)}{h(f(x))} = g(x).$$

It is convenient to represent this in the form

$$\frac{dy}{h(y)} = g(x)dx, \tag{4.2.1}$$

keeping in mind, as ever, that these expressions have only symbolic character. Now we integrate:

$$\int \frac{f'(x)}{h(f(x))}\, dx = \int g(x)\, dx.$$

By the substitution rule, we have

$$\int \frac{f'(x)}{h(f(x))}\, dx = \int \frac{dy}{h(y)},$$

so

$$\int \frac{dy}{h(y)} = \int g(x)\, dx.$$

(This is what we also get by just writing the integral sign in front of the expressions in (4.2.1).) Assuming that we can evaluate these indefinite integrals, say

$$\int g(x)\, dx = G(x) + C_1$$

and

$$\int \frac{dy}{h(y)} = H(y) + C_2,$$

we obtain

$$H(y) = G(x) + C.$$

(Here $C = C_1 - C_2$ is another generic constant.) Thus we have turned a differential equation into an algebraic equation, which we can now try to solve for y. There is still an undetermined constant C in that equation, which reflects that fact that differential equations normally have many solutions. We will obtain one solution for every value of C.

Example 4.2.1. Solve the differential equation $f' = f$.
Solution. First we introduce more convenient notation. Write $y = f(x)$. Then the equation becomes $\frac{dy}{dx} = y$, which gives rise to

$$\frac{dy}{y} = dx.$$

Integration gives

$$\int \frac{dy}{y} = \int dx.$$

As

$$\int dx = x + C_1 \quad \text{and} \quad \int \frac{dy}{y} = \ln|y| + C_2,$$

we obtain

$$\ln|y| = x + C.$$

Solving for y, we conclude that

$$y = e^{x+C} = e^C e^x \quad \text{or} \quad y = -e^{x+C} = -e^C e^x.$$

Writing $c = e^C$ or $c = -e^C$, we obtain the solution

$$y = ce^x.$$

We can check that this is a solution for any constant c.

Example 4.2.2. Solve the equation $\frac{dy}{dx} = \frac{x}{y}$.
Solution. We first separate variables:

$$y\,dy = x\,dx.$$

Thus

$$\int y\,dy = \int x\,dx.$$

We have

$$\int y\,dy = \frac{y^2}{2} + C_1 \quad \text{and} \quad \int x\,dx = \frac{x^2}{2} + C_2.$$

Hence

$$\frac{y^2}{2} = \frac{x^2}{2} + C.$$

Solving for y, we obtain

$$y = \sqrt{x^2 + 2C} \quad \text{or} \quad y = -\sqrt{x^2 + 2C}.$$

Again we can check that these are indeed solutions.

Example 4.2.3. Solve the initial value problem

$$\frac{dy}{dx} = e^{x+y}, \quad y(0) = 8.$$

Solution. Using the fact that $e^{x+y} = e^x e^y$, we separate variables:

$$e^{-y} dy = e^x\,dx.$$

Thus

$$\int e^{-y}\,dy = \int e^x\,dx.$$

We have

$$\int e^{-y}\,dy = -e^{-y} + C_1 \quad \text{and} \quad \int e^x\,dx = e^x + C_2.$$

Therefore,

$$-e^{-y} = e^x + C.$$

Solving for y, we obtain

$$y = -\ln(-e^x - C).$$

Now we use the initial condition to determine C. We must have

$$8 = -\ln(-1 - C).$$

Hence $C = -e^{-8} - 1$. We finally obtain

$$y = -\ln(e^{-8} + 1 - e^x).$$

4.3 Linear equations

Here we study differential equations that can be written in the form

$$p(x)\frac{dy}{dx} + q(x)y(x) = r(x) \tag{4.3.1}$$

for certain functions p, q, and r. If either $q(x) = 0$ or $r(x) = 0$, then this is a separable equation and we can normally solve it with the method from the previous section. Otherwise, we need a different method.

We say that equation (4.3.1) is *homogeneous* if $r(x) = 0$ for all x.

4.3.1 Integrating factors

The main idea behind this method is that the equation may become easier if we multiply y by another function, say m. Setting $z = my$, we obtain, using the product rule

$$\frac{dz}{dx} = \frac{dm}{dx}y(x) + m(x)\frac{dy}{dx}.$$

If y is to solve (4.3.1), then

$$\frac{dz}{dx} = \frac{dm}{dx}y(x) + \frac{r(x)}{p(x)}m(x) - \frac{q(x)}{p(x)}m(x)y(x).$$

If we choose m such that is solves the equation

$$\frac{dm}{dx} = \frac{q(x)}{p(x)}m, \tag{4.3.2}$$

then two of these terms cancel each other, and we have

$$\frac{dz}{dx} = \frac{r(x)}{p(x)}m(x). \tag{4.3.3}$$

This is then an equation that is easy to solve.

