Structural Form Finding using Zero-Length Springs with Dynamic Mass

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Summary

This paper describes a new method for the form-finding of funicular structures in two or three dimensions using a zero-length spring system with dynamic nodal masses. The resulting found geometry consists of purely axial forces under self-weight, with zero bending moment at nodes for both shells and tension net forms. A real-time solver using semi-implicit Euler integration with viscous damping is used to achieve system equilibrium. By using a real-time solver, the designer is able to alter the gravitational field or apply new point loads without re-starting the analysis, leading to an interactive experience in generating design options. The advantages of this method over existing approaches are discussed, with its successful application in a recent real case-study project also shown.

Keywords: *Form Finding; Particle-Spring Systems; Dynamic Mass Method; Zero-Length Springs.*

1. Introduction

The form-finding of funicular structures has long been advantageous for designers of compression shells and tension nets due to their zero out-of-plane bending moment property under self-weight. There are many examples from the past of attempts by designers to generate such efficient funicular forms within certain boundary constraints including both physical and computational models or combinations of both. Antoní Gaudi's physical hanging chain models have inspired many engineers to use similar methods in funicular form-finding including Heinz Isler, [1] Felix Candella [2] and Frei Otto [3] to name but a few. Such methods give a real-life understanding of the behaviour of material subject to self-weight but at the cost of having to model each design option individually which can be extremely time consuming and constraining, especially if the boundary conditions are complex or the exact requirements of the design are not known *a priori.*

1.1 Computational methods

Recent advances in computing power have made design exploration possible using direct simulation. Active statics [4] for example is a development of Culmann's graphic statics [5] that allows the user to interact with the system and directly see the feedback from various actions, although it is limited to solving problems in two dimensions.

In three dimensions, Killian and Ochsendorf [6] have employed particle spring systems with stiff springs in order to approximate rigid links. By using real-time integration the designer is able to explore different funicular designs quickly by changing boundary conditions and adding or subtracting additional links. Such software aims to mimic simulations of natural systems such as Gaudi's hanging chain models but allows for more flexibility in terms of design exploration than its physical counterpart.

More recently Block and Ochsendorf [7] have developed Thrust-Network Analysis (TNA) using one-step linear optimisation in order to explore funicular forms in real-time, in particular vaulted masonry structures with additional rib and web elements.

The goal with these last two approaches to funicular design is not to optimise for one objective but to instead apply constraints to free-form surface generation in order to maintain structural logic. It will be shown that the dynamic mass approach presented here is similar to the previous two methods but instead extends such processes beyond what can be modelled in the real world, in order to allow fast design exploration

1.2 Zero-length springs

In 1932, the physicist Lucien LaCoste [8] first discovered zero-length springs and applied them to the design of seismographs and gravimeters. The zero-length spring obeys Hooke's law of elasticity but has zero natural length. This means that the force exerted by such a spring is exactly proportional its length which is also its extension.

$$F = -kl \quad (1)$$

This paper explains how zero-length springs may be used in particle spring system in order to form-find funicular structures through the innovative step of carefully varying the lumped mass applied at each node in real-time. We begin by introducing the approach to solving simple two-dimensional systems with known results using zero-length springs and then extend the method in sections three and four to account for variable mass and three dimensional situations.

2. Two-dimensional Systems

2.1 Overview of method

A simple particle-spring system is constructed with nodes joined with zero-length springs (Fig.1). An equal mass is lumped at each node and stays constant for each node in the system. The springs are equally pre-stressed at the initial condition (a) with nodes equally spaced along the x-axis. The property of zero-length springs mean that we then need only solve for the y component of the force when a gravitational field is applied in the negative y direction as shown for the system in (b). For each node we therefore have the residual force in y given as:

$$F_i = k \cdot (y_{i-1} + y_{i+1} - 2y_i) - mg \quad (2)$$

where y_{i+1} and y_{i-1} are the coordinates of nodes adjacent to n_i, k is a global stiffness constant equal for all the springs in the system and g is a gravitational constant. The equation of motion thus becomes (3) with static equilibrium reached by finding both sides of the equation equal to zero representing no residual force and acceleration. This is basically simple harmonic motion with viscous damping term within a bound that avoids an over-damped system.

