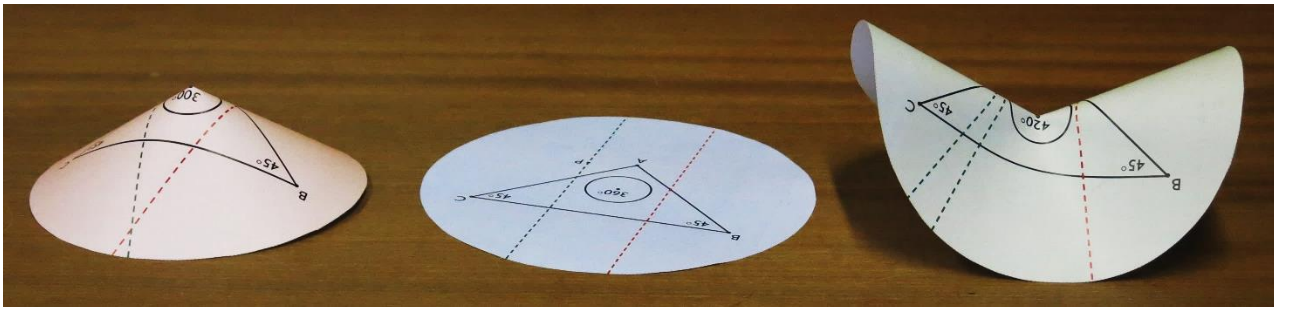
Simple non-Euclidean paper models

OK, let's start by admitting that anything you make out of flat paper will have zero intrinsic* curvature, except at vertices (where the paper isn't continuous anyway). But we can make some instructive models where the curvature is concentrated at one point, for example the apex of a cone.

Print just the last two pages of this document (on separate sheets!) and follow the instructions to make the three models. If you must, save some pennies by printing in black-and-white, and/or spend extra pennies by persuading your pdf reader and printer driver to print them bigger, on A3 paper. You will be cutting and sticking (I used Pritt glue because it's handy to use, dries quickly and isn't too messy), but do not fold creases in any of the models.

Before you start gluing, admire how straight all the lines (that aren't circles) are. They will still be straight lines (or *geodesics*, strictly speaking) when the models are finished. Committed sceptics can also check that the 45° angles really are 45°.

Your models should look something like this:

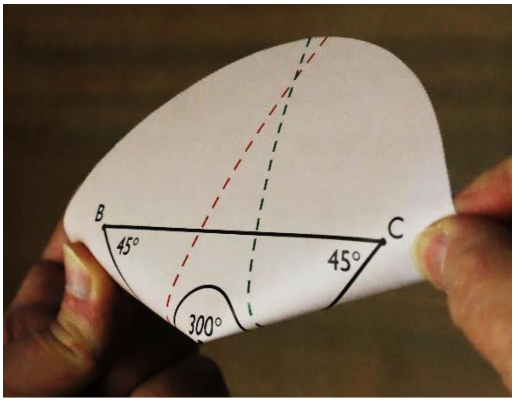


Positive curvature (spherical)

Zero curvature (flat)

Negative curvature (hyperbolic)

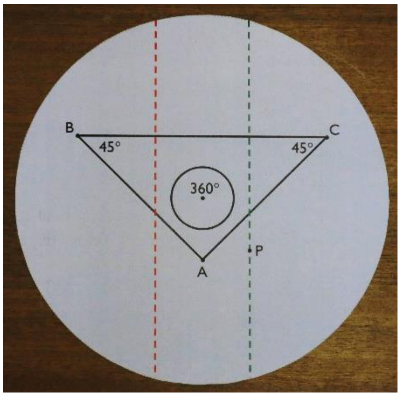
You can verify that the lines are still straight by gently pulling (without creasing) part of a model to make it *extrinsically** flat for a moment:



* *Extrinsic curvature is the way a surface is curved in the 3-D space it's embedded in; intrinsic curvature depends only on geometrical measurements confined to the surface.*

Flat space (zero curvature)

The first model is quite trivial - it's just a flat Euclidean disc. It represents flat Euclidean 2-D space, and it does it very well.



