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Prestressed gridshell structures

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Abstract

This paper describes a method for the form finding of shell structures composed of both compression and tension members which may lie in one layer or two layers. The length of some of the members can be constrained to a fixed length yielding some control of the resulting form found shape. The form finding is accomplished by adjusting the nodal positions until an equilibrium state is reached using dynamic relaxation. If part of a structure is unstable due to compression forces, then a *negative mass* must be used in the dynamic relaxation. The length constraint is met by adjusting the force density during form finding, again using dynamic relaxation. Finally, case studies are presented where the applied load and the prestress is used to govern the form found shape.

Key words: form finding, dynamic relaxation, force density, prestress, shell structures

1. Introduction

Prestressed structures require a combination of compression and tension elements and in this paper we introduce a numerical procedure for form finding prestressed structures which produces only compressive and tensile axial forces in structural members with no bending moment in the form found state.

Most numerical methods for structural form finding simulate a physical model which has to be stable in order to achieve equilibrium. The physical model might involve hanging chains which will be inverted to form a compression structure as explained by Hooke [3], or a combination of a soap film and cotton threads in tension as well as masts in compression described by Otto and Rasch [6]. In the later case the tension elements stabilise the masts. Michael Barnes used a technique for the form finding of tension structures supported by arches in which he reversed the forces on the arches so that they became stable tension elements and our technique takes this idea a step further.

We shall see that it is possible to numerically form find structures which are in equilibrium but still are unstable mechanisms. Of course, the real structure must be stable, but this will be ensured by 'freezing' the form found structure either by adding bending stiffness or bracing.

2. Theory

The geometric stiffness of a pin-ended member is equal to the tension divided by its length. This is because if one end of a member of length l undergoes a small lateral displacement δ it will rotate by the small angle δ/l radians. Thus, if T is the tension in the member there will now be a lateral component of force equal to $\pm(T/l)\,\delta$ applied to the member end nodes in a direction *perpendicular to the original direction* of the member.

The quotient T/l is often known as the tension coefficient in English, but following Linkwitz and Schek [5] and Schek [8] it is now also commonly called the force density from the German *Kraftdichte*.

If the member is in compression, then the force density is negative and so is its geometric stiffness. Thus, if the form finding model contains compression elements, then without bending stiffness the structure may be unstable, meaning that the global stiffness matrix is no longer positive definite. The square symmetric global stiffness matrix contains both elastic stiffness and geometric stiffness and it has to be invertible in order to find an equilibrium configuration. But a matrix can be invertible, that is nonsingular, without being positive definite. If a matrix is positive definite, it *must* be nonsingular, but if a matrix is not positive definite it *may* be nonsingular [4]. However, in the following theory we shall use Alistair Day's dynamic relaxation [1], so we will not need to formulate the entire global stiffness matrix.

Since we do not want any bending moment in the form found state, the static equilibrium equation of node a in a structure is

$$\mathbf{p}_a + \sum_b \left[\frac{T_{ab}}{l_{ab}} (\mathbf{x}_b - \mathbf{x}_a) \right] = 0 \tag{1}$$

in which $\mathbf{p}_a$ is the sum of the applied load and support reaction, if the node is restrained. T_{ab} and l_{ab} are the tension in the member ab and its *current* length. The summation extends to all the nodes b that are connected to node a by a structural member ab. $\mathbf{x}_a$ and $\mathbf{x}_b$ are the position vectors of nodes a and b and $(\mathbf{x}_b - \mathbf{x}_a)/l_{ab}$ is a unit vector in the direction from node a to node b. Thus $T_{ab}(\mathbf{x}_b - \mathbf{x}_a)/l_{ab}$ is the force that the member ab exerts on node a and the summation extends to all the members meeting at node a.

It should be noted that eq. (1) is purely about static equilibrium and it makes no assumption regarding the material properties of the members which might be linear or non-linear elastic or be subject to creep or plastic deformation. The structure may be statically determinate or indeterminate or even a mechanism, provided that it is in equilibrium. The structure may have undergone a large deformation from some initial state.

In order to determine the form found geometry we need further information regarding the tensions T_{ab} and their relationship with the current lengths l_{ab}. The simplest relationship is linear elastic,

$$T_{ab} = k_{ab}(l_{ab} - l_{0,ab}) \tag{2}$$

in the case of a member with unstressed length $l_{0,ab}$. The constant $k_{ab} = (EA)_{ab}/l_{0,ab}$ in which $(EA)_{ab}$ is equal to the Young's modulus times the cross-sectional area of the member. However, during form finding we can postulate any relationship between tension and length, including inextensible members whose length cannot change and members with a constant tension.