$$\ddot{y}_i + \frac{c}{m}\dot{y}_i = \frac{F_i}{m} \qquad (0<c/m<1) \quad (3)$$

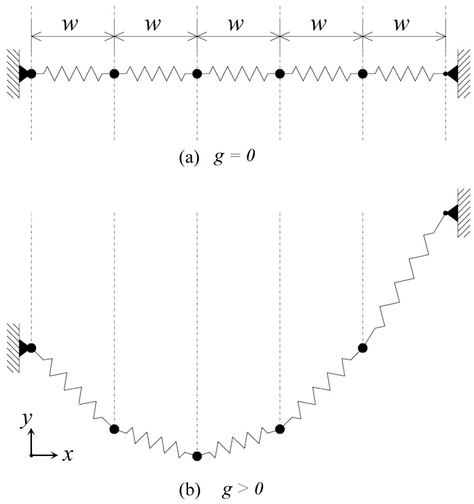

Fig. 1. Zero-length springs form a parabola with equal mass applied at each node.

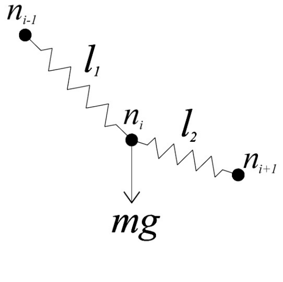

Fig. 2. Free body diagram.

2.2 Solving for static equilibrium

An analytical solution is of course known but we use a numerical approach here as it will be useful later. Although only first-order, the semi-implicit Euler method does not lead to instabilities that are present using the Euler method for harmonic oscillators, and so is a good choice for this problem as it is also suited to large time steps. The semi-implicit Euler is therefore:

$$\dot{y}_i^{\,s+1} = \dot{y}_i^{\,s} + \frac{F_i}{m}\Delta t \quad (4) \qquad\qquad y_i^{\,s+1} = y_i^{\,s} + \frac{c}{m}\dot{y}_i^{\,s+1} \quad (5)$$

(Where $t_s = t_0 + s\Delta t$)

This is then iterated for each node in the system with an appropriate time step until the velocities and residual force tend to zero. The particle-system as a whole acts as a series of coupled oscillators with an additional term for the gravitational load for each node. The result is known to form a parabola, a well known real-world example being a slinky supported from each end under self-weight. An analytical solution to this problem can be found by setting $F_i = 0$ in equation (3) and defining a length for the first spring. The resulting parabola shows exact correlation with results from the numerical approach.

2.2.1 Comment on solving process

By using an iterative process to solve the equations of motion for each node independently we are effectively employing a dynamic relaxation method [9] in order to reach static equilibrium. The iterative force density method is another such example [10]. Such techniques have been traditionally applied to the form-finding of cable net, membrane and pneumatic structures [11] whereby an equal tension field is usually required, mainly because finite element methods do not converge for systems initially far from equilibrium. By assuming we wish to interact with our system in the generation of funicular forms (of which there are theoretically an infinite number for each boundary condition), a distributed approach such as dynamic relaxation therefore seems an appropriate choice.

2.3 Extension to funicular forms

An extension of the approach described in section 2.2 can now include a dynamic mass that is relative to the length of the adjoining springs. Such a modification gives rise to a discretised catenary once static equilibrium is reached.

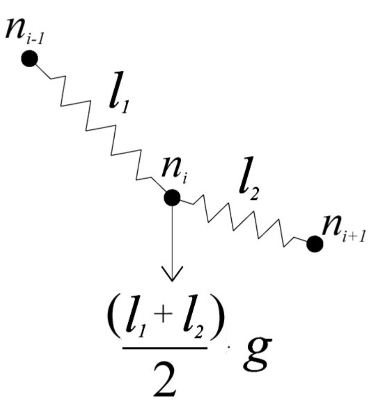

Fig. 3. Dynamic mass free body.