The dashed lines are parallel, and illustrate Playfair's statement of Euclid's parallel postulate (features specific to the model are in parentheses):

"Given a (red dashed) line and a point (P) not on it, exactly one (green dashed) line can be drawn through the point (P) that does not intersect the given (red dashed) line."

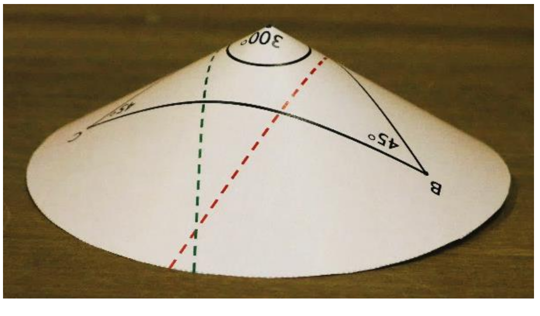
(It's assumed that all lines are straight and extend to infinity in both directions.)

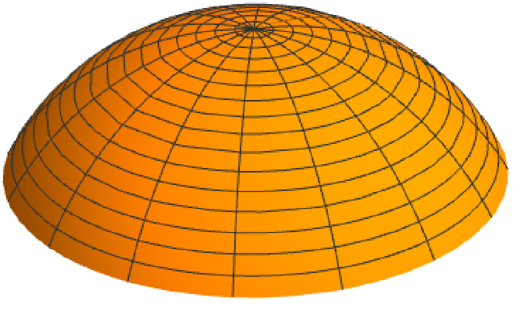
The 45° isosceles triangle ABC has angles that add up to 180°. The angle at A is pretty-obviously a right angle.

The total angle around the centre of the disc is 360°. There is no *angular defect* there. The circle around it has a circumference that is 2π times the radius.

Spherical space (positive curvature)

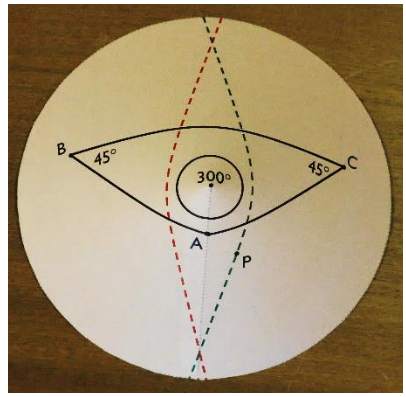
The second model is a cone, approximating 2-D space that is curved like a sphere, except that all the curvature is concentrated at the centre (the apex of the cone):





There's more space in the radial direction, compared to the tangential direction, than in the Euclidean case ⇒ the surface is heaped in the radial direction.

If we look at it from above, we see geometrical features that are typical of spherical curvature. Remember that those bent-looking lines are in fact intrinsically "straight" (geodesic) on the surface.



The dashed lines illustrate one way to deny Euclid's parallel postulate by changing the underlined text:

"Given a (red dashed) line and a point (P) not on it, <u>no</u> (green dashed) line can be drawn through the point (P) that does not intersect the given (red dashed) line."

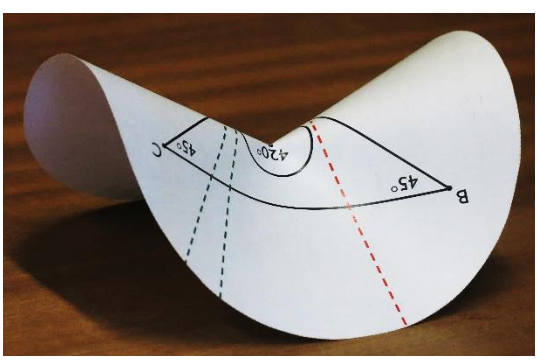
No matter how you swivel the green line about P, it will intersect the red line somewhere.

