The following questions form the coursework element of MA40059 Mathematical Methods 2 [M&P]. They concern the application of Green's function methods for solving PDEs in the analysis of advanced physical problems, and a physical problem involving calculus of variations.

Answers to these questions should be handed in to the box marked "MA40059 – Maths Methods" to be found just inside the door of the Undergraduate Physics Laboratory (3W3.19) by 12.15 on Thursday 9th May 2019. Following standard practice in the Department of Physics, work handed in late will lose 20% of all available marks for each 24 hours (or part thereof) past the deadline.

Note that some marks are awarded for presentation. Full marks will be awarded for work which correctly answers the questions, with sketches that are suitable drawn and annotated, and with answers logically structured with a concise explanation of the mathematical steps performed. Marks will be lost for work which just gives intermediate and/or final results, or consists of pages of algebra without explanation. <u>Each problem carries equal weight</u>.

Note that whilst I encourage you to discuss the problems with fellow students, the work that you hand in should be substantially your own. Implicit in the handing in of your work is the declaration that it is your own, and where you have used other sources of assistance that these are appropriately acknowledged and referenced.

If you do have questions/problems that you would like help with, ask in one of the scheduled drop-in sessions.

The marks available for this piece of assessed work are 20% of the total mark for this unit.

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Problem 1

This is a problem in electrostatics, concerning the electrostatic potential $\Phi(\mathbf{r})$ due to a point charge outside a grounded conducting sphere, radius R. Electrons are free to move in the conductor, and in the presence of the external charge a charge distribution is generated which itself creates an electric field, such that when equilibrium is reached (in typically 10^{-14} seconds) the charge distribution is static and the electric field in the sphere vanishes. Hence the potential inside the sphere is constant, and the potential at the surface is an equipotential surface, which by definition is $\Phi = 0$.[1]

The starting point for determining the electrostatic potential outside the conductor is Poisson's equation, which relates the charge density $\rho(\mathbf{r})$ and the electrostatic potential $\Phi(\mathbf{r})$, which in SI units is given by

$$\nabla^2\Phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_0 \tag{1}$$

where ε_0 is the dielectric constant of free space. The solution of (1) in volume V with bounding surface S may be written

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')dV + \frac{1}{4\pi}\oint_S \left[G(\mathbf{r},\mathbf{r}')\frac{\partial\Phi(\mathbf{r}')}{\partial n'} - \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\Phi(\mathbf{r}')\right] dS' \tag{2}$$

where $\partial/\partial n$ denotes the derivative in the direction normal to the surface and G is the Green function satisfying

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{3}$$

where $\delta(\mathbf{r})$ is the Dirac-delta function

$$\int_V \delta(\mathbf{r})dV = \begin{cases} 1 & \mathbf{r}\in V \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

a) Write down (2) when Φ vanishes on S and the Green function is chosen to satisfy Dirichlet boundary conditions on S. Hence show that in the present problem of the grounded sphere and where the charge density in V is that of a single electron at $\mathbf{r}_0 \in V$ [i.e. the charge density is $\rho(\mathbf{r}) = -|e|\delta(\mathbf{r}-\mathbf{r}_0)$], the electrostatic potential is directly related to the Dirichlet Green function G_D.

b) Now find G_D, assuming the sphere is centered on the origin. To do so, set

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r},\mathbf{r}') \tag{5}$$

and determine the equation satisfied by F and the corresponding boundary conditions.[2] Expand the angular dependence of F using spherical harmonics[3]

$$F(\mathbf{r},\mathbf{r}') = \sum_{\ell=0}^{\infty}\sum_{m=-l}^{l} F_{\ell,m}(r,\mathbf{r}')Y_{\ell,m}(\vartheta,\varphi), \tag{6}$$

substitute, and solve. Hence obtain an expression for the electrostatic potential outside the sphere.

c). Use your result to show that the electron outside the sphere feels an attractive force due to the induced screening charge, and obtain it's asymptotic ($r\to\infty$) form. [you may find it useful to recall $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}, \quad |x|<1$.]

d). By making appropriate substitutions/taking appropriate limits, **use your answer to b)** to determine the exact form for the classical force on an electron a distance z above a plane surface of an ideal conductor.