2.3.1 Dynamic Mass Method

By assigning a variable lumped mass proportional to the length of the zero-length spring adjoining each node we essentially keep the mass density of all the springs constant, much like a catenary cable. This mass in turn affects the length of the spring itself and hence a coupled relationship is formed between the two. With reference to Fig.3, the lengths are now given by our x-axis spacing: *w*.

$$l_1 = \sqrt{(y_{i-1} - y_i)^2 - w^2} \quad (6) \qquad l_2 = \sqrt{(y_{i+1} - y_i)^2 - w^2} \quad (7)$$

With the residual force for each node now being:

$$F_i = k \cdot (y_{i-1} + y_{i+1} - 2y_i) - \frac{(l_1 + l_2)}{2} g \quad (8)$$

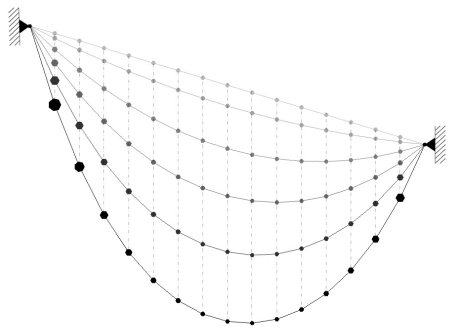

Fig. 4. Exploration of catenaries by varying the gravitational constant.

With $F_i = 0$ required for equilibrium, this equation is again non-linear and this time has no closed form. We may still use the method described in section 2 in order to solve for static equilibrium by substituting equation (8) into (4) & (5). Once again, we need only solve for the y component because the zero-length springs are all equally stiff and posses the same extension leading to equal spacing in the *x*-direction. In practice however we may also solve for *x* to allow for an initial condition that is not equally spaced.

2.3.2 Real-time exploration

Various forms may now be explored within the two pinned boundary conditions simply by varying the gravitational constant. Fig. 4 shows different length discretised catenaries formed quickly in real-time with the user varying *g*.

2.3.3 Convergence

The mass term will overpower the spring forces should g > k as the nodes continue to accumulate mass too quickly for the springs to resist the extra force. Even if this condition is met, the system can also be too stiff and not converge. In practice this is rectified by choosing a stiffness constant appropriate for the time-step. This is easily done manually even for the complex 3D forms discussed later because *k* is global and identical for all springs. This is another advantage of using zero-length springs.

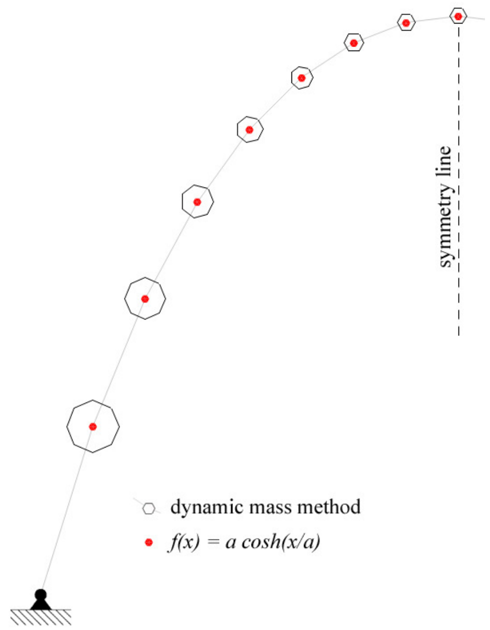

Fig. 5. 17 node system in comparison to actual catenary.

2.3.4 Generating forms

Once static equilibrium is achieved we can now replace the springs with rigid links that give exactly the correct mass for a funicular form under self-weight assuming the mass per unit length is equal for all members. It may seem like the springs are pre-stressed and hence by replacing them with rigid links the form will somehow change - but this is not the case because each spring has zero natural length and will therefore - when converted to a rigid member – balance the x and y component forces at each node in the same proportion and hence direction.