2.1. Constant force densities

The simplest case is when members have constant force density, that is the force density in a particular member is constant, but the constant may vary from member to member. A constant force density corresponds to $l_{0,ab} = 0$ in eq. (2) and it is possible to manufacture 'zero-length springs', such as those used in the Anglepoise lamp [2]. If we know all the T_{ab}/l_{ab} in eq. (1) then we have a system of linear equations in the nodal coordinates which are the components of the position vectors $\mathbf{x}_a$. Clearly we will have the same number of equations as unknowns, if a node is restrained, then the unknown coordinates are replaced by the unknown support reactions. The matrix of coefficients in these equations may be singular in which case there is more than one equilibrium configuration.

Let us write the force density in the member ab as

$$q_{ab} = T_{ab}/l_{ab}, \tag{3}$$

so that the 'out of balance force' or ' residual force' on node a is

$$\mathbf{r}_a = \mathbf{p}_a + \sum_b \left[q_{ab} \left(\mathbf{x}_b - \mathbf{x}_a \right) \right]. \tag{4}$$

Thus, if the force density q_{ab} remains constant, the end stiffness of the member ab is q_{ab} in *all directions*. Therefore $\sum_b q_{ab}$ is the stiffness of node a if all the other nodes remain fixed.

It follows that the fictitious mass of node a to be used for dynamic relaxation should be

$$m_a = \lambda \left(\delta t \right)^2 \sum_b q_{ab} \tag{5}$$

in which δt is the time step and λ is a positive constant chosen to get the best convergence.

The dynamic relaxation algorithm to solve eq. (1) with eqs. (3) to (5) inserted is shown in the pseudo code in algorithm 1. The damping constant μ must be less than 1 and super dot denotes time derivatives.

Algorithm 1: Dynamic relaxation algorithm for the case with constant force densities.

for each node a **do**
  $m_a \leftarrow 0$;
  $\dot{\mathbf{x}}_a \leftarrow \mathbf{0}$;
end
for each member ab **do**
  $m_a \leftarrow m_a + \lambda \cdot (\delta t)^2 \cdot q_{ab}$;
end
while nodes are still moving **do**
  for each node a **do**
    $\mathbf{r}_a \leftarrow \mathbf{p}_a$;
  end
  for each member ab **do**
    $\mathbf{r}_a \leftarrow \mathbf{r}_a + q_{ab} \left(\mathbf{x}_b - \mathbf{x}_a \right)$;
    $\mathbf{r}_b \leftarrow \mathbf{r}_b + q_{ab} \left(\mathbf{x}_a - \mathbf{x}_b \right)$;
  end
  for each node a which is not constrained **do**
    $\ddot{\mathbf{x}}_a \leftarrow \dfrac{\mathbf{r}_a}{m_a}$;
    $\dot{\mathbf{x}}_a \leftarrow \mu \cdot \dot{\mathbf{x}}_a + \delta t \cdot \ddot{\mathbf{x}}_a$;
    $\mathbf{x}_a \leftarrow \mathbf{x}_a + \delta t \cdot \dot{\mathbf{x}}_a$;
  end
end

If all the members meeting at node a are in tension, then m_a will be positive. If they are all in compression m_a will be negative. If there are both tension and compression members, the sign of m_a will depend upon which dominate. The procedure will fail if $m_a = 0$, but that applies whatever solution technique is used since there is no solution to the equations.

A negative value of m_a means that *the node is moved in the opposite direction to the out of balance force*. This means that a compression structure will move towards equilibrium in the same way as a tension structure and a structure with both tension and compression elements will move the nodes as appropriate.

2.2. Required member length

A more complex case is when some members should have a required length $l_{req,ab}$ at equilibrium. This implies that for these members $l_{ab} \to l_{req,ab}$ as $\mathbf{r}_a, \mathbf{r}_b \to \mathbf{0}$ and thus the force densities can no longer be constant throughout the dynamic relaxation procedure. We can use dynamic relaxation to adjust the force densities of members with required length in a similar way to that in which we move the nodes. The pseudo code is shown in algorithm 2 and the code itself is shown in listing 1 with particular values for the constants.