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[1] Because it is grounded. In theory this can be achieved by connecting the object to earth using an infinitesimally thin perfectly conducting wire (available at all good electrical shops) so as not to otherwise disturb the electrostatic field.

[2] You may quote the results

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}'), \qquad \text{and} \qquad \frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell,m}(\vartheta,\varphi)Y^*_{\ell,m}(\vartheta',\varphi')$$

where $Y_{\ell,m}(\vartheta,\varphi)$ is a spherical harmonic[3] and $r_>$ ($r_<$) is the greater (lesser) of r, r'.

[3] e.g. `http://en.wikipedia.org/wiki/Spherical_harmonics#Spherical_harmonics_expansion` – I don't normally recommend wikipedia, but I've read the info and it looks sound. See also e.g. Arfken, *Mathematical methods for physicists*, Academic Press. Useful properties of the spherical harmonics that you may quote are

$$\nabla^2 Y_{\ell,m}(\vartheta,\varphi) = -\frac{\ell(\ell+1)}{r^2}Y_{\ell,m}(\vartheta,\varphi), \qquad \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\vartheta,\varphi)Y^*_{\ell,m}(\vartheta,\varphi) = \frac{2\ell+1}{4\pi}, \qquad \text{and} \qquad \int_S Y_{\ell,m}(\vartheta,\varphi)Y^*_{\ell',m'}(\vartheta,\varphi)dS = \begin{cases} 1 & \ell=\ell', m=m' \\ 0 & \text{otherwise.} \end{cases}$$

where S is the surface of a sphere.

Problem 2

Let $\mathcal{L}$ be a differential operator of Sturm-Liouville form: $\mathcal{L}y(x) \equiv \dfrac{d}{dx}\left(p(x)\dfrac{dy(x)}{dx}\right) + q(x)y(x)$.

(a) If $y_1(x)$ and $y_2(x)$ are two different solutions of the homogeneous Sturm-Liouville equation $\mathcal{L}y = 0$, show that the quantity

$$\Delta[y_1, y_2] = p(x)\left(y_1(x)y_2'(x) - y_2(x)y_1'(x)\right)$$

(where $' \equiv d/dx$) is a constant. Δ is called the Wronskian. [Hint: show $d\Delta/dx = 0$.]

(b) Let $G(x,\tilde{x})$ be the Green function corresponding to $\mathcal{L}$, i.e. G satisfies

$$\mathcal{L}G(x,\tilde{x}) = \frac{d}{dx}\left(p(x)\frac{dG(x,\tilde{x})}{dx}\right) + q(x)G(x,\tilde{x}) = \delta(x-\tilde{x}), \qquad \text{Eqn. 1}$$

where the right hand side is the Dirac delta function. Let $y_a(x)$ and $y_b(x)$ be solutions of $\mathcal{L}y = 0$ that satisfy the boundary conditions at a and b.

(i) Write down expressions for $G(x,\tilde{x})$ valid for $x < \tilde{x}$ and $x > \tilde{x}$ in terms of y_a, y_b, and $\tilde{x}$-dependent coefficients.

(ii) Integrating Eqn. 1 across the delta function, identify matching conditions at $x = \tilde{x}$, and use these to establish expressions for the coefficients.

(iii) Hence show $G(x,\tilde{x})$ can be written in terms of solutions of the homogeneous problem satisfying boundary conditions at a and b, and their Wronskian.

(c) The solution of $Ly = f$ where L is a linear second order differential operator is

$$y(x) = y_0(x) + \int G(x,\tilde{x})f(\tilde{x})d\tilde{x}$$

where y_0 is the solution of the homogeneous equation $Ly = 0$, and the Green function satisfies

$$LG(x,\tilde{x}) = \delta(x-\tilde{x}).$$

L contains derivatives with respect to x.