2.3.5 Catenary comparison.

An analytical solution to the continuous catenary in two dimensions is well known and is given by the hyperbolic cosine function. The dynamic mass method generates accurate funicular forms for discretised catenaries with straight members, however by increasing the number of nodes in the system we can very closely approximate a continuous system even for a relatively small amount of nodes. Figure 5 shows the form-finding of a 16m wide by 14 meter high catenary arch with nodes at 1m spacing in the x-axis.

With this problem, the maximum error was found to be only 3mm where the curvature of the catenary was greatest. By increasing nodes this error converges to zero monotonically (Figure 6) but at the cost of calculation time.

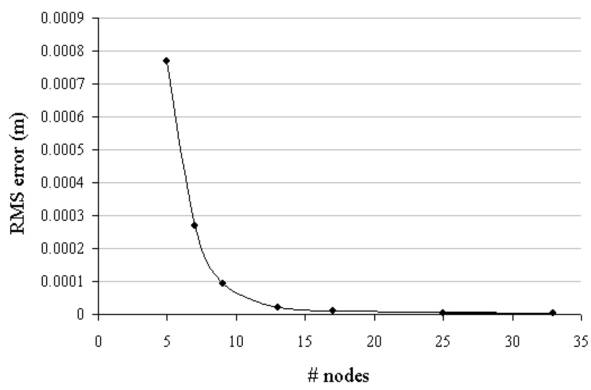

Fig. 6. Increasing of nodes reduces error

2.3.6 Discussion

In practice, at the design exploration stage of a project a particle system with only a small number of nodes is quicker to solve and hence feedback to the designer is faster. As a design is settled upon, if a curved form is desired, it is then possible to increase the node number for example let 10 nodes represent 10 positions on one member as it curves between connections.

For most real projects however, cost constraints or even aesthetics often dictate that a discretised solution is required anyway.

3. Three-Dimensional Systems

3.1 Introduction

So far approaches to 2D problems with known analytical solutions have been investigated and hence have no additional value in terms of finding form. However, the method applied to 3D systems has much more use for the design of funicular shells in compression or tension. The method can also be applied to statically indeterminate structures due to the low stiffness requirement of the springs during the iterative process. As has been discussed, transferring from 2D to the complexity of doubly curved structures embedded in 3d space can be challenging for some other approaches. However if the initial condition of the zero-length springs form an equally spaced grid when projected vertically onto the ground plane it will now be in equilibrium in both x and y directions and we need only solve for the *z* coordinate.

3.2 Extension to 3D

Figure 7 shows a particular node in a spring system with 4 adjoining springs. The residual force in the z direction now given as (9) with the spring lengths also a function of *z* if a square grid is used when projected to a plane perpendicular to the gravitational force (i.e. the ground plane).

$$F_i = k \cdot \left(z_{i+1,j} + z_{i,j+1} + z_{i+1,j} + z_{i,j-1} - 4z_{ij}\right) - \frac{\sum_{n=1}^{4} l_n}{4} g \quad (9)$$

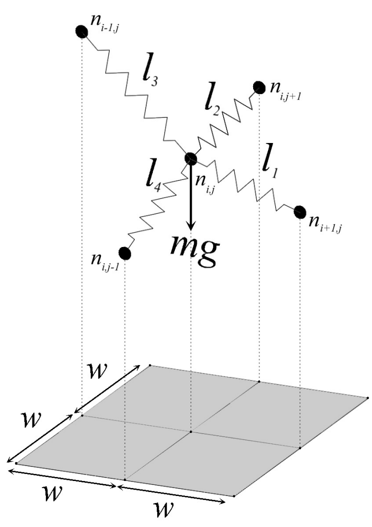

Fig. 7. Node in a 3d system

Once again the system is solved for static equilibrium until all the residual forces and velocities are zero. As before, varying the gravitational constant *g* allows the designer to explore many potential funicular forms.