In order to implement this idea, we need to solve (4.3.2). We can do this by separation of variables, as we can write the equation in the form

$$\frac{dm}{m} = \frac{q(x)}{p(x)}\,dx,$$

which gives

$$\ln m = \int \frac{q(x)}{p(x)}\,dx + C.$$

For our purpose here, it is sufficient to find one solutions. So we may set $C = 0$. Then

$$m = e^{\int q(x)/p(x)\,dx}.$$

Once we have determined m, we can solve (4.3.3) just by integration. We have

$$z = \int \frac{r(x)}{p(x)} m(x)\, dx.$$

From this we now easily obtain y.

So the method works as follows.

Step 1 Evaluate

$$\int q(x)/p(x)\, dx.$$

Step 2 Set

$$m = e^{\int q(x)/p(x)\, dx}.$$

Step 3 Evaluate

$$z = \int \frac{r(x)}{p(x)} m(x)\, dx.$$

Step 4 The desired solution is $y = z/m$.

Example 4.3.1. Solve the equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2.$$

Solution. We first note that

$$\int \frac{dx}{x} = \ln x + C.$$

Our integrating factor is $m = e^{\ln x} = x$. Next we set

$$z = \int x^3\, dx = \frac{x^4}{4} + C.$$

So

$$y = \frac{\frac{x^4}{4} + C}{x} = \frac{x^3}{4} + \frac{C}{x}.$$

At the end it is usually a good idea to check if this is really a solution.

Example 4.3.2. Solve the equation

$$\frac{dy}{dx} + 2y = x.$$

Solution. We use the integrating factor

$$m = e^{\int 2\, dx} = e^{2x}$$

here. Then

$$z = \int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

and

$$y = e^{-2x}z = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}.$$

4.3.2 The method of undetermined coefficients

This method typically works only for equations of the specific form

$$a\frac{dy}{dx} + by = r(x), \tag{4.3.4}$$

where a and b are constants, and only when the function r has a special structure.[1] The idea is to find solutions that mimic the structure of $r(x)$. This is easiest explained with some examples.

Example 4.3.3. Consider the equation

$$\frac{dy}{dx} + 2y = x.$$

The right-hand side is a polynomial of degree 1. We conjecture that there is a solution that is also a polynomial of degree 1, so $y(x) = ax + b$. Inserting this into the equation, we obtain

$$a + 2(ax + b) = x.$$

Rearrange the terms:

$$(2a - 1)x + a + 2b = 0.$$

This is satisfied if we choose a and b such that

$$\begin{aligned} 2a &= 1, \\ a + 2b &= 0. \end{aligned}$$

This system has a unique solution, which is $a = \frac{1}{2}$ and $b = -\frac{1}{4}$. So we obtain the solution

$$y = \frac{x}{2} - \frac{1}{4}$$

of the differential equation.

Example 4.3.4. Consider the equation

$$2\frac{dy}{dx} + 3y = e^{2x}.$$

[1] It also works for certain equations involving higher order derivatives, but this is outside the scope of this course.

Here we conjecture that there may be a solution of the form $y(x) = ce^{2x}$ for some constant c. Inserting this into the equation, we obtain

$$4ce^{2x} + 3ce^{2x} = e^{2x}.$$

That is,

$$(7c - 1)e^{2x} = 0.$$

This is satisfied for $c = \frac{1}{7}$. So we have the solution

$$y = \frac{1}{7}e^{2x}.$$

The following table gives expressions that may fruitfully be tried for different forms of $r(x)$.

$r(x)$	$y(x)$
$a_n x^n + \cdots + a_0$	$b_n x^n + \cdots + b_0$
$ae^{\alpha x}$	$be^{\alpha x}$
$(a_n x^n + \cdots + a_0)e^{\alpha x}$	$(b_n x^n + \cdots + b_0)e^{\alpha x}$
$a\cos(\alpha x)$	$b\cos(\alpha x) + c\sin(\alpha x)$
$a\sin(\alpha x)$	$b\cos(\alpha x) + c\sin(\alpha x)$
$(a_n x^n + \cdots + a_0)\cos(\alpha x)$	$(b_n x^n + \cdots + b_0)\cos(\alpha x) +$ $+(c_n x^n + \cdots + c_0)\sin(\alpha x)$
$(a_n x^n + \cdots + a_0)\sin(\alpha x)$	$(b_n x^n + \cdots + b_0)\cos(\alpha x) +$ $+(c_n x^n + \cdots + c_0)\sin(\alpha x)$

But even if this works, we are not done yet. We have found just one solution out of infinitely many, and it is unlikely to be the one we are looking for. Fortunately, it is now easy to generate all other solutions as well. Here the idea is to solve the corresponding homogeneous equation, i.e.,

$$a\frac{dz}{dx} + bz = 0.$$

This can be done by separation of variables, and because we have an equation of a specific form, we can immediately write down the general solution:

$$z(x) = Ce^{-\frac{bx}{a}}.$$

The general solution of the original equation (4.3.4) is the sum of this and the particular solution found earlier.

