Out-of-gap Bose-Einstein solitons in optical lattices

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We study the existence and the spectral and collisional properties of bright and dark solitons in a Bose-Einstein condensate (BEC) in optical lattices. Both types of these solitons are located on the background of a matter wave of finite amplitude with characteristic frequency lying outside the forbidden gap. Thus we term our solitons out-of-gap solitons. We discuss feasibility of practical preparation of the background wave to observe these solitons and make connections with recent experiments studying BEC in optical lattices. We prove that the dark out-of-gap solitons are robust against perturbations and numerically verify the possibility of their excitation using a phase imprinting technique.

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I. INTRODUCTION

It is well known that dispersion and diffraction of waves having different physical origin can be efficiently controlled by the periodic potentials. After recent breakthroughs in experimental observations of atomic Bose-Einstein condensates (BEC) in periodic optical fields (optical lattices), see, e.g., Refs. [1–6], there is a growing theoretical interest in properties of periodic and localized states of coherent matter waves in periodic potentials [7–10]. In particular, detailed studies of the stability properties of the nonlinear Bloch states were recently reported in Refs. [7,8]. Atomic Bragg or gap solitons and their well-known optical analogs, see Refs. [11–13] and references therein, have been studied quite comprehensively in the situation when characteristic frequency of the soliton-state belongs to the gap, where propagation of any linear excitations is forbidden. These solitons are spatially localized structures having exponentially decaying tails, which ensure their existence inside the gap.

Quite recently, dark atomic solitons with nondecaying tails asymptotically approaching a periodic Bloch state were reported to exist in optical lattices [8]. Characteristic spatial frequency of these solitons belongs to the frequency bands open for propagation of small amplitude waves, therefore, they can be termed out-of-gap solitons. Much earlier, apparently the same results were found analytically within a framework of the coupled-mode approach in the context of nonlinear optics [14]. Note that findings of Refs. [8,14] have not been supported so far by any spectral stability analysis of the out-of-gap solitons, which is important for understanding the feasibility of their experimental observation. Theoretical study of the details of existence and spectral properties of the out-of-gap solitons in BEC appears to be an open and interesting problem, which is addressed in this work.

Conditions for observation of the localized states of BEC in optical lattices on nonzero background (out-of-gap solitons) are different from the conditions required for observation of the solitons with exponentially decaying tails (in-gap solitons). In particular, to observe the out-of-gap solitons it is necessary to prepare condensate in the Bloch state, which belongs to the first excited energy band, i.e., to the band located straight above the first forbidden gap, and has such momentum that the group velocity dispersion of the matter waves is the same as in the free space. Experimental methods to prepare the condensate with required energy and momenta has been recently demonstrated by the NIST group [6]. After the excited state has been prepared, one needs to flip the phase of one half of the condensate using optical phase imprinting, which was successfully applied to create dark solitons in BECs without optical lattices, see Ref. [15]. Observation of the bright in-gap solitons requires preparing a wave packet with energy inside the first forbidden gap, which might prove to be a more complex practical task. Note, that one of the advantages of atomic gap solitons is that they can easily have zero group velocity, which simplifies their visual observation, while optical gap solitons, in the most common configurations, travel with speed of light with no testing methods available to observe what happens inside the fiber grating.

II. MODEL EQUATIONS

We start our analysis from the one-dimensional Gross-Pitaevskii equation describing evolution of the macroscopic matter wave function $\Psi$ of the zero-temperature Bose-Einstein condensate interacting with off-resonant standing electro-magnetic wave in one-dimensional geometry, see, e.g., Refs. [5–10,12]:

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_x^2 \Psi + aE_r \sin^2(k_{\perp}x)\Psi + \frac{4\pi \hbar^2 a}{m A_{\text{eff}}} |\Psi|^2 \Psi.$$  (1)