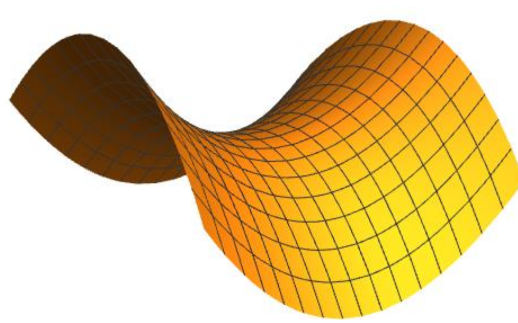
The 45° isosceles triangle ABC has angles that add up to more than 180°. The angle at A is pretty-obviously obtuse; in fact it's 150°.

The total angle around the centre is 300°. There is an angular defect of 60° there, because we removed a 60° sector to make the model. The circle around it has a circumference that is less than 2π times the radius, because we removed 60° of its circumference with the sector.

Hyperbolic space (negative curvature)

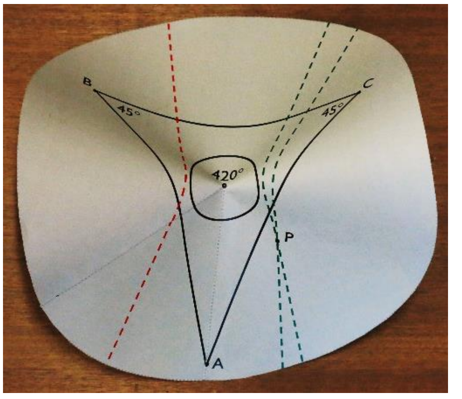
The third model approximates 2-D space that is curved like a saddle, except that all the curvature is concentrated at the centre (the "apex" of the "cone"):





There's more space in the tangential direction, compared to the radial direction, than in the Euclidean case ⇒ the surface is heaped in the tangential direction.

Again, if we look at it from above, we see geometrical features that are typical of hyperbolic curvature.



The dashed lines illustrate the opposite way to deny Euclid's parallel postulate by changing the underlined text:

"Given a (red dashed) line and a point (P) not on it, more than one (green dashed) line can be drawn through the point (P) that does not intersect the given (red dashed) line."

Both green lines diverge from the red line at the edges of the model, and there's an infinity of other such lines that can be drawn through P.

The 45° isosceles triangle ABC has angles that add up to less than 180°. The angle at A is pretty-obviously acute; in fact it's 30°.

The total angle around the centre is 420°. There is an *angular excess* of 60° there, because we inserted a 60° sector to make the model. The circle around it has a circumference that is more than 2π times the radius, because we inserted 60° of extra circumference with the sector.

How our curved models are different from smoothly-curved 2-D surfaces

Our models are only curved at the centre, so parallels, triangles and circles that do not surround the centre obey Euclidean geometry. You can press the flat model against any part of the other models apart from their centres. This is not true of smoothly-curved surfaces, like an actual sphere. On the other hand, small-enough areas of smoothly-curved surfaces are *locally flat* (tending to Euclidean) with no angular defect anywhere (always 360° at a point), whereas the tip of a cone is just as pointy however much you zoom in on it.

