Inverting A Matrix

- Swap rows or columns
- Multiply a whole row or column by a number
- Add or subtract one row or column from another
- Whatever we do to the Left matrix we do to the Right one

$$\mathbf{k} \qquad\qquad \mathbf{I}$$

$$\left[\begin{array}{ccccccc} k_{x_1x_1} & k_{x_1y_1} & k_{x_1x_2} & k_{x_1y_2} & \cdots & k_{x_1x_n} & k_{x_1y_n} \\ k_{y_1x_1} & k_{y_1y_1} & k_{y_1x_2} & k_{y_1y_2} & \cdots & k_{y_1x_n} & k_{y_1y_n} \\ k_{x_2x_1} & k_{x_2y_1} & k_{x_2x_2} & k_{x_2y_2} & \cdots & k_{x_2x_n} & k_{x_2y_n} \\ k_{y_2x_1} & k_{y_2y_1} & k_{y_2x_2} & k_{y_2y_2} & \cdots & k_{y_2x_n} & k_{y_2y_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{x_nx_1} & k_{x_ny_1} & k_{x_nx_2} & k_{x_ny_2} & \cdots & k_{x_nx_n} & k_{x_ny_n} \\ k_{y_nx_1} & k_{y_ny_1} & k_{y_nx_2} & k_{y_ny_2} & \cdots & k_{y_nx_n} & k_{y_ny_n} \end{array}\right] \Bigg\| \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

$$
\begin{bmatrix}
k_{x_1x_1} & k_{x_1y_1} & k_{x_1x_2} & k_{x_1y_2} & \cdots & k_{x_1x_n} & k_{x_1y_n} \\
k_{y_1x_1} & k_{y_1y_1} & k_{y_1x_2} & k_{y_1y_2} & \cdots & k_{y_1x_n} & k_{y_1y_n} \\
k_{x_2x_1} & k_{x_2y_1} & k_{x_2x_2} & k_{x_2y_2} & \cdots & k_{x_2x_n} & k_{x_2y_n} \\
k_{y_2x_1} & k_{y_2y_1} & k_{y_2x_2} & k_{y_2y_2} & \cdots & k_{y_2x_n} & k_{y_2y_n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{x_nx_1} & k_{x_ny_1} & k_{x_nx_2} & k_{x_ny_2} & \cdots & k_{x_nx_n} & k_{x_ny_n} \\
k_{y_nx_1} & k_{y_ny_1} & k_{y_nx_2} & k_{y_ny_2} & \cdots & k_{y_nx_n} & k_{y_ny_n}
\end{bmatrix}
$$

Divide Row1 through by 1st Diagonal ($k_{x_1x_1}$)

$$\begin{bmatrix} k_{x_1x_1} & k_{x_1y_1} & k_{x_1x_2} & k_{x_1y_2} & \cdots & k_{x_1x_n} & k_{x_1y_n} \\ k_{y_1x_1} & k_{y_1y_1} & k_{y_1x_2} & k_{y_1y_2} & \cdots & k_{y_1x_n} & k_{y_1y_n} \\ k_{x_2x_1} & k_{x_2y_1} & k_{x_2x_2} & k_{x_2y_2} & \cdots & k_{x_2x_n} & k_{x_2y_n} \\ k_{y_2x_1} & k_{y_2y_1} & k_{y_2x_2} & k_{y_2y_2} & \cdots & k_{y_2x_n} & k_{y_2y_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{x_nx_1} & k_{x_ny_1} & k_{x_nx_2} & k_{x_ny_2} & \cdots & k_{x_nx_n} & k_{x_ny_n} \\ k_{y_nx_1} & k_{y_ny_1} & k_{y_nx_2} & k_{y_ny_2} & \cdots & k_{y_nx_n} & k_{y_ny_n} \end{bmatrix}$$

$$
\begin{bmatrix}
1 & ? & ? & ? & \cdots & ? & ? \\
k_{y_1x_1} & k_{y_1y_1} & k_{y_1x_2} & k_{y_1y_2} & \cdots & k_{y_1x_n} & k_{y_1y_n} \\
k_{x_2x_1} & k_{x_2y_1} & k_{x_2x_2} & k_{x_2y_2} & \cdots & k_{x_2x_n} & k_{x_2y_n} \\
k_{y_2x_1} & k_{y_2y_1} & k_{y_2x_2} & k_{y_2y_2} & \cdots & k_{y_2x_n} & k_{y_2y_n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{x_nx_1} & k_{x_ny_1} & k_{x_nx_2} & k_{x_ny_2} & \cdots & k_{x_nx_n} & k_{x_ny_n} \\
k_{y_nx_1} & k_{y_ny_1} & k_{y_nx_2} & k_{y_ny_2} & \cdots & k_{y_nx_n} & k_{y_ny_n}
\end{bmatrix}
$$

Subtract $k_{y_1x_1}$ times Row1 from Row2

$$
\begin{bmatrix}
1 & ? & ? & ? & \cdots & ? & ? \\
0 & ? & ? & ? & \cdots & ? & ? \\
k_{x_2x_1} & k_{x_2y_1} & k_{x_2x_2} & k_{x_2y_2} & \cdots & k_{x_2x_n} & k_{x_2y_n} \\
k_{y_2x_1} & k_{y_2y_1} & k_{y_2x_2} & k_{y_2y_2} & \cdots & k_{y_2x_n} & k_{y_2y_n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{x_nx_1} & k_{x_ny_1} & k_{x_nx_2} & k_{x_ny_2} & \cdots & k_{x_nx_n} & k_{x_ny_n} \\
k_{y_nx_1} & k_{y_ny_1} & k_{y_nx_2} & k_{y_ny_2} & \cdots & k_{y_nx_n} & k_{y_ny_n}
\end{bmatrix}
$$