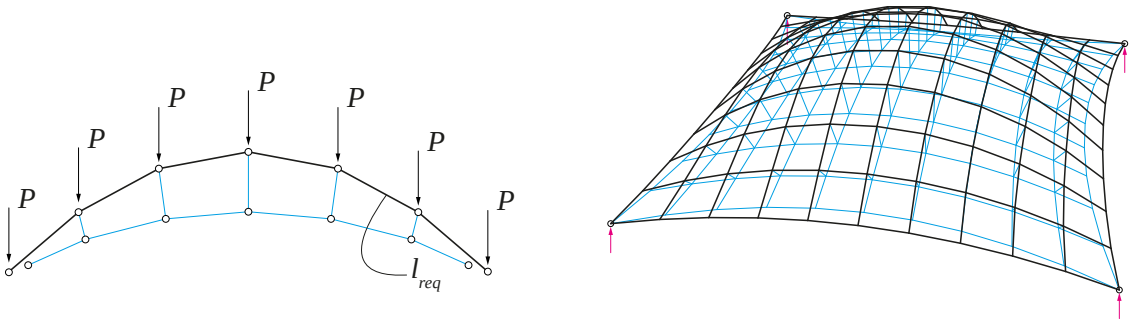
Algorithm 2: Dynamic relaxation algorithm for the case with prescribed required length for some members.

for each node a **do**
  $\dot{\mathbf{x}}_a \leftarrow \mathbf{0}$;
  $\dot{q}_{ab} \leftarrow 0$;
end
while nodes are still moving **do**
  for each node a **do**
    $m_a \leftarrow 0$;
    $\mathbf{r}_a \leftarrow \mathbf{p}_a$;
  end
  for each member ab **do**
    $m_a \leftarrow m_a + \lambda \cdot (\delta t)^2 \cdot q_{ab}$;
    $\mathbf{r}_a \leftarrow \mathbf{r}_a + q_{ab}(\mathbf{x}_b - \mathbf{x}_a)$;
    $\mathbf{r}_b \leftarrow \mathbf{r}_b + q_{ab}(\mathbf{x}_a - \mathbf{x}_b)$;
    if member ab has prescribed length $l_{req,ab}$ **then**
      $(l_{ab})^2 \leftarrow (\mathbf{x}_b - \mathbf{x}_a) \cdot (\mathbf{x}_b - \mathbf{x}_a)$;
      $\dot{q}_{ab} \leftarrow \beta \cdot \dot{q}_{ab} + \alpha \cdot \dfrac{(l_{ab})^2 - (l_{req,ab})^2}{(l_{req,ab})^2} \cdot q_{ab}$;
      $q_{ab} \leftarrow q_{ab} + \dot{q}_{ab}$;
    end
  end
  for each node a which is not constrained **do**
    $\ddot{\mathbf{x}}_a \leftarrow \dfrac{\mathbf{r}_a}{m_a}$;
    $\dot{\mathbf{x}}_a \leftarrow \mu \cdot \dot{\mathbf{x}}_a + \delta t \cdot \ddot{\mathbf{x}}_a$;
    $\mathbf{x}_a \leftarrow \mathbf{x}_a + \delta t \cdot \dot{\mathbf{x}}_a$;
  end
end

We now have 4 constants λ, μ, α and β to choose. For fast convergence, we would like to specify our damping constants, μ and β in such a way that we obtain *critical damping*, something discussed by Rezaiee-pajand et al. [7]. However, since we are also changing the length of all members we are adding energy to the system. This means that it becomes difficult to determine how to choose the damping constants so that we obtain critical damping; the behaviour of the structure is highly nonlinear and unpredictable. Instead, trial and error is suggested when choosing the constants. As a rule of thumb starting with small values and trying to increase them is a good strategy.

3. Case studies

Figure 1 shows a double layer pretensioned shell structure made up of two square grids of equal topology where the upper grid contains compression elements with required length and the lower grid tension elements. At each interior intersection, a tension member connects the two grids. All elements have a prescribed initial force density, negative for the upper layer and positive for the lower, and all nodes in the upper grid are loaded with a downwards vertical load. Pinned supports are provided at the four corner nodes for both the upper and the lower grids. The pin jointed structure is clearly a mechanism and linear static methods would not be possible to use for design and analysis.