Example 4.3.5. The general solution of

$$\frac{dy}{dx} + 2y = x.$$

is

$$y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}.$$

(Cf. Example 4.3.2.)

Example 4.3.6. The general solution of

$$2\frac{dy}{dx} + 3y = e^{2x}.$$

is

$$y = \frac{1}{7}e^{2x} + Ce^{-\frac{3x}{2}}.$$

Example 4.3.7. Solve the differential equation

$$\frac{dy}{dx} - 2y = 3x\cos x.$$

Solution. We want to find the general solution, but first we try to find one particular solution of the form

$$y = (ax + b)\cos x + (cx + d)\sin x.$$

This gives

$$\begin{aligned}\frac{dy}{dx} &= a\cos x - (ax + b)\sin x + c\sin x + (cx + d)\cos x \\ &= (cx + a + d)\cos x + (-ax + c - b)\sin x.\end{aligned}$$

Inserting this into the equation, we obtain

$$((c - 2a)x + a + d - 2b)\cos x + (-(a + 2c)x + c - b - 2d)\sin x = 3x\cos x.$$

We now want to solve the system

$$\begin{aligned} -2a + c &= 3, \\ a - 2b + d &= 0, \\ a + 2c &= 0, \\ -b + c - 2d &= 0. \end{aligned}$$

This has the unique solution $a = -\frac{6}{5}$, $b = -\frac{9}{25}$, $c = \frac{3}{5}$, $d = \frac{12}{25}$. Thus a solution of the equation is

$$y = -\frac{30x + 9}{25}\cos x + \frac{15x + 12}{25}\sin x.$$

The general solution of the homogeneous equation is $z = Ce^{2x}$, and so the general solution of the full equation is

$$y = Ce^{2x} - \frac{30x + 9}{25}\cos x + \frac{15x + 12}{25}\sin x.$$

4.4 Numerical methods

For more complicated differential equations, we typically cannot find exact solutions and therefore we have to resort to numerical approximations. Consider an initial value problem of the form

$$\frac{dy}{dx} = F(x, y(x)), \quad y(x_0) = y_0, \tag{4.4.1}$$

where F is an arbitrary function in two variables. (We will discuss functions in two variables in more detail later. For the moment, just think of some quantity depending on x and y.)

The basic idea for almost all numerical methods is the same as for integration: divide the x-axis (or an interval) into small intervals and find an approximation in each interval. But in contrast to integration, we now have to compute the approximations in the right order and use the output from one computation as an input for the next.

4.4.1 Euler's method

The following is a very simple method, called *Euler's method* (or sometimes called *forward Euler method* to distinguish it from the backward version discussed below). Here we divide the interval $[x_0, \infty)$ into small intervals of the same length, say h. Define $x_n = x_0 + nh$; then $[x_n, x_{n+1}]$ is one of these intervals for $n = 0, 1, 2, \ldots$. Set $y_n = y(x_n)$. Now use a difference quotient to approximate the derivative in (4.4.1). More precisely,

$$\frac{dy}{dx}(x_n) \approx \frac{y(x_{n+1}) - y(x_n)}{h} = \frac{y_{n+1} - y_n}{h}.$$

Inserting this into the equation, we obtain

$$\frac{y_{n+1} - y_n}{h} \approx F(x_n, y(x_n)) = F(x_n, y_n).$$

Replace the approximation by equality and solve for y_{n+1}:

$$y_{n+1} = y_n + hF(x_n, y_n).$$

The equation $y_{n+1} = y(x_{n+1})$ is no longer true exactly, but we expect that it is true approximately; so $y_{n+1} \approx y(x_{n+1})$. Therefore, we can use this idea to compute approximations for $y(x_n)$ recursively. Using the value y_0 from (4.4.1), we set

$$y_{n+1} = y_n + hF(x_n, y_n), \quad n = 0, 1, 2, \ldots .$$

In the end we expect that

$$y(x_n) \approx y_n.$$

The results will of course depend on the choice of h, which is called the *step size*, and we will get better approximations the smaller the values of h.

Example 4.4.1. Consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{y}, \quad y(0) = 1.$$

In this case, we have the formula

$$y_{n+1} = y_n + \frac{h}{y_n}$$

and we choose $y_0 = 1$. The following table gives the approximations for the values of y up to $y(2)$ for $h = 0.5$ and $h = 0.25$.

h	$y(0)$	$y(0.25)$	$y(0.5)$	$y(0.75)$	$y(1)$	$y(1.25)$	$y(1.5)$	$y(1.75)$	$y(2)$
0.5	1		1.5		1.8333		2.1061		2.3435
0.25	1	1.25	1.45	1.6224	1.7765	1.9172	2.0476	2.1697	2.2849

In this case, the equation is separable and we can calculate the exact solution, namely $y(x) = \sqrt{2x+1}$, with $y(2) = \sqrt{5} \approx 2.2361$.