Here $m$ is mass of an atom, $a$ is a two-body scattering length of atoms, $k_{\perp}$ is the wave number of a photon, $\alpha$ is a dimensionless parameter proportional to the laser intensity, $E_r = \hbar^2 k_{\perp}^2/2m$ is the recoil energy. Three-dimensional harmonic potential, which is often presented in practice is not explicitly included into our model equations. This can be justified, e.g., for the experimental setup using an optical dipole trap, which has been used in the ongoing series of the experiments on observation of the nonlinear dynamics of BEC in optical lattices by the Konstanz group [5]. Let us assume that condensate is initially prepared in the region of intersection of...
the focal areas of two Gaussian beams, say propagating along \(x\) and \(y\) directions. Then, the \(y\) beam is switched off and condensate expands along \(x\) direction until it reaches some equilibrium length determined by the decrease of the on-axis intensity of the Gaussian beam. The Rayleigh range of the beam can be chosen to be sufficiently large, such that the resulting holding potential along \(x\) direction is broad and shallow compare to the potential along \(z\) and \(y\) directions and its influence is disregarded in this work. This approximation suits our purposes particularly well providing that the width of the potential along \(x\) accommodates hundreds or thousands of periods of the standing wave, while characteristic width of the solitonic structures discussed below is order of 10 of periods. On the other hand, confinement along \(z\) and \(y\) is assumed to be so tight that only lowest-order transverse mode of the trap is excited, which is taken into account through the effective area of this mode, \(A_{\text{eff}}\), [12]. Then intensity of the standing wave along \(x\) direction is turned on, which creates a situation described by model (1).

Introducing new dependent and independent variables we rescale Eq. (1) to the dimensionless form

\[
i\partial_x \psi = -\partial_x^2 \psi - \beta \psi \cos 2\xi + s |\psi|^2 \psi, \tag{2}\]

which is convenient for the following analysis. Here \(\tau = \hbar k x^2 / (2m)\), \(\xi = k x\), \(s = \text{sgn}(a)\), \(\beta = a E, ml (2\hbar^2 k_x^2) = a/4\), and \(\psi = \sqrt{8 \pi} |a| / (k_x^2 A_{\text{eff}}) \Psi \exp[-i\beta \tau]\).

A substantial part of the subsequent analysis will be done in the approximation of the coupled-mode approach [11,13] and supplemented, where appropriate, by the direct numerical modeling of Eq. (2). The coupled-mode approximation is valid provided that \(\beta\) is small enough and that the optical lattice has noticeable influence only on the wave packets with momenta close to \(\hbar k\), which scatter resonantly. Dispersion characteristic of the free, \(\alpha = 0\), small amplitude, i.e., linear, phonon \(\Psi \sim e^{-i\omega \tau + i K \xi}\) of the Eq. (1) is given by \(\hbar \omega = \hbar^2 k_x^2 / (2m)\). Parameter \(\beta\) measures the width of the gap opened by the periodic potential relative to the energy \(\hbar^2 k_x^2 / (2m)\) corresponding to the momenta \(\pm \hbar k\) and lying at the center of the first forbidden gap in the spectrum.

Under the above assumptions the wave function \(\psi\) can be presented as a superposition of two counterpropagating waves with momenta \(\pm \hbar k\) and amplitudes varying slowly in \(x\). Characteristic length on which change of these amplitudes becomes noticeable is given by \(2/(\beta k)\). Thus we should require that the Rayleigh length of the holding Gaussian beam is at least order of \(2/(\beta k)\). In our dimensionless variables coupled-mode equations are derived by taking the ansatz

\[
\psi = \frac{2}{\beta} [A_+(X, T) e^{i\xi} + A_-(X, T) e^{-i\xi}] e^{-i(\Omega + 1)T}. \tag{3}\]

Here we once again introduced new independent variables, which are \(T = \beta \tau / 2\) and \(X = \beta (\xi - v \tau) / 2\), and new parameters \(v\) and \(\Omega\). The latter are proportional, respectively, to the group velocity and frequency shift of the atomic wave packets. Both, \(v\) and \(\Omega\) are free parameters, so far. They have been brought into play at this stage for the convenience of the parametrization of the solitary waves, which are studied below.

