Given a function f, the Taylor series of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Often there is an interpretation of this as a limit:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(provided that the limit exists). If so, we may hope that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k,$$

at least if x is close to a. This is not always the case, but for many functions it is true.

If a function is represented by its Taylor series, then you can manipulate the terms of the series similarly to polynomials.

<u>Example</u>

$$\sin(2x) = 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} - \cdots$$

$$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$$

for all x

__Example__

$$\sin x \cos x = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)$$

$$= \left(x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^3}{6} + \frac{x^5}{12} + \frac{x^5}{120} + \dots\right)$$

$$= x - \frac{2}{3}x^3 + \left(\frac{1}{24} + \frac{1}{12} + \frac{1}{120}\right)x^5 + \dots$$

for all x

__Example__

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left(1 - x + x^2 - x^3 + \dots\right)$$

$$= -1 + 2x - 3x^2 + \dots$$

for $-1 < x < 1$.

2.6 Numerical Differentiation

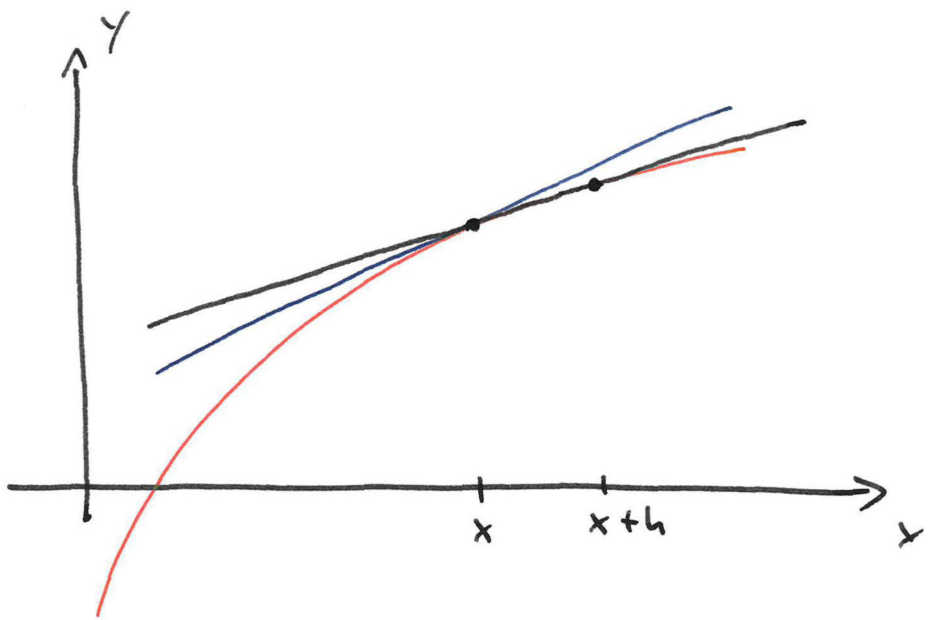
Suppose that we want to find $f'(x)$ (for a given function f) numerically.

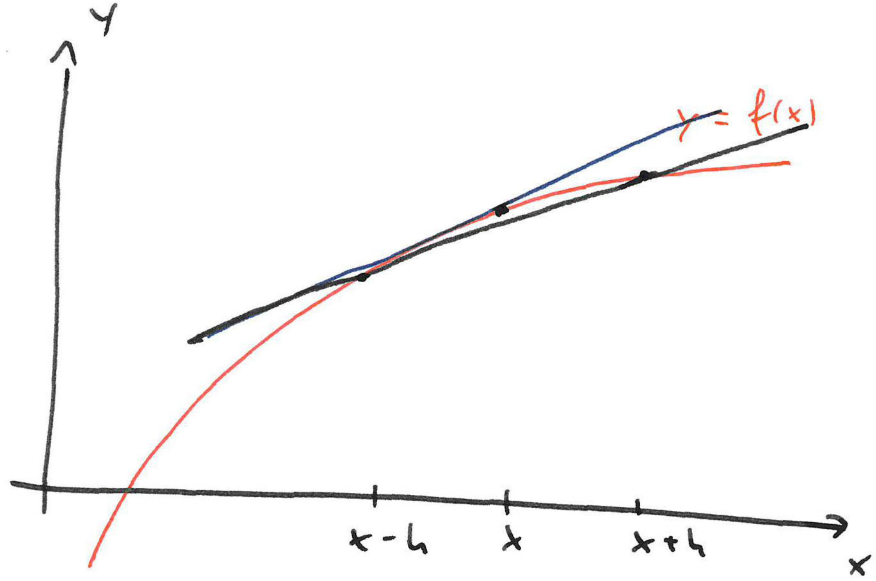
One way is to use

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

as long as h is small. This works but the following formula works better:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$





Suppose that f is represented by its Taylor series. Then

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \dots$$

So

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{f''(x)}{2}h + \cdots$$

The error in the formula

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

is

$$\frac{f''(x)}{2}h + \cdots$$

This is of order h.
We also find that

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \cdots$$

So

$$\frac{f(x+h)-f(x-h)}{2h} = \frac{2f'(x)h}{2h} + \frac{\frac{1}{3}f'''(x)h^3}{2h} + \ldots$$

$$= f'(x) + \frac{f'''(x)h^2}{6} + \ldots$$

The error in

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

is

$$\frac{f'''(x)}{6}h^2 + \ldots,$$

which is of order h^2.

Example of a Taylor series

Let $f(x) = \sin x$. Let $a = 3$. We compute

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x,$$

$$f^{(4)}(x) = \sin x.$$

So

$$T_{n,3}(x) = \frac{\sin 3}{0!} + \frac{\cos 3}{1!}(x-3) + \frac{-\sin 3}{2!}(x-3)^2 + \frac{-\cos 3}{6}(x-3)^3 + \frac{\sin 3}{24}(x-3)^4 + \cdots + \frac{f^{(n)}(x)}{n!}(x-3)^n$$

The Taylor series is

$$\sin 3 + \cos 3\,(x-3) + \frac{-\sin 3}{2}(x-3)^2 + \ldots$$