In order to understand more sophisticated methods, we now have a look at Euler's method from a different point of view. If y is a solution of (4.4.1), then the fundamental theorem of calculus implies that

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} F(x, y(x))\, dx. \tag{4.4.2}$$

If we imagine that we know what $F(x, y(x))$ is and approximate the integral by the left point rule, we obtain

$$y(x_{n+1}) \approx y(x_n) + hF(x_n, y(x_n)).$$

This is exactly the approximation formula underpinning the method. We may now replace the left point rule by something else and see what happens.

4.4.2 Backward Euler method

As the name suggests, this method is closely related to Euler's method. The relationship between the two is similar to the relationship between the left point rule and the right point rule for numerical integration. But when we solve differential equations, the fact that we need to treat the points $x_1, x_2, \ldots$ consecutively, gives rise to some new complications.

Again we divide the interval $[x_0, \infty)$ into small intervals of the same length h, giving rise to $x_n = x_0 + nh$ for $n = 1, 2, \ldots$, and set $y_n = y(x_n)$. But now we use the approximation formula

$$\frac{dy}{dx}(x_{n+1}) \approx \frac{y_{n+1} - y_n}{h}.$$

Using (4.4.1), this gives rise to

$$\frac{y_{n+1} - y_n}{h} \approx F(x_{n+1}, y_{n+1})$$

If we replace the approximation by equality and rewrite the formula, we obtain

$$y_{n+1} = y_n + hF(x_{n+1}, y_{n+1}).$$

The next step is to solve this equation for $y(x_{n+1})$. But since it appears on both sides of the equation, this may be difficult to do. The way out of this difficulty is to solve the equation *numerically*. For example, we may use Newton's method at this step.

Assuming that we have successfully done this, we expect that

$$y(x_{n+1}) \approx y_{n+1}.$$

Just like for the previous method, we then feed y_{n+1} into the next step and construct an approximate solution step by step.

Because this method does not provide explicit formulas, but rather requires an additional numerical method as an intermediate step, it is called an *implicit method* (in contrast to the forward Euler method, which is explicit).

Example 4.4.2. Consider the initial value problem

$$\frac{dy}{dx} = -y, \quad y(0) = 1.$$

If we want to solve this with the backward Euler method, we obtain the formula

$$y_{n+1} = y_n - hy_{n+1}.$$

In this case, we can solve for y_{n+1} exactly:

$$y_{n+1} = \frac{y_n}{1+h}.$$

Thus for $h = 0.25$, we obtain the following values.

h	$y(0)$	$y(0.25)$	$y(0.5)$	$y(0.75)$	$y(1)$	$y(1.25)$	$y(1.5)$	$y(1.75)$	$y(2)$
0.25	1	0.8	0.64	0.512	0.4096	0.3277	0.2621	0.2097	0.1678

The exact solution is $y(x) = e^{-x}$ and $y(2) \approx 0.1353$.

This is example is not typical; in general we will need to use Newton's method or something similar in each step, which means that this method will require a lot more computation than the forward Euler method. The reason why implicit methods are used nevertheless is that they are more robust. Certain differential equations require an extremely small step size h for explicit methods, making them impractical. In contrast, implicit methods typically work well for these equations.

4.4.3 Other methods

We have seen that for numerical integration, the trapezium rule is better than the left or right point rule. If we use it to approximate the integral in (4.4.2), it gives rise to

$$y_{n+1} = y_n + \frac{h}{2}(F(x_n, y_n) + F(x_{n+1}, y_{n+1})). \tag{4.4.3}$$

This is the *trapezium method*. It is an implicit method, because in order to solve (4.4.3) for y_{n+1}, we typically have to use numerical methods again.

There are many other methods. For example, we could have approximated the integral by Simpson's rule. This would have given one of a family of methods called *Runge-Kutta methods*. These methods are among the most commonly used, because they are very efficient, but as they are also slightly complicated, we do not discuss them in detail.

Chapter 5

Functions of several variables

5.1 Background

So far we have considered functions of one variable, where the input is a single quantity. But some quantities depend on several things. For example, the volume of a box depends on the length, width, and height; so this corresponds to a function of three variables, say ℓ, w, h, which is determined by the formula

$$V = \ell w h.$$

If we denote this specific function by f, then we can write $V = f(\ell, w, h)$.

A function of several variables still has a graph. For a function of two variables, this is a surface in a three-dimensional space. For example, if we have $z = f(x, y)$, then the graph of f comprises all points with coordinates (x, y, z) satisfying this equation. This can still be visualised, even though it is more difficult to draw. The graph of a function of three variables is an object in a four-dimensional space, which is difficult to imagine.