Now we substitute ansatz (3) into Eq. (2). Using our assumption about smallness of \(\beta\) and following from it slowness of the amplitudes \(A_{\pm} \) we can disregard the terms containing second derivatives in \(X\). Conditions of the resonant scattering allow us to neglect the terms containing spatial harmonics higher then the first one. As a result we derive a well-known nonlinear optics [13] system of the couple-mode equations

\[
i \partial_T A_{\pm} = i(v + 1) \partial_X A_{\pm} - \Omega A_{\pm} - A_{\mp} + s |A_{\pm}|^2 A_{\mp}. \tag{4}\]

Using Eq. (4) we find that dispersion of the small amplitude phonons, \(A_{\pm} \sim e^{iKX}\), propagating in the resting, \(v = 0\), i.e., laboratory, frame, is given by \(\Omega^2 = K^2 + 1\). This dispersion has a forbidden gap for \(\Omega \in (-1, 1)\). In practice the width of the forbidden zone is proportional to the depth of the optical potential but our variables are normalized so that the width is equal to 2. The upper \((\Omega = \sqrt{K^2 + 1} > 0)\) branch of the lattice dispersion characteristic has a group velocity dispersion sign, \(\text{sgn}(d^2 \Omega / dK^2) > 0\), equal to that of the free space, while lower branch has the opposite sign. Practical techniques for population of the upper and lower allowed bands by the Bose-Einstein condensate with selected values of the momenta, are reported, e.g., by the NIST [6] and Konstanz [5] groups.

Zero order, i.e., lowest, allowed band in the energy spectrum of the lattice can be efficiently populated through the adiabatic switching of the lattice potential [5,6]. The first excited band, i.e., the band with \(\Omega > 0\), has corresponding Bloch states dominated by the plane waves with symmetry opposite to the symmetry of the lattice itself and of the eigenmodes of the zero band [6]. Therefore, condensate population can be transferred from the zero-order band up to the first excited band through shaking potential along the \(x\) direction, which creates symmetry breaking perturbations [6].

It is interesting to note that changing sign of \(s\), i.e., changing two-body interaction from repulsive \(s = +1\) to attractive \(s = -1\), together with simultaneous substitution \((A_+, A_-, \Omega) \rightarrow (A_+^*, -A_-^*, -\Omega)\) leave Eq. (4) unchanged. Change of the sign of \(\Omega\), which is in any case a matter of our choice, see ansatz (3), implies that one jumps from the first excited band \((\Omega > 1)\), to the zero-order band \((\Omega < -1)\). It means that all results obtained for atoms with repulsive interaction in the first excited band can be directly applied for the atoms with attractive interaction in the zero-order band. Below we have chosen \(s = +1\), which corresponds to the reported experiments with BECs in optical lattices [1–5].

III. EXISTENCE OF SOLITONS IN OPTICAL LATTICES

Though analytical solutions for the in-gap [13] and out-of-gap [14] solitons in the coupled-mode approach have been found in the past, they represent very little insight in the later case due to their cumbersome and semiexplicit form. For this
reason we present below simple eigenvalue analysis, that gives analytical expressions for the existence regions of both types of the solitons.

Profiles of the solitary solutions \( A_\pm(T,X) = U_\pm(X) \) obey Eq. (4) with \( \partial_x = 0 \), i.e.,

\[
0 = i(U + 1) \partial_x U_\pm - \Omega U_\pm - U_\pm + (|U_\pm|^2 + 2|U_\pm|^2) U_\pm ,
\]

which can be considered as a dynamical system in \( X \). Below we will consider solitary waves connecting different equilibrium points of this system, i.e., solutions such that \( \partial_x = 0 \). We naturally assume that the necessary condition for solitons to exist is that they should have tails exponentially decaying towards the equilibrium points of the phase flow determined by Eq. (5).

We identify three equilibrium states of Eq. (5) as follows:

\[
U_+^{(e)} = 0, \tag{6}
\]

\[
U_-^{(e)} = \pm \sqrt{\frac{\Omega - 1}{3}} e^{i\theta}, \tag{7}
\]

\[
U_+^{(e)} = \sqrt{\frac{\Omega + 1}{3}} e^{i\theta}, \tag{8}
\]

where \( \theta \) is an arbitrary constant phase. Solution (6) is a trivial linear eigenmode, which exists for all \( \Omega \)’s, and solutions (7) and (8) are two nonlinear eigenmodes existing, respectively, for \( \Omega > 1 \) and for \( \Omega > -1 \). Note that Eq. (7) is the odd nonlinear eigenmode, see Eq. (3), and thus its symmetry is opposite to the symmetry of the lattice potential, \( \sin^2(kx) \), while Eq. (8) is the even eigenmode. As we will show in Sec. IV B, solution (8) is always dynamically unstable, while solution (7) can be stable under the quite general conditions.