Other craft ideas for exploring non-Euclidean surfaces

Search the web for hyperbolic footballs, quilts, crochet and games, for example:

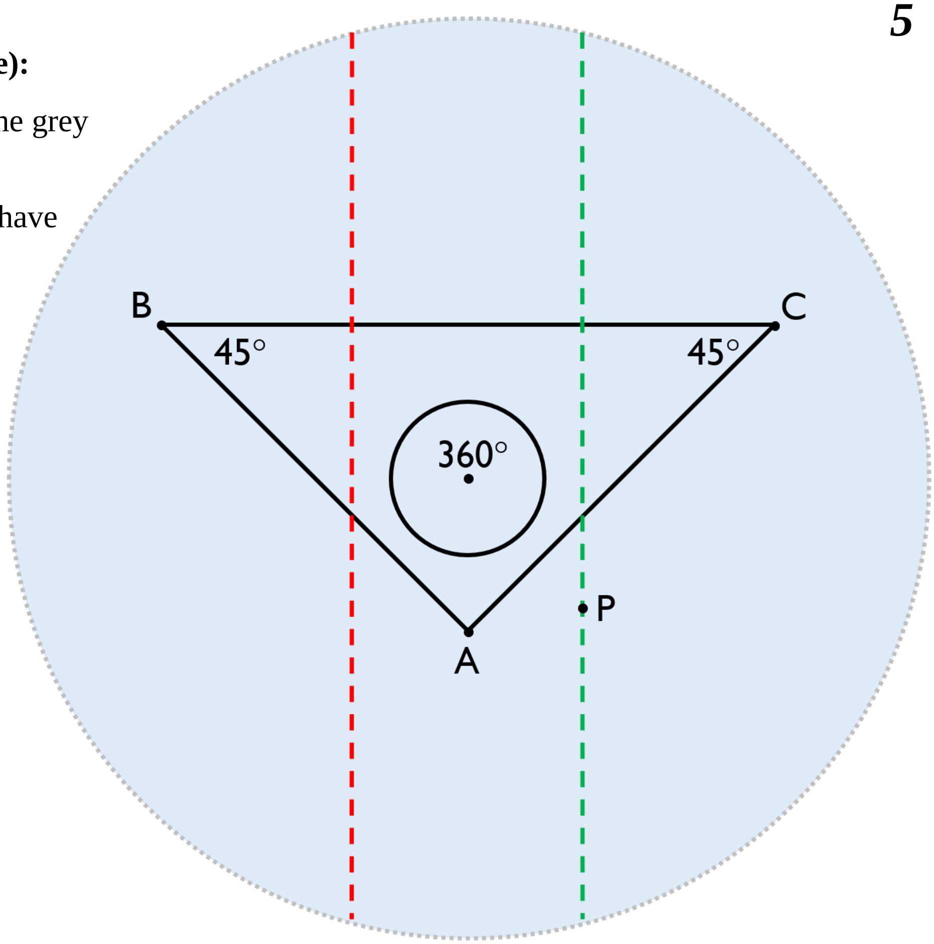
https://www.math.tamu.edu/~sottile/research/stories/hyperbolic_football/index.html
http://theiff.org/images/IFF_HypSoccerBall.pdf
https://blog.doublehelix.csiro.au/hyperbolic-paper-craft
http://geometrygames.org/HyperbolicBlanket
https://pi.math.cornell.edu/~dtaimina/hypplanes.htm
https://blog.doublehelix.csiro.au/hyperbolic-crochet
https://www.goldenlucycrafts.com/2015/07/16/crochet-hyperbolic-coral
https://www.roguetemple.com/z/hyper

No warranties - these links have not been checked for malware.

Flat space (zero curvature):

(a) Cut out the disc along the grey dotted line.

(b) Err... that's it. You now have a flat Euclidean disc.