Once a suitable shape has been found, the springs are replaced with rigid links. As with the 2D system, since the lumped masses are coupled to the length of the adjoining springs for each node, the form is perfectly balanced for the self-weight of the structure assuming all members are of equal mass density.

Figure 8 shows such a design whereby the first step (a) in the process is to find static equilibrium with the zero-length springs with no gravity applied. This forms a doubly curved tension net structure. Step (b) shows how increasing *g* allows the shell to begin to find a form in arching action. The results of an analysis of the final design are shown in (c) – in this case scaled to a 16x16x6m structure under self-weight with a 76.1x4mm

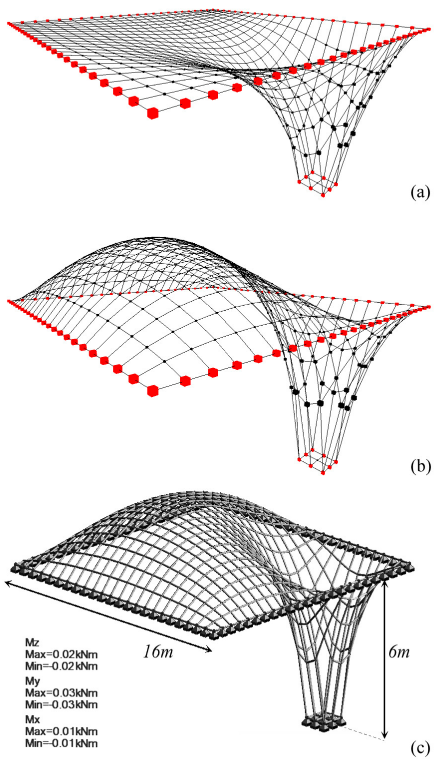

Fig. 8. Discretised funicular surface.

CHS used for each member.

All axial forces were compressive with magnitudes at the boundary edge were in the region of 4kN, however the maximum moment in the structure was minimal in comparison (approximately 0.03kNm) solely due to the self-weight bending of the members exerting a small local moment at the joints. Buckling of compressive struts thus becomes the most important factor with such structures and future ways of extending the method to include these considerations are discussed in section five.

3.3 Boundary conditions

At present the method requires a continuous pinned boundary around the edge of the springs to keep the nodes located on a grid throughout the process. This boundary can vary in z height however, and additional supports internal to the outside boundary may be included.

3.4 Additional Loads

So far, we have been solely concerned with a linear relationship between the length of the spring and the mass applied at each node. This assumes that all members will be of equal mass per unit-length when the springs are replaced with rigid links which for large shells of varying axial force, this may not be the case.

3.4.1 Varying Member Cross-Sections

An example of the above is where the cross-sectional area of each member should vary according to stress to ensure an efficient utilisation of material. This requirement can be handled by setting up a non-linear relationship for the amount of dynamic mass applied at the node. The residual force now becomes:

$$F_i = k \cdot \left(z_{i+1,j} + z_{i,j+1} + z_{i+1,j} + z_{i,j-1} - 4z_{ij}\right) - \frac{\sum_{n=1}^{4} l_n^2}{4} g \quad (10)$$

In practice this system still converges so long as $g<k^2$ and still gives very fast calculation for static equilibrium, certainly enough for real-time results for systems of around 2000 nodes.

3.4.2 Area elements

It is also possible to form-find continuous shells using this method by approximating the self-weight of an area element and applying it at the node. This is achieved by calculating the cross product of the four adjacent area elements and taking the mean average as the nodal lumped mass. Like the size of the links, this area changes dynamically as the process runs but still converges upon static equilibrium for a similar choice of *g and k.*

3.4.3 Additional point loads

The designer can further modify the shape within the current boundary conditions then by adding additional point loads in addition to the dynamic masses at each node. The form can then be manipulated whilst still maintaining funicularity. This occurred in the following real case-study project, whereby the range of funicular forms did not suit the requirements of the architect. Interestingly, although the addition of load increased the axial force, as the bending moment was greatly reduced at each node the connection details became much simpler as a result.