Now we investigate spectral properties of equilibrium states (6)–(8). This will provide us with knowledge about the spatial profiles of the tails of the possible solitary waves. If an equilibrium has only real or complex eigenvalues, then trajectories in the phase space of Eq. (5) connecting two such equilibria correspond to the solitary solutions. If an equilibrium has purely imaginary eigenvalues, then phase trajectories in its vicinity are limited cycles and we do not expect existence of the soliton families nesting on the top of such equilibria.

The following analysis consists of two steps: first we linearize Eq. (5) in the vicinity of an equilibrium and then seek solutions of the linear system in the exponential form \( -e^{\kappa X} \). Corresponding solvability conditions give us an algebraic equation for \( \kappa \), which determines local behavior of the phase trajectories. For trivial solution (6) we find that

\[
\kappa_{1,2,3,4} = \pm \left( \frac{i\Omega v \pm \sqrt{1 - \Omega^2 - v^2}}{v^2 - 1} \right) . \tag{9}
\]

All \( \kappa \)’s are purely imaginary if \( |\Omega| > 1 \). It means that in this case \( U_\pm = 0 \) is a center and, therefore, outside the forbidden gap, all stationary perturbation on the zero background are oscillatory in \( X \). However, if \( \Omega \) shifts into the gap \( |\Omega| < 1 \)

FIG. 1. BEC densities, \( |U_+|^2 + |U_-|^2 \), corresponding to the dark (a) and bright (b) solitons shown nesting on the equilibrium state \( U_\pm^{(e)} = \pm \sqrt{(\Omega - 1)/3} \). Other parameters are \( \Omega = 1.4, v = 0.2 \).

then \( U_\pm^{(e)} = 0 \) has four complex \( \kappa \)’s, of which two have positive and two have negative real parts, providing that \( \Omega^2 < 1 - v^2 \). The later condition thereby determines the existence region of the in-gap bright solitary waves with exponentially decaying tails.

Equilibrium (7) generates the following values of \( \kappa \)’s:

\[
\kappa_{1,2} = 0, \quad \kappa_{3,4} = \pm \frac{2}{|1 - v^2|} \sqrt{(\Omega - 1) + \frac{v^2}{3}(\Omega + 2)} . \tag{10}
\]

\( \kappa_{3,4} \) are purely real provided that

\[
\Omega > 1, \quad v^2 < 3|\Omega - 1|/|\Omega + 2| . \tag{11}
\]

Thus we expect the existence of moving and resting solitons nesting on background (7) with different values of \( \theta \) for \( X \) approaching \( +\infty \) and \( -\infty \). Numerically solving (4), using the shooting method, we have been able to find two types of such solutions, which are bright and dark out-of-gap solitons. Envelopes of the condensate densities given by \( |U_+|^2 + |U_-|^2 \) and corresponding to these are shown in Fig. 1. For equilibrium state (8) we find that

\[
\kappa_{1,2} = 0, \quad \kappa_{3,4} = \pm \frac{2}{|1 - v^2|} \sqrt{-(\Omega + 1) + \frac{v^2}{3}(\Omega - 2)} . \tag{12}
\]
where $\kappa_{3,4}$ are purely real providing that

$$\Omega > 2, \quad v^2 > 3|\Omega + 1|/|\Omega - 2|.$$  \quad (13)

Thus existence of moving solitary waves on the nonzero background connecting equilibria (8) with different values of $\Theta$ can be expected and was verified numerically. Below, it will be shown that background solution (8) is always dynamically unstable, therefore solitons nesting on the top of it do not represent substantial physical interest and we do not consider them here.

A $(v,\Omega)$-diagram showing regions of existence of the in-gap and out-of-gap solitons in the $(v,\Omega)$ plane is shown in Fig. 2. It is clearly seen that bright and dark solitons located on background states (7) and (8) are out-of-gap solitons. Note that solutions (6), (7), and (8) never become saddle points simultaneously, therefore solitary waves interconnecting backgrounds with different values of $|U_{\pm}|$ do not exist in optical lattices.