Example 5.1.1. The graph of the function given by $z = \frac{x^2y^2}{1000}$ is depicted in Fig. 5.1.1.

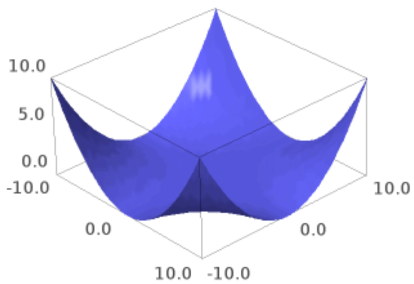

Figure 5.1.1: The graph of the function in two variables given by $z = \frac{x^2y^2}{1000}$.

Most of the things we have seen so far have generalisations for functions in several variables. We will discuss differentiation in particular here.

5.2 Partial derivatives

The easiest way to differentiate a function in several variables is to treat it like a function in one variable. That is, we single out one variable and pretend that the other variables are constants. This has the advantage that we can use all the techniques seen earlier, but it has the disadvantage that we will get only partial information, as we have ignored all the other variables.

Suppose, for example, that we have a function f, depending on the variables x and y. If we 'freeze' y, then we have a function depending only on x. The derivative of this function is called the *partial derivative* of f with respect to x and denoted by $\frac{\partial f}{\partial x}$. This is still a function of x and y, so if we evaluate it at a specific point, we write $\frac{\partial f}{\partial x}(x, y)$. More formally, we have

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, if we 'freeze' x and then differentiate, we obtain the partial derivative with respect to y, denoted $\frac{\partial f}{\partial y}$. Thus

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

For functions of more than two variables, we can define partial derivatives the same way, 'freezing' all variables but one. For a function of n variables, this gives rise to n different partial derivatives.

Example 5.2.1. For $f(x, y) = xy^2$, we have

$$\frac{\partial f}{\partial x}(x, y) = y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 2xy.$$

Example 5.2.2. For $f(x, y, z) = xe^y + \cos z$, we have

$$\frac{\partial f}{\partial x}(x, y, z) = e^y, \quad \frac{\partial f}{\partial y}(x, y, z) = xe^y, \quad \text{and} \quad \frac{\partial f}{\partial z}(x, y, z) = -\sin z.$$

5.3 Tangent plane and linear approximation

We restrict our attention to functions of two variables here, because it is useful to visualise the following concepts geometrically. Nevertheless, similar statements are also valid for functions of more than two variables.

Just like the graph of a function of one variable can be approximated by the tangent line near a specific point, for functions of two variables, we have an approximation by tangent planes. Suppose that we have $z = f(x, y)$ for a certain function f. Fix a point (x_0, y_0) in the plane. Then for $z_0 = f(x_0, y_0)$, the point (x_0, y_0, z_0) belongs to the graph of f. If we want to determine the tangent plane at this point, we can again 'freeze' one variable. If we fix y_0, then effectively we consider the intersection of the graph with the plane given by $y = y_0$. The intersection of the tangent plane with the plane $y = y_0$ corresponds to the tangent line to the graph of the function given by $z = f(x, y_0)$, which is now a function in one variable (see Fig. 5.3.1). This tangent line is given by the equations

$$\begin{aligned} z &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0), \\ y &= y_0. \end{aligned}$$

If we 'freeze' $x = x_0$, we obtain another line, given by

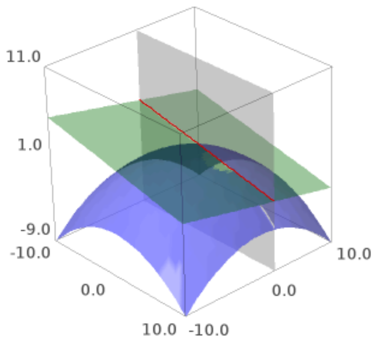

Figure 5.3.1: The graph of the function f with $f(x, y) = 1 - \frac{x^2+y^2}{20}$ (blue) together with its tangent plane at $(2, 2, \frac{3}{5})$ (green), the plane given by $y = 2$ (grey) and the intersection of these planes (red)

$$\begin{aligned} z &= f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0), \\ x &= x_0. \end{aligned}$$

The tangent plane is the plane containing both of these lines. It is given by the equation

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Why is this useful? Sometimes we want to compute approximate values of a function using a linear approximation, and for a function in two variables, the tangent plane tells us what the appropriate linear approximation is. Recall the Taylor polynomials for functions in one variable from Sect.