4. Application in Practice

4.1 Interactive applet

The method has been recently applied on a design competition at Ramboll Engineers. An interactive Java applet was developed for use by the project team in a competition design for a 360m long timber roof for a new terminal at Riga airport, Latvia. Figure 9 shows a screenshot from the applet. Initial conditions came from the design brief and the architect's initial internal design options. Interestingly, these internal layouts were not fixed from day one and hence the applet was able to adapt to new conditions as the design progressed.

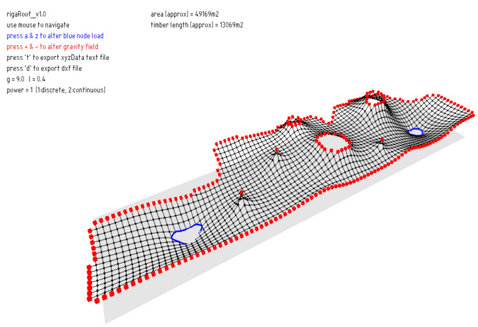

Fig. 9.Java applet allowing various forms to be explored by altering the gravitational constant.

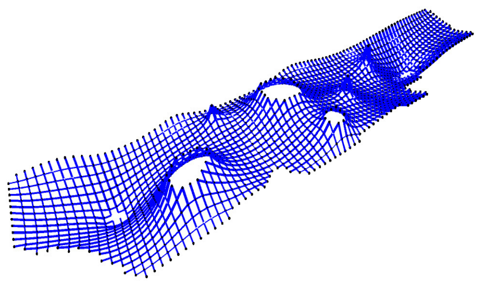

Fig. 10.Analysis of the found-form showing all tension only components. Thickness of the lines is proportional to stress.

4.2 Design process implementation

By using a real-time solver, the architect was again able to alter the gravitational field and apply new point loads without re-starting the analysis, leading to a interactive experience in generating design options. Information on material cost was also then fed-back in real-time via the applet.

By maintaining the nodes on a plan grid throughout the form-finding process, column locations for the internal supports were able to be preserved. There were also further advantages to maintaining a grid at the construction stage whereby temporary props and construction tolerance datum points were easily located on a regular grid on the ground plane.

Figure 10 shows a finite element analysis conducted with all hinged joints. The structure remains in its found form shape indicating zero-residual moment and is made up of tension only components with a sound distribution of force.

Due to the zero-length spring calculation being reasonably simple, the process is very fast: running 2000 nodes with ease on a standard desktop computer. In the opinion of the authors it is this speed and real-time feedback made possible by the *low-stiffness* springs that makes this approach very appealing in encouraging structural logic at the concept design stage. By doing so, the requirement to post-rationalise the complex doubly curved roof shape later becomes far less likely.

5. Discussion

This paper has presented a new approach for developing funicular forms, both discrete and continuous in a way that is not easily re-creatable using physical models alone. In applying the approach in practice it has also been shown how the development of such tools can lead to more efficient outcomes later on in the design process. This is a response to the proliferation of post-rationalised free-form surfaces in architecture as modelling tools become more widely available to designers.

There are limitations in the boundary conditions for this method and that a regular grid is required as an initial condition. Current research indicates that this may not be necessary although this will need to be investigated further, as does the possible inclusion of Euler buckling criteria as an additional term for the dynamic mass. With the recent increase in computing power allowing for real-time interaction of structures, the authors hope that by using such tools designers can embed funicularity into the design process right from the start, and reduce today's preference for free-form surfaces that require extensive post-rationalisation.

Acknowledgements

This research has been supported by the Bath Industrial Doctorate Centre in Systems, Ramboll Engineers and the Engineering and Physical Sciences Research Council, UK. The authors would also like to thank Chris Williams of the University of Bath for helpful discussions throughout.

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