(i) Find $G(x,\tilde{x})$ corresponding to $L = -(\hbar^2/2m)d^2/dx^2 - E$, where E is a constant with infinitesimal positive imaginary part, and requiring that G be well behaved as $x \to \pm\infty$. Explain carefully your working.

(ii) Hence show that the solution $\psi(x)$ of the one-dimensional time-independent Schrödinger equation for arbitrary potential $V(x)$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

can be written as

$$\psi(x) = \psi_0(x) - \frac{im}{\hbar^2 k}\int e^{ik|x-\tilde{x}|}V(\tilde{x})\psi(\tilde{x})dx$$

where $k = \sqrt{2mE/\hbar^2}$ and $\psi_0(x)$ is a free-space ($V = 0$) solution.

Aside: Written formally the solution is $\psi = \psi_0 - GV\psi$. Rearranging $\psi = (1+GV)^{-1}\psi_0$, or $\psi = T\psi_0$ where T is the transition operator. We have $T = (1+GV)^{-1}$, so $(1+GV)T = 1$ or explicitly in one-dimension

$$T(x,\tilde{x}) = \delta(x-\tilde{x}) - \int G(x,\tilde{x}')V(\tilde{x}')T(\tilde{x}',\tilde{x})d\tilde{x}',$$

an integral equation for T. The transition operator plays as important role in scattering problems. There, we are interested in how an incident free particle described by a wave function ψ_0 is scattered by a localised object, described by potential V. $\psi = T\psi_0$ tells us just that.

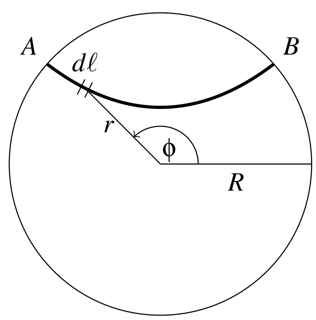
Problem 3

a). The gravitational potential energy $U(\boldsymbol{r})$ of a mass m due to a mass density $\rho(\boldsymbol{r})$ satisfies $\nabla^2 U = 4\pi G m \rho$, where G is the gravitational constant. If the earth is considered to be a uniform sphere of mass M, radius R, show that the gravitational potential energy of a mass m *inside* the earth a distance r from the center is

$$U(r) = \frac{mg}{2R}(r^2 - 3R^2)$$

where $g = GM/R^2 = 9.81$ m/s^2.

b.) A tunnel is to be constructed through the earth, along which a frictionless train will run between cities at A and B. The track is described by the curve $r(\phi)$ (polar coordinates).



Use energy conservation to derive an expression for the speed v of a train which starts at rest at A as it passes through segment $d\ell$ of the curve at $r(\phi)$, and hence show that the journey time is

$$T = \int dt = \int \frac{d\ell}{v} = \sqrt{\frac{R}{g}} \int_{\phi_A}^{\phi_B} \sqrt{\frac{r^2 + r'^2}{R^2 - r^2}} d\phi.$$

where $r' = dr/d\phi$.

c). The tunnel is to be constructed so as to minimise the journey time. Taking care to note the nature of the integrand, write down an Euler-Lagrange equation for the extremal curve $r(\phi)$ [you are not asked solve this].

Let a be the minimum distance of the tunnel from the center of the earth, where $dr/d\phi = 0$. Use this to obtain an expression for $dr/d\phi$ in terms of r, R, and a.

d). Noting $d\phi = dr/(dr/d\phi)$, reexpress T as an integral

$$T = 2\int_a^R \ldots dr$$

and evaluate. (You may quote tables or use `https://www.wolframalpha.com/calculators/integral-calculator` where necessary).

e). If $R = 6,400$ km determine the journey time in minutes through the center of the earth.