2.5. In its simplest form, Taylor's theorem says that for a function g of one variable, we have

$$g(x) \approx g(x_0) + g'(x_0)(x - x_0),$$

provided that x is near x_0. For a function f of two variables, the corresponding statement is:

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Again this holds if x is near x_0 and y is near y_0. Sometimes it is more convenient to write the formula in terms of the differences $\delta x = x - x_0$ and $\delta y = y - y_0$. Then it becomes

$$f(x_0 + \delta x, y_0 + \delta y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\delta x + \frac{\partial f}{\partial y}(x_0, y_0)\delta y.$$

This is valid for small values of δx and δy.

For a function of three variables, the corresponding formula is:

$$\begin{aligned} f(x, y, z) \approx f(x_0, y_0, z_0) &+ \frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) \\ &+ \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) \end{aligned}$$

or

$$\begin{aligned} f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z) \approx f(x_0, y_0, z_0) &+ \frac{\partial f}{\partial x}(x_0, y_0, z_0)\delta x \\ &+ \frac{\partial f}{\partial y}(x_0, y_0, z_0)\delta y + \frac{\partial f}{\partial z}(x_0, y_0, z_0)\delta z \end{aligned}$$

For functions of more than three variables, we have to modify it accordingly.

5.4 Total derivatives

Suppose that $z = f(x, y)$ for a certain function f, and the quantities x and y both depend on another quantity, say t (so they correspond to certain functions, too). This makes z depend on t, and we may ask what $\frac{dz}{dt}$ is. This is called the *total derivative* of z with respect to t.

In order to answer this question, we need a version of the chain rule from Sect. 2.2.4 for functions in several variables. For the situation described above, this is

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{5.4.1}$$

Similar rules hold for functions of three or more variables. For example, if $w = f(x, y, z)$, and x, y, and z all depend on t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

A situation that is very common, especially if t stands for time, is that we have $z = f(t, x, y)$ (possibly involving other variables as well). Then we can use the same chain rule and observe in addition that $\frac{dt}{dt} = 1$. This gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

We can see why we have these formulas using the linear approximations from the preceding section. For example, in the situation $z = f(x, y)$, we have

$$f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

Moreover,

$$\delta x \approx \frac{dx}{dt}\delta t \quad \text{and} \quad \delta y \approx \frac{dy}{dt}\delta t,$$

so that

$$\frac{f(x + \delta x, y + \delta y) - f(x, y)}{\delta t} \approx \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

The left-hand side is a difference quotient and will converge to $\frac{dz}{dt}$ when we let $\delta t \to 0$. This gives rise to formula (5.4.1).

Example 5.4.1. A particle moves in a circular motion on the plane, so that its coordinates at time t are given by $x = \cos t$, $y = \sin t$. The concentration c of some chemical at time t at the point (x, y) is given by $e^x y \sin t$. What is the rate of change of the concentration experienced by the particle?
Solution. We have

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x}\frac{dx}{dt} + \frac{\partial c}{\partial y}\frac{dy}{dt} = e^x y \cos t - e^x y \sin^2 t + e^x \sin t \cos t.$$

5.5 Second derivatives

If we compute the partial derivatives of a function $f(x, y)$ of two variables, we obtain two new functions $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$. Both of them are functions of two variables again. So we can try to differentiate them as well. If successful, we get the second order partial derivatives of f. These are

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x, y)\right), \qquad \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x, y)\right),$$
$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x, y)\right), \qquad \frac{\partial^2 f}{\partial y^2}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x, y)\right).$$

Example 5.5.1. If $f(x, y) = x^3 \cos y$, then

$$\frac{\partial f}{\partial x}(x, y) = 3x^2 \cos y \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = -x^3 \sin y.$$

Hence

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x\cos y, \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -3x^2 \sin y,$$
$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -3x^2 \sin y, \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = -x^3 \cos y.$$

It is no coincidence that two of the expressions in this example are the same. It is in fact always true that

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

Thus we only need to calculate three second derivatives for a function of two variables.

For a function $f(x,y,z)$ of three variables, we may differentiate the partial derivatives $\frac{\partial f}{\partial x}(x,y,z)$, $\frac{\partial f}{\partial y}(x,y,z)$, and $\frac{\partial f}{\partial z}(x,y,z)$ once more and we obtain

$$\begin{array}{lll} \frac{\partial^2 f}{\partial x^2}(x,y,y), & \frac{\partial^2 f}{\partial x \partial y}(x,y,z), & \frac{\partial^2 f}{\partial x \partial z}(x,y,z), \\ \frac{\partial^2 f}{\partial y \partial x}(x,y,z) & \frac{\partial^2 f}{\partial y^2}(x,y,z), & \frac{\partial^2 f}{\partial y \partial z}(x,y,z), \\ \frac{\partial^2 f}{\partial z \partial x}(x,y,z) & \frac{\partial^2 f}{\partial z \partial y}(x,y,z), & \frac{\partial^2 f}{\partial z^2}(x,y,z). \end{array}$$

In practice, we don't have to calculate all of these, as

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}.$$

Of course we can differentiate more than twice, and we use similar notation for these higher order partial derivatives, e.g.,

$$\frac{\partial^3 f}{\partial y^3} \quad \text{or} \quad \frac{\partial^5 f}{\partial x^3 \partial y^2}.$$

5.6 Applications

5.6.1 Maxima and minima

Just as for functions of one variable, a function of two variables, say $z = f(x,y)$, has a *local maximum* at a given point (c,d) if $f(x,y) \leq f(c,d)$ for all points (x,y) in the immediate neighbourhood of (c,d). Similarly, we say that f has a *local minimum* at (c,d) if $f(x,y) \geq f(c,d)$ for all points (x,y) in the immediate neighbourhood of (c,d).