Characteristic width of the solitonic structures calculated in the coupled-mode approximation typically covers around 50 of the lattice periods. These are the numbers, which are large enough to satisfy approximation of the slowly varying amplitudes, which was made to derive Eq. (4). On the other hand they are small enough in a way that the solitonic structures are insensitive to the presence of the weak holding potential along $x$, which was neglected in Eq. (1).

IV. STABILITY OF OUT-OF-GAP SOLITONS

A. General theory

To carry out the Bogolyubov-type stability analysis we transform Eq. (4) to the Hamiltonian form

$$i\partial_t \tilde{\phi} + \frac{\delta H}{\delta \tilde{\phi}^*} = 0,$$  \quad (14)

where $\tilde{\phi} = (A_+, A_+^*, A_-, A_-^*)^T$,

$$H = \int dX \left\{ \left(-iA_+^* \partial_X A_+ + iA_-^* \partial_X A_- + A_A + A_A^* \right) + c.c. \right\}$$  
$$+ \frac{1}{2} (|A_+|^2 + |A_-|^2) + \frac{1}{2} \Omega N + vM$$  \quad (15)

is the generalized Hamiltonian, $N = \int dX(|A_+|^2 + |A_-|^2)$ is the number of particles, $M = \int dX\{iA_+^* \partial_X A_+ + iA_-^* \partial_X A_- \} + c.c.$ is the momentum of the condensate and

$$\tilde{\eta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta H/\delta A_+^* \\ \delta H/\delta A_+ \\ \delta H/\delta A_-^* \\ \delta H/\delta A_- \end{bmatrix}.$$  

Let us assume that $\tilde{\phi}_0 = (U_+, U_+^*, U_-, U_-^*)^T$ is a $T$ independent solution of Eq. (14) obeying $\delta H/\delta \tilde{\phi}_0 = 0$. In order to study its stability properties, we take the ansatz

$$\tilde{\phi} = \tilde{\phi}_0 + \tilde{p}(X,T),$$  \quad (16)

where $\tilde{p}$ is the perturbation vector, and substitute it into Eq. (14). Neglect all the terms nonlinear in $\tilde{p}$, we find that dynamical evolution of $\tilde{p}$ is governed by the system

$$i\partial_t \tilde{p} + \hat{B}\tilde{p} = 0,$$  \quad (17)

where $\hat{B}$ is the matrix of the quadratic Bogolyubov Hamiltonian $E_B$. Here

$$\hat{B} = \begin{bmatrix} i(v-1)\partial_X + Q & U_+^* & 2U_-U_+ & 2U_-U_+ \\ U_+ & -i(v-1)\partial_X + Q & 2U_-U_+ & 2U_-U_+ \\ 2U_-U_+ & 2U_-U_+ & i(v+1)\partial_X + Q & U_- \\ 2U_+U_+ & 2U_+U_+ & U_+ & -i(v+1)\partial_X + Q \end{bmatrix}.$$  \quad (18)
The absolute value of $e$ components of which properly takes into account that the second and fourth components of $\tilde{p}$ are its complex conjugated first and third. Here

$$
\tau = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

is the transposition matrix and $\omega$ is the eigenfrequency of the perturbations $\tilde{\xi}(X)$ obeying

$$
\hat{\eta}\hat{B}\tilde{\xi} = \omega\tilde{\xi}.
$$

Equation (21) determines eigenmodes and eigenfrequencies of the condensate in the vicinity of the equilibrium $\phi_0$. We define energy $\epsilon$ carried by any particular eigenmode $\tilde{\xi}$ of the operator $\hat{\eta}\hat{B}$ as

$$
\epsilon = \langle \tilde{\xi} | \hat{B} \tilde{\xi} \rangle = \omega \langle \tilde{\xi} | \hat{\eta}\tilde{\xi} \rangle.
$$

The absolute value of $\epsilon$ obviously depends on the strength and shape of the perturbations, but its sign does not. If $\hat{\eta}\hat{B}$ has an eigenmode with negative energy it has twofold consequences. First is that a solution with negative-energy modes can be neither a local nor a global minimum of the Hamiltonian $H$ and, therefore, it is expected to be unstable in the presence of dissipation. Second is that the negative-energy eigenmodes can resonate with positive-energy ones generating quartets of complex eigenvalues leading to dynamical instabilities.