Subtract $k_{x_2x_1}$ times Row1 from Row3 … and so on

$$
\begin{bmatrix}
1 & ? & ? & ? & \cdots & ? & ? \\
0 & ? & ? & ? & \cdots & ? & ? \\
0 & ? & ? & ? & \cdots & ? & ? \\
0 & ? & ? & ? & \cdots & ? & ? \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & ? & ? & ? & \cdots & ? & ? \\
0 & ? & ? & ? & \cdots & ? & ?
\end{bmatrix}
$$

Divide Row2 through by 2nd Diagonal

$$\begin{bmatrix} 1 & ? & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & ? & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & ? & ? & ? & \cdots & ? & ? \\ 0 & 1 & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & ? & ? & ? & \cdots & ? & ? \\ 0 & ? & ? & ? & \cdots & ? & ? \end{bmatrix}$$

Subtract multiples of Row2 from the rows below

$$\begin{bmatrix} 1 & ? & ? & ? & \cdots & ? & ? \\ 0 & 1 & ? & ? & \cdots & ? & ? \\ 0 & 0 & ? & ? & \cdots & ? & ? \\ 0 & 0 & ? & ? & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ? & ? & \cdots & ? & ? \\ 0 & 0 & ? & ? & \cdots & ? & ? \end{bmatrix}$$

$$
\begin{bmatrix}
1 & ? & ? & ? & \cdots & ? & ? \\
0 & 1 & ? & ? & \cdots & ? & ? \\
0 & 0 & 1 & ? & \cdots & ? & ? \\
0 & 0 & 0 & ? & \cdots & ? & ? \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & ? & \cdots & ? & ? \\
0 & 0 & 0 & ? & \cdots & ? & ?
\end{bmatrix}
$$

$$\begin{bmatrix} 1 & ? & ? & ? & \cdots & ? & ? \\ 0 & 1 & ? & ? & \cdots & ? & ? \\ 0 & 0 & 1 & ? & \cdots & ? & ? \\ 0 & 0 & 0 & 1 & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & ? & ? \\ 0 & 0 & 0 & 0 & \cdots & ? & ? \end{bmatrix}$$

$$
\begin{bmatrix}
1 & ? & ? & ? & \cdots & ? & ? \\
0 & 1 & ? & ? & \cdots & ? & ? \\
0 & 0 & 1 & ? & \cdots & ? & ? \\
0 & 0 & 0 & 1 & \cdots & ? & ? \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & ? \\
0 & 0 & 0 & 0 & \cdots & 0 & ?
\end{bmatrix}
$$

Now it is Upper-Triangular

$$\begin{bmatrix} 1 & ? & ? & ? & \cdots & ? & ? \\ 0 & 1 & ? & ? & \cdots & ? & ? \\ 0 & 0 & 1 & ? & \cdots & ? & ? \\ 0 & 0 & 0 & 1 & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & ? \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Subtract multiples of LastRow from the rows above

$$\begin{bmatrix} 1 & ? & ? & ? & \cdots & ? & 0 \\ 0 & 1 & ? & ? & \cdots & ? & 0 \\ 0 & 0 & 1 & ? & \cdots & ? & 0 \\ 0 & 0 & 0 & 1 & \cdots & ? & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$
\begin{bmatrix}
1 & ? & ? & ? & \cdots & 0 & 0 \\
0 & 1 & ? & ? & \cdots & 0 & 0 \\
0 & 0 & 1 & ? & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & ? & ? & 0 & \cdots & 0 & 0 \\
0 & 1 & ? & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

$$\begin{bmatrix} 1 & ? & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\mathbf{I} \qquad\qquad \mathbf{k}^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \Bigg\| \begin{bmatrix} f_{x_1x_1} & f_{x_1y_1} & f_{x_1x_2} & f_{x_1y_2} & \cdots & f_{x_1x_n} & f_{x_1y_n} \\ f_{y_1x_1} & f_{y_1y_1} & f_{y_1x_2} & f_{y_1y_2} & \cdots & f_{y_1x_n} & f_{y_1y_n} \\ f_{x_2x_1} & f_{x_2y_1} & f_{x_2x_2} & f_{x_2y_2} & \cdots & f_{x_2x_n} & f_{x_2y_n} \\ f_{y_2x_1} & f_{y_2y_1} & f_{y_2x_2} & f_{y_2y_2} & \cdots & f_{y_2x_n} & f_{y_2y_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{x_nx_1} & f_{x_ny_1} & f_{x_nx_2} & f_{x_ny_2} & \cdots & f_{x_nx_n} & f_{x_ny_n} \\ f_{y_nx_1} & f_{y_ny_1} & f_{y_nx_2} & f_{y_ny_2} & \cdots & f_{y_nx_n} & f_{y_ny_n} \end{bmatrix}$$

Summary

$$[K] \quad \| \quad [I]$$

- Multiply a whole row by a number
- Subtract multiples of one row from another
- Whatever we do to the LHS we do to the RHS

- Work in a systematic way down the lower triangle
- Then work back up the upper triangle

$$[I] \quad \| \quad [K^{-1}]$$