For functions of one variable, we can typically find the local minima and maxima by considering the derivative. For functions of several variables, we can apply similar principles.

Suppose that f has a local maximum at the point (c, d). Then consider the function g obtained by 'freezing' the second variable. That is, $g(x) = f(x, d)$. This function automatically has a local maximum at c, and if a derivative exists, then we know that it has to vanish at c. But of course the derivative of g is the partial derivative of f. So

$$\frac{\partial f}{\partial x}(c, d) = 0.$$

Similarly, 'freezing' the first variable, we see that

$$\frac{\partial f}{\partial y}(c, d) = 0.$$

Also, the same applies to local minima. Therefore, if we look for local maxima or minima, we can get a set of candidates by solving this pair of equations for c and d.

Example 5.6.1. Find the local maxima and minima of $z = f(x, y) = (1 - x^2)^2 + y^2$.
Solution. We compute

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= -4x(1 - x^2), \\ \frac{\partial f}{\partial y}(x, y) &= 2y. \end{aligned}$$

Thus we need to solve the system of equations

$$\begin{aligned} -4x(1 - x^2) &= 0, \\ 2y &= 0. \end{aligned}$$

The second equation is easy to solve: $y = 0$. The first one we can factorise, as

$$-4x(1 - x^2) = -4x(1 - x)(1 + x).$$

We obtain the solutions $x = 0$, $x = 1$, and $x = -1$. For the system consisting of both equations, we have the solutions $(0, 0)$, $(1, 0)$, and $(-1, 0)$.

It is not difficult to see that $(1, 0)$ and $(-1, 0)$ are local minima. This is because $f(\pm 1, 0) = 0$, while $f(x, y) \geq 0$ everywhere. The other point, $(0, 0)$, is less obvious. Here we need other ways to decide.

As for the case of functions of one variable, we can usually (but not always) distinguish between local maxima, local minimia, and other candidates using second derivatives. The following criteria apply

- If $\frac{\partial f}{\partial x}(c,d) = 0$ and $\frac{\partial f}{\partial y}(c,d) = 0$, and in addition,
$$\frac{\partial^2 f}{\partial x^2}(c,d) + \frac{\partial^2 f}{\partial y^2}(c,d) < 0$$
and
$$\frac{\partial^2 f}{\partial x^2}(c,d)\frac{\partial^2 f}{\partial y^2}(c,d) - \left(\frac{\partial^2 f}{\partial x \partial y}(c,d)\right)^2 > 0,$$
then f has a local maximum at (c,d).
- If $\frac{\partial f}{\partial x}(c,d) = 0$ and $\frac{\partial f}{\partial y}(c,d) = 0$, and in addition,
$$\frac{\partial^2 f}{\partial x^2}(c,d) + \frac{\partial^2 f}{\partial y^2}(c,d) > 0$$
and
$$\frac{\partial^2 f}{\partial x^2}(c,d)\frac{\partial^2 f}{\partial y^2}(c,d) - \left(\frac{\partial^2 f}{\partial x \partial y}(c,d)\right)^2 > 0,$$
then f has a local minimum at (c,d).
- If $\frac{\partial f}{\partial x}(c,d) \neq 0$ or $\frac{\partial f}{\partial y}(c,d) \neq 0$ or
$$\frac{\partial^2 f}{\partial x^2}(c,d)\frac{\partial^2 f}{\partial y^2}(c,d) - \left(\frac{\partial^2 f}{\partial x \partial y}(c,d)\right)^2 < 0,$$
then f has neither a local maximum nor a local minimum at (c,d).

Example 5.6.2. In Example 5.6.1, we compute
$$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 4, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 0, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2.$$
Thus
$$\frac{\partial^2 f}{\partial x^2}(0,0)\frac{\partial^2 f}{\partial y^2}(0,0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right) = -8.$$
So we have neither a local minimum nor a local maximum at $(0,0)$.

The method is not restricted to two variables. For example, for a function $f(x,y,z)$ of three variables, we would solve the system
$$\begin{aligned}\frac{\partial f}{\partial x}(x,y,z) &= 0,\\ \frac{\partial f}{\partial y}(x,y,z) &= 0,\\ \frac{\partial f}{\partial z}(x,y,z) &= 0,\end{aligned}$$
in order to find candidates for local maxima or minima. It is still possible to use second derivatives in order to distinguish between local maxima and local minima, but this is rather more complicated for functions in three variables and is not discussed here.