Note that if $\omega$ is imaginary or complex then conservation of the Hamiltonian requires that $\langle \tilde{\xi} | \hat{\eta}\tilde{\xi} \rangle = 0$, and therefore any such eigenmode carries zero energy [16]. Note also that all real and imaginary $\omega$'s are presented in pairs, $\pm \omega$, and all complex $\omega$'s in quadruplets $\pm \omega, \pm \omega^*$, Thus presence of any $\omega$ with $\text{Im} \omega \neq 0$ unavoidably signals dynamical instability.

B. Stability of the background solutions

Stability of the zero background is irrelevant for our purposes, and therefore we start our analysis from solution (7). We seek eigenmodes of $\hat{\eta}\hat{B}$ in the exponential form, i.e., $\tilde{\xi}(X) = e^{i\lambda X}$ and find that the spectrum of Eq. (7) is given by the solution of the following biquadratic equation:

$$
\begin{align*}
(\omega - kv)^4 - &\frac{4}{3} \left[ k^2 + \Omega + 2 \right] (\omega - kv)^2 + k^2[k^2 + 4\Omega - 4] \\
= &0.
\end{align*}
$$

Condition for all $\omega$'s to be real for any $k$ is that $\Omega < 5/2$, which together with the existence condition $\Omega > 1$ gives a range of $\Omega$'s, where solution (7) is dynamically stable. Computing eigenvectors corresponding to all eigenvalues found from Eq. (23) shows that the eigenvectors corresponding to the two pairs of $\omega$'s carry energy with opposite signs, see Fig. 3. For $\Omega > 5/2$ this solution becomes dynamically unstable due to resonance of the positive- and negative-energy modes (Hamiltonian Hopf instability) generating zero-energy spectral branches with $\text{Im} \omega \neq 0$. This instability is due to growth of the short-wavelength excitations with $k^2 > |\Omega + 2|/2\Omega - 5/3|/\Omega$.

Similar analysis carried out for solution (8) gives the characteristic polynomial as

$$
\begin{align*}
(\omega - kv)^4 - &\frac{4}{3} \left[ k^2 - \Omega + 2 \right] (\omega - kv)^2 + k^2[k^2 - 4\Omega - 4] \\
= &0.
\end{align*}
$$

One can show that for $k^2 < 4(\Omega + 1)$ there always exist purely imaginary $\omega$'s, therefore the background solution (8)
is dynamically unstable with respect to the long-wavelength excitations. Providing that $V$ is taken within the forbidden gap, where Eq. (8) also exists, this instability is expected to generate trains of the bright in-gap solitons on the zero background.

Thus results of the stability analysis of the background solutions agree with intuitively clear conjecture that the modulationally stable state of the condensate of repelling atoms embedded into optical lattice should exist only under the conditions, when sign of the overall effective group-velocity dispersion coincide with that in the no lattice case, i.e., for $\Omega > 1$.

**C. Stability of the solitons**

Thus out of the two nontrivial equilibria only Eq. (7) provides dynamically stable background for the solitary waves, therefore from now on we do not consider solitons on background (8). Solving eigenvalue problem (21) numerically for $\tilde{\phi}_0$ being either bright or dark soliton on background (7), we found that the dark solitons are dynamically stable for the wide range of checked values of $\nu$ and $\Omega$. We also found that the negative-energy eigenmodes come only from the spectrum of the background. Thus instability of the dark solitons can arise only in the presence of dissipation due to negative-energy branches in the spectrum of the background, see Fig. 3. In contrary, the bright out-of-gap solitons on stable background (7) were found to be dynamically unstable due to a new solitonic instability. The instability is generated by the quartet of complex eigenfrequencies found in the spectrum of $\tilde{\eta} B$ through out the wide range of $\nu$ and $\Omega$. The eigenmodes governing this instability are weakly localized and are close to the spatially unbounded excitations, see Fig. 4. To verify existence of this instability we numerically solved Eq. (2) initializing it with the bright soliton found from the coupled-mode approach. The instability exhibits itself through emission of the linear waves out from the soliton core towards its periphery and gradual decay of the soliton to the distorted stable solution (7), see Fig. 5. It was checked that the spatial profiles of the growing perturbation and its growth rate were very close to those found from eigenvalue problem (21).

