Class Test

Solutions

1. Using the logarithm rules, we can transform this equation as follows:
$$\ln(x+1) = 1.$$
This means that $x + 1 = e$ and thus $x = e - 1$.

2. We want to find r and θ such that
$$\cos t + 30\sin t = r\cos(t+\theta) = r\cos t\cos\theta - r\sin t\sin\theta.$$
This means solving the system
$$r\cos\theta = 1, \quad -r\sin\theta = 30,$$
or equivalently, finding the polar coordinates of the point $(1, -30)$. The solution is $r = \sqrt{901}$ and $\theta = 2\pi - \arctan 30$ (since the point is in the fourth quadrant). Hence
$$\cos(t + 2\pi - \arctan 30) = \frac{15}{\sqrt{901}}.$$
The solutions satisfy
$$t + 2\pi - \arctan 30 = \pm\arccos\frac{15}{\sqrt{901}} + 2\pi n,$$
where n stands for an arbitrary integer, and after solving for t, we have the solutions
$$t = \arctan 30 \pm \arccos\frac{15}{\sqrt{901}} + 2\pi m,$$
where $m = n - 1$.

3. A combination of the quotient rule and the product rule gives
$$\frac{d}{dx}\left(\frac{e^x}{x\ln x}\right) = \frac{xe^x\ln x - e^x(\ln x + 1)}{x^2(\ln x)^2} = \frac{e^x}{x^2(\ln x)^2}(x\ln x - \ln x - 1).$$

4. Differentiating both sides of the equation, we obtain
$$(2y - x^2e^y)\frac{dy}{dx} - 2xe^y = -e.$$
Hence
$$\frac{dy}{dx} = \frac{2xe^y - e}{(2y - x^2e^y)}.$$
If $x = 1$ and $y = 1$, then
$$\frac{dy}{dx} = \frac{e}{2-e}.$$

5. We have
$$f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x).$$
Since $e^x > 0$ for all x, the equation $f'(x) = 0$ holds true exactly when when $\cos x = \sin x$, which is the case when $x = \frac{\pi}{4} + \pi n$ for an integer n.

Now note that $f''(x) = -2e^x \sin x$, which is negative at $x = \frac{\pi}{4} + 2\pi n$ and positive at $x = \frac{\pi}{4} + \pi + 2\pi n$ for any integer n. Therefore, we have a local maximum $x = \frac{\pi}{4} + 2\pi n$ and a local minimum at $x = \frac{\pi}{4} + \pi + 2\pi n$ for any integer n.

6. We use the substitution $u = \cos x - x$, so that $du = -(\sin x + 1)\,dx$. Then
$$\int \frac{\sin x + 1}{\cos x - x}\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x - x| + C.$$

7. It's convenient to first write
$$\log_5 x = \frac{\ln x}{\ln 5}.$$
Now we use integration by parts, giving
$$\begin{aligned}\int_1^5 x \log_5 x\,dx &= \frac{1}{\ln 5}\int_1^5 x \ln x\,dx \\ &= \frac{1}{\ln 5}\left[\frac{x^2}{2}\ln x\right]_1^5 - \frac{1}{\ln 5}\int_1^5 \frac{x}{2}\,dx \\ &= \frac{25}{2} - \frac{1}{\ln 5}\left[\frac{x^2}{4}\right]_1^5 = \frac{25}{2} - \frac{6}{\ln 5}.\end{aligned}$$

8. We first differentiate the function 3 times, using the product rule each time. This gives
$$\begin{aligned}f'(x) &= \cosh x + x \sinh x, \\ f''(x) &= 2\sinh x + x\cosh x, \\ f'''(x) &= 3\cosh x + x \sinh x.\end{aligned}$$
Now by the formula for the Taylor polynomials,
$$\begin{aligned}T_{3,e}(x) &= f(e) + f'(e)(x-e) + \frac{f''(e)}{2}(x-e)^2 + \frac{f'''(e)}{6}(x-e)^3 \\ &= e\cosh e + (\cosh e + e\sinh e)(x-e) + \left(\sinh e + \frac{e}{2}\cosh e\right)(x-e)^2 \\ &\quad + \left(\frac{1}{2}\cosh e + \frac{e}{6}\sinh e\right)(x-e)^3\end{aligned}$$
is the answer to the question.

RM, 06/11/2017