(a) Cross-section and loading

(b) Perspective view

Figure 1: Principle layout of the double layer shell structure. Black members in compression, blue members in tension and pin supports at the corners.

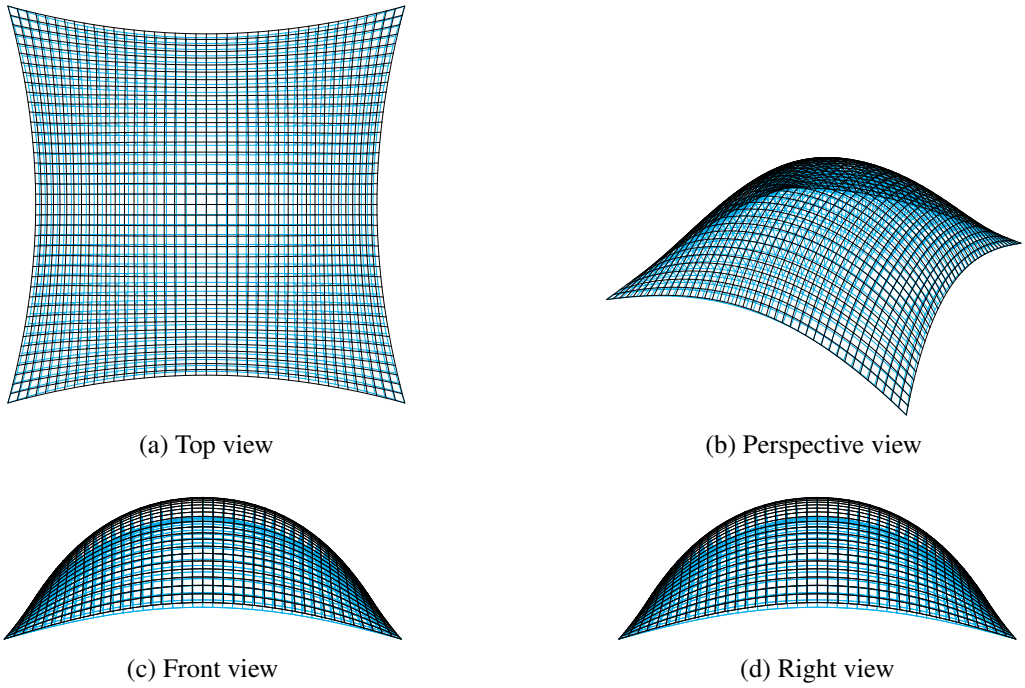
Two load cases were investigated using the same principal layout as in fig. 1. In both cases, the required length was set to $l_{req} = 2.2a/m$ where $a = 400$ is the length of the square and $m = 40$ the number of cells in each direction of the grid. The force density was taken as -1.0 for the upper grid, 1.0 for the lower grid, $-1.0m$ for edge elements of the upper grid, $1.01m$ for the edge elements of the lower grid, and -2.0 for the upper-lower grid links. In the first load case a uniformly distributed load was applied with $P = 0.2$ giving a doubly symmetric form found structure as shown in fig. 2. In the second case a load quadratically increasing in one direction was applied with $P_{ij} = 0.2i^2/m$ where i, j are the row and column numbers of the upper grid. This gave a shape that is symmetric about only one plane as shown in fig. 3. Figure 4 shows the same structure as in the first case, but with a finer grid ($m = 50$) and rendered to show a timber gridhell prestressed by a cable net below.

Finally, fig. 5 shows a gridshell all in one layer with compression members in brown and tension members blue - green. The topology of the grid is a Cairo tiling consisting only of pentagons, but with random paths across the grid assigned to be compression members whose length remains the same as in the randomised initial geometry. The model contains 360,000 nodes and 600,000 members. The computer program was written in C++ using OpenGL for the rendering and took about 2 minutes to converge running on a 4 year old Apple iMac, including numerous renderings during the convergence.

4. Conclusions

In this paper, we have implemented a procedure for form finding of single or double layer prestressed shell structures. The method works for structures that are statically determinate, statically indeterminate as well as for mechanisms.

Structural members in tension or compression can be prescribed to be inextensible or extensible. In both cases the force densities and a nodal configuration in force equilibrium is found by use of dynamic relaxation. The method is demonstrated on a double layer gridshell and a single layer randomised Cairo tiling gridshell.