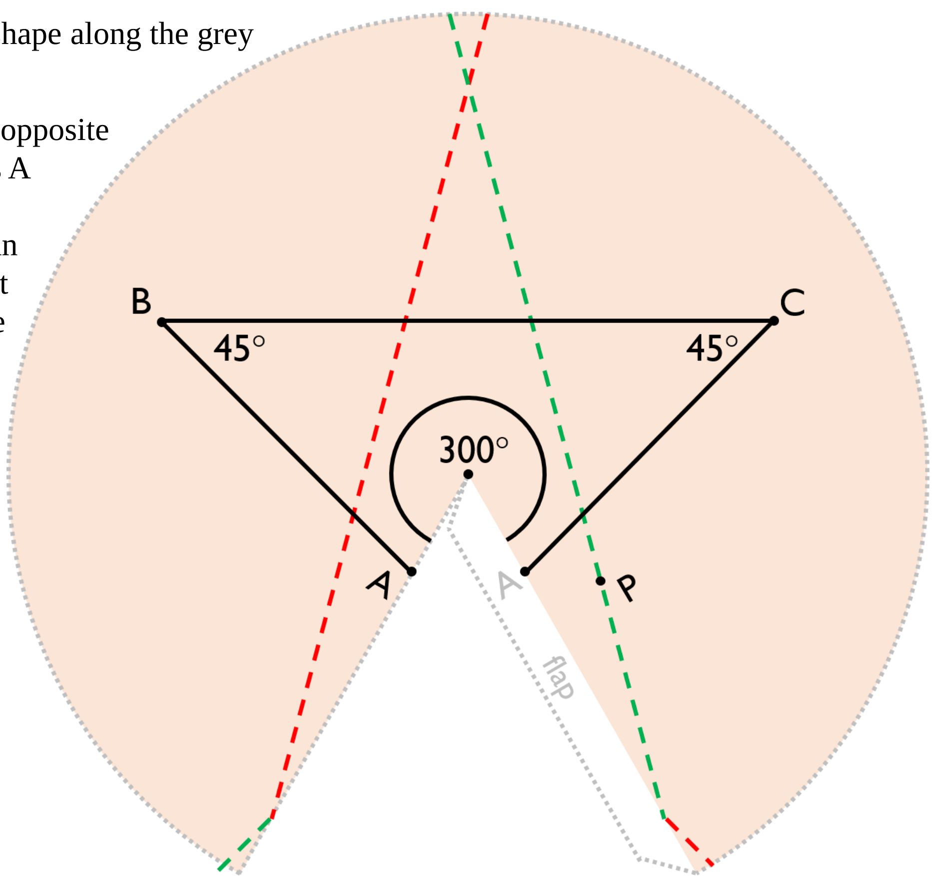


Spherical space (positive curvature):

(a) Cut out the "pac-man" shape along the grey dotted lines.

(b) Glue the flap under the opposite edge, so that the two points A coincide. (If you turn the model upside down, you can firmly press the join against a table top to make the glue stick better.)

(c) You now have a paper cone.