5.6.2 Exact differential equations

Occasionally problems for functions of one variable become easier when we reformulate it in terms of functions of several variables. Here we consider ordinary differential equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \tag{5.6.1}$$

where M and N are two given functions in two variables.

Example 5.6.3. The differential equation

$$y\frac{dy}{dx} = \sin(x+y)$$

can be written in this form, where $M(x,y) = -\sin(x+y)$ and $N(x,y) = y$. (But there are other ways to do it. For example, the choice $M(x,y) = -\frac{\sin(x+y)}{y}$ and $N(x,y) = 1$ would have produced the required form, too.)

Suppose that we look for solutions $y(x)$ of (5.6.1) that are given *implicitly* through an equation of the form $h(x,y) = 0$. Then the total derivative of $z = h(x,y)$ with respect to x is

$$\frac{dz}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx}.$$

But since $z = h(x,y) = 0$, we must have the equation $\frac{dz}{dx} = 0$. So

$$\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx} = 0.$$

(This is in fact just an example of implicit differentiation. If we consider a function $y = y(x)$ given implicitly by the equation $h(x,y) = 0$, then implicit differentiation gives this formula.) Comparing this equation with (5.6.1), we see that we will get a solution if we can find h such that

$$\frac{\partial h}{\partial x}(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial h}{\partial y}(x,y) = N(x,y).$$

It is not always possible to solve this system of equations. In fact, if we have a solution, then it will necessarily satisfy

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x},$$

as we have seen in Sect. 5.5. In terms of M and N, this is expressed as follows:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{5.6.2}$$

If (5.6.2) is satisfied, then we say that the differential equation (5.6.1) is *exact*. If (5.6.2) is *not* satisfied, then there is no chance to solve (5.6.1) with this method. On the other hand, if our equation *is* exact, then it turns out that we can solve the above system for h, and this will give a solution of (5.6.1) in implicit form.

Example 5.6.4. Consider the differential equation

$$2x + y^2 + (2xy + 1)\frac{dy}{dx} = 0.$$

This is of the form (5.6.1) with $M(x, y) = 2x + y^2$ and $N(x, y) = 2xy + 1$. We first check that the equation is exact, computing

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y.$$

Thus we may attempt the method, to which end we need to solve

$$\frac{\partial h}{\partial x}(x, y) = 2x + y^2 \quad \text{and} \quad \frac{\partial h}{\partial y}(x, y) = 2xy + 1.$$

In order to find h, we first consider the first equation only. If we fix y, then this just means that we need to find an antiderivative of the function $2x + y^2$ (with respect to the variable x). Hence

$$h(x, y) = x^2 + xy^2 + C(y)$$

for some integration 'constant' $C(y)$, which is important here and may depend on y. (So it is a function of y rather than a proper constant.) We now need to determine $C(y)$, and to this end, we consider the second equation for h. Inserting the expression that we already have, we obtain

$$2xy + \frac{dC}{dy} = 2xy + 1.$$

So we find that $\frac{dC}{dy} = 1$, which implies that $C(y) = y + c$ for another integration constant c. This finally gives

$$h(x, y) = x^2 + xy^2 + y + c.$$

That is, we have found implicit solutions of the differential equations in terms of the equation

$$x^2 + xy^2 + y + c = 0.$$

If we want the solution explicitly, we have to solve this for y. It is a quadratic equation in y, so we can do that and we get

$$y = \frac{-1 \pm \sqrt{1 - 4x(x^2 + c)}}{2x}.$$

Example 5.6.5. Consider the equation

$$y \cos x + 2xe^y + (\sin x + x^2 e^y - y^2)\frac{dy}{dx} = 0.$$

Here we set $M(x, y) = y \cos x + 2xe^y$ and $N(x, y) = \sin x + x^2 e^y - y^2$ and check that

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y = \frac{\partial N}{\partial x}.$$

So the equation is exact. We need to solve

$$\frac{\partial h}{\partial x} = y \cos x + 2xe^y \quad \text{and} \quad \frac{\partial h}{\partial y} = \sin x + x^2 e^y - y^2.$$

The first equation implies that

$$h(x, y) = y \sin x + x^2 e^y + C(y)$$

for some function $C(y)$. The second equation implies that

$$\sin x + x^2 e^y + \frac{dC}{dy} = \sin x + x^2 e^y - y^2.$$

Hence $C(y) = c - \frac{y^3}{3}$, and we have solutions of the differential equation given implicitly in terms of the equation

$$y \sin x + x^2 e^y - \frac{y^3}{3} + c = 0.$$

In this case, it's difficult to find an explicit representation, so we leave it in this form.

Index