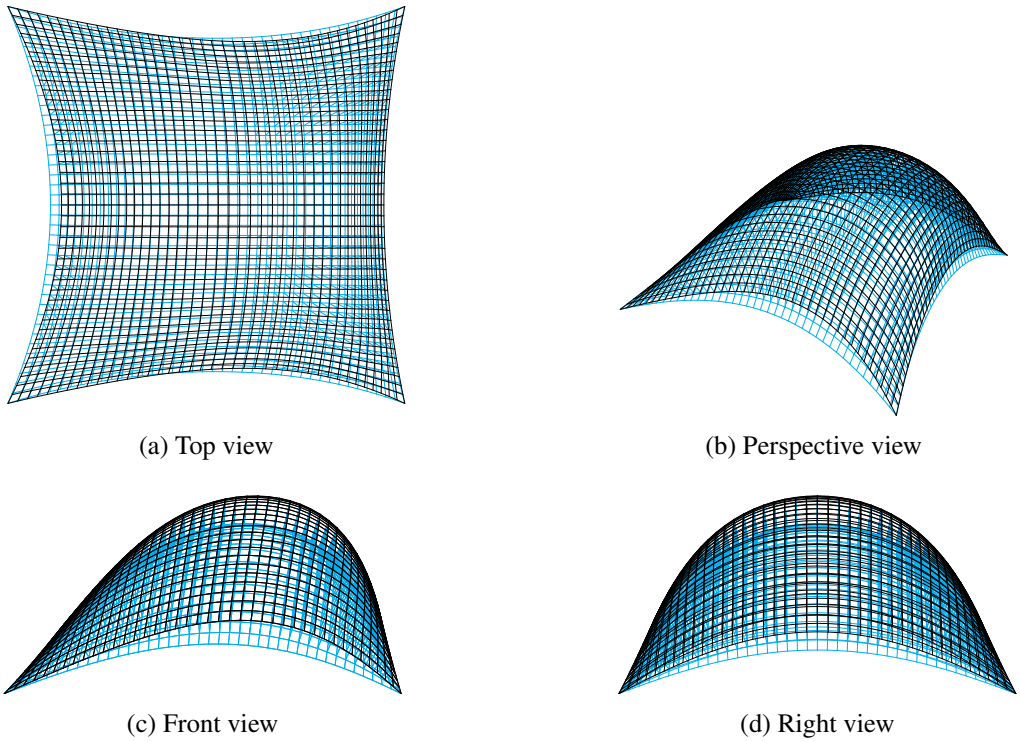
(a) Top view

(b) Perspective view

(c) Front view

(d) Right view

Figure 2: Model with uniformly distributed loading.



(a) Top view

(b) Perspective view

(c) Front view

(d) Right view

Figure 3: Model with quadratically increasing load in one direction.