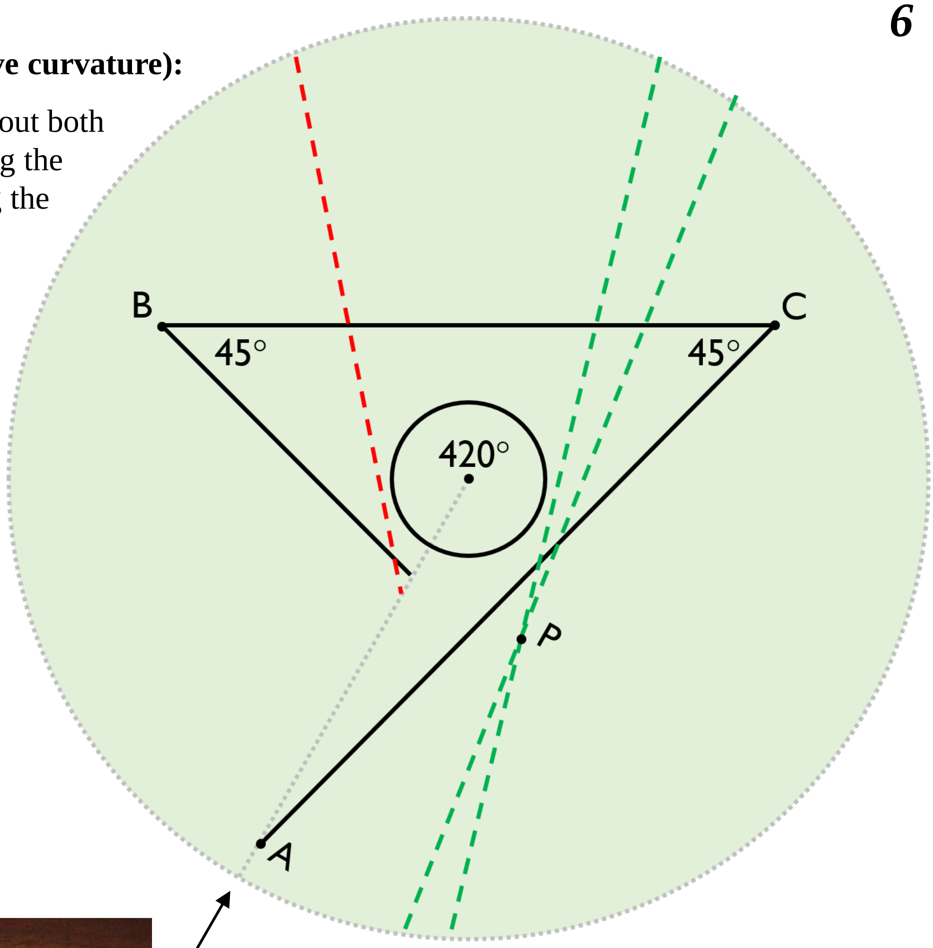


Hyperbolic space (negative curvature):

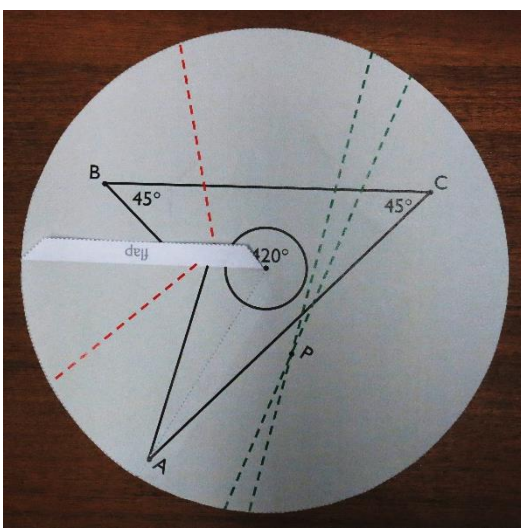
(a) This has two parts. Cut out both the disc and the sector along the grey dotted lines, including the radial slit indicated by the arrow.

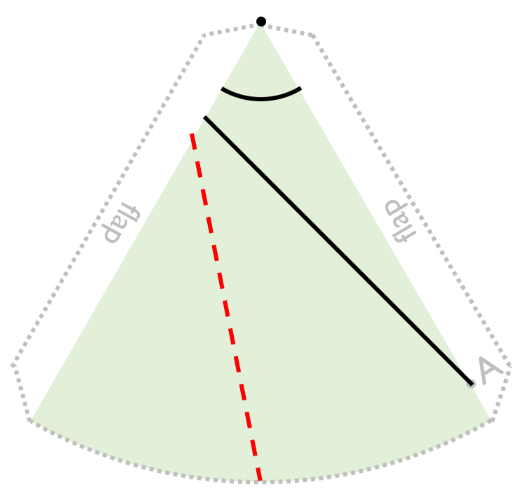
(b) Apply glue to both of the sector's flaps.

(c) Glue the right flap under the anti-clockwise edge of the slit in the disc, so that the two points A coincide.



The model after step (c):





(d) Glue the left flap under the clockwise edge of the slit in the disc. The disc will need to warp, but be sure to avoid creasing it. (Again, if you turn the model upside down, you can firmly press the join against a table top to make the glue stick better.)

(e) You now have the "opposite of a cone", made from a disc by inserting a 60° sector instead of removing one.