We have also verified stability of the dark solitons within framework of Eq. (2) and checked their robustness against collisions. We have found that two dark solitons repel one another, providing that their relative velocity is small.
The minimal approach distance depends on the initial soliton velocities and above some critical velocity one cannot distinguish the solitons at the moment of their collision, see Fig. 6a. Nevertheless outcome of this high-speed collision is the same two solitons moving with the same velocities. Let us stress that interaction does not lead to any noticeable radiative effects and solitons emerge from the collision in their original undistorted form.

FIG. 6. Numerical modeling of Eq. (2) showing spatiotemporal evolution of the condensate density $|\psi|^2$ during collisions of two dark solitons moving with velocities $v = \pm 0.07$ (a) and $v = \pm 0.4$ (b). $\beta = 0.024$ and $\Omega = 1.7$.

FIG. 7. Velocity $v$ of the dark out-of-gap soliton vs the difference $\delta \theta$ between the values of $\theta$ approached on the soliton tails for $X \rightarrow \pm \infty$. Solid line shows results obtained from Eq. (5) for $\Omega = 1.6$ and dots correspond to the direct numerical modeling of Eq. (2).

FIG. 8. Numerical modeling of Eq. (2) showing spatiotemporal evolution of the condensate density $|\psi|^2$ during excitation of the resting (a) and moving (b) dark solitons by phase imprinting technique. Pulse parameters are $A = 2.72$, (a) $\tau_0 = 1.156$ and (b) $\tau_0 = 4.624$. Background (7) was prepared for $\Omega = 1.6$.

FIG. 9. Numerical modeling of Eq. (2) showing spatiotemporal evolution of the condensate density $|\psi|^2$ during decay of the dark out-of-gap soliton, when periodic potential is switched off. $\beta = 0.087$ and $\Omega = 1.6$. 

D. Phase imprinting for out-of-gap solitons

Finally, we demonstrate numerically how dark out-of-gap solitons in optical lattices can be excited using well-known phase imprinting technique [15]. Assuming that dynamically stable state (7) of the condensate is achieved in the first place, we send a short optical pulse over one half of the condensate, flipping its phase with respect to the other half. Practically it was done assuming that \( \mu \) in Eq. (1) and hence \( \beta \) in Eq. (2) has the following dependence on \( \tau \) and \( \xi \): \( \beta = \lambda \Theta(\xi - \xi_0) \Theta(\tau - \tau_0 - \tau) \), where \( \Theta \) is the Heaviside function, \( \xi_0 \) is the position, where we aim to excite the solitary wave, \( \tau_0 \) is the pulse duration, and \( \lambda \) is proportional to the pulse amplitude. It is important to note here, that the difference \( \partial \theta \) between the values of \( \theta \) approached on the soliton tails for \( X \rightarrow \pm \infty \) has unique correspondence with velocity of the soliton, see Fig. 7. Pulse area \( A \tau_0 \) determines the phase difference \( \partial \theta \) between the right and left halves of the condensates acquired during the excitation process and thereby controls velocity of the solitons. Excitation of resting and moving dark solitons is demonstrated in Fig. 8.

Note, that out-of-gap dark solitons are different from the dark Bose-Einstein solitons existing without optical lattices [15] in several aspects. First, during the excitation of the out-of-gap solitons we observe strong symmetric ripples emitted from the soliton core, see Fig. 8. Excitation of the dark solitons without periodic potential leads to the emission of a similar wave, but only to the one side of the soliton core [15]. Also, if, after the dark out-of-gap soliton is created, the periodic potential is switched off, we observe its immediate decay, see Fig. 9. This decay happens through emission of the two dips in the condensate density, which propagate symmetrically out of the core of the initially resting soliton. These dips are reminiscent to the conventional dark solitons. However, our computational window and any practical size of the condensate will hardly be enough to verify it reliably.

V. SUMMARY

We studied the existence and stability properties of different classes of out-of-gap Bose-Einstein solitons in optical lattices. We have found that dark out-of-gap solitons are dynamically stable and robust against collisions among themselves. We also demonstrated that these solitons can be created by means of phase imprinting technique.

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