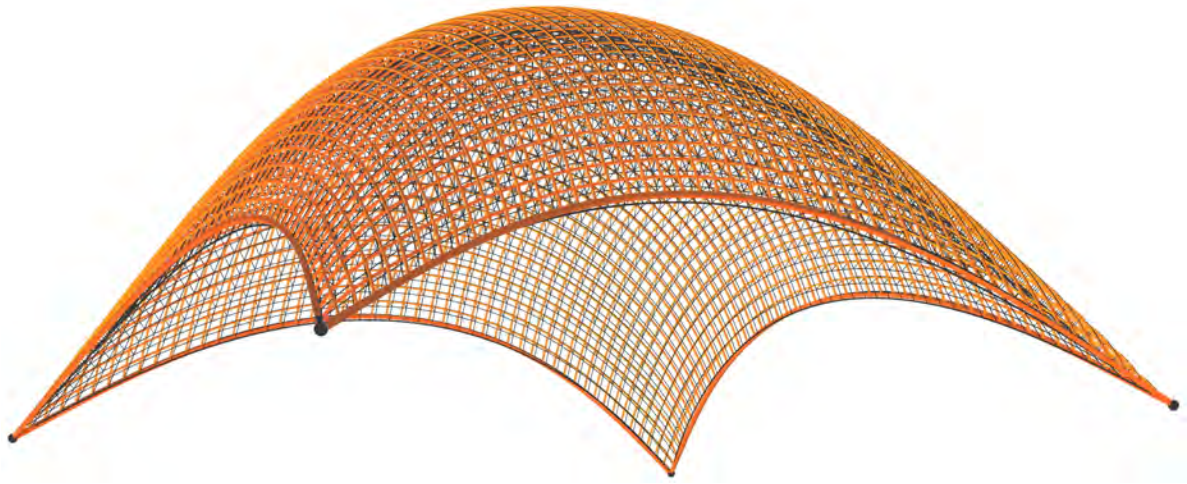
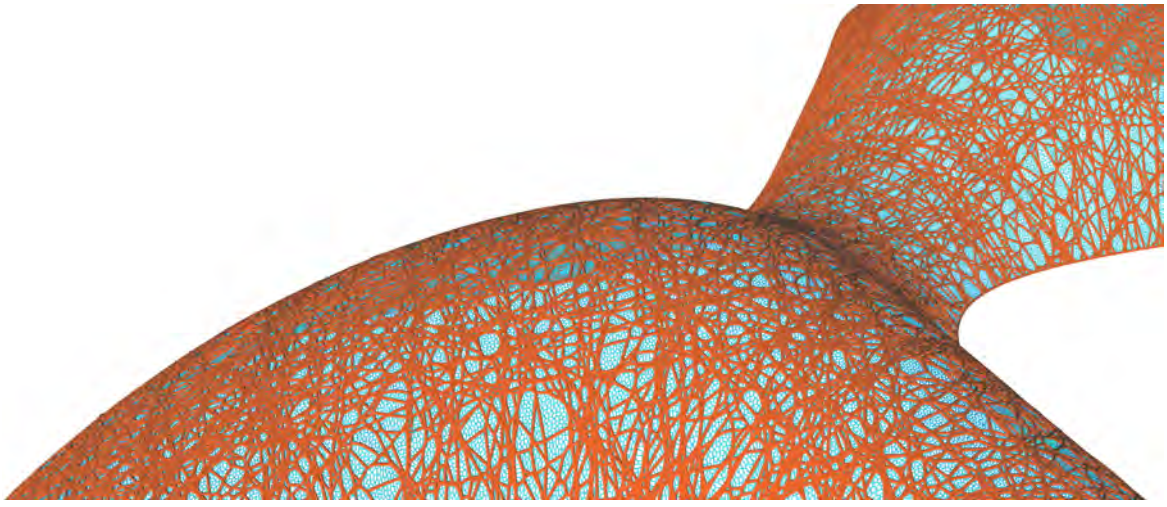


Figure 4: Timber gridshell prestressed by a cablenet.

(a) Overall view



(b) Detail

Figure 5: Gridshell in one plane. Compression members brown, tension members blue - green.

Listing 1: C++ solver step. The step-function is computed until no nodes are moving any more.

```
void step() {
  // Reset residual and mass for all nodes
  for(int node = 0; node <= lastNode; node ++) {
    for(int xyz = 0; xyz <= 2; xyz ++) residual[node][xyz] =
     load[node][xyz];
    mass[node] = 0.0;
  }

  // Compute mass, residual and update force density if needed
  for(int member = 0; member <= lastMember; member ++) {
    mass[memberEnd[member][0]] += dt * dt * forceDensity[member];// Start
     node
    mass[memberEnd[member][1]] += dt * dt * forceDensity[member];// End
     node

    lengthSquared = 0.0;

    for(int xyz = 0; xyz <= 2; xyz ++) {
      delta = x[memberEnd[member][1]][xyz] - x[memberEnd[member][0]][xyz];
      component = forceDensity[member] * delta;

      lengthSquared += delta * delta;

      residual[memberEnd[member][0]][xyz] += component;  // Start node
      residual[memberEnd[member][1]][xyz] -= component;  // End node
    }

    if(fixedLengthMember[member]) {
      forceDensityRate[member] = 0.9 * forceDensityRate[member] + 0.0001 *
       forceDensity[member] * (lengthSquared -
       requiredLengthSquared[member]) / requiredLengthSquared[member];
      forceDensity[member] += forceDensityRate[member];
    }
  }

  // Update position
  for(int node = 0; node <= lastNode; node ++) {
    for(int xyz = 0; xyz <= 2; xyz ++) {
      if(freeToMove[node][xyz]) {
        acceleration = force[node][xyz] / mass[node];
        velocity[node][xyz] = 0.9 * velocity[node][xyz] + 1.0 * dt *
         acceleration;
        x[node][xyz] += dt * velocity[node][xyz];
      }
    }
  }
}
```

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