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Inventory-Constrained Structural Design: New Objectives and Optimization Techniques

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Abstract

Increasing pressure to reduce the life-cycle impact and material consumption of new structures is driving interest in the use of minimally processed or second-hand structural materials in new construction. Conventional structural design methods, however, assume supply chains which can deliver effectively infinite supplies of structural elements in a wide range of dimensions. The use of minimally-processed or reused structural materials requires a new design approach which is tolerant of highly constrained inventories, in which elements exist in strictly finite quantities and are of predetermined dimensions. There is little literature or industry experience concerning this kind of inventory-constrained design. This paper presents theoretical contributions which are intended to provide a groundwork for future researchers and designers to develop and evaluate methods for inventory-constrained structural design. The concept of an "assignment" of inventory elements to structural elements is introduced. An integer program for the optimization of assignments of bar-type inventory elements to a statically determinate pin-jointed truss is formulated. New objectives for the optimization of assignments are introduced. Finally, analytical and numerical techniques for computing the "Offcut Ratio" objective are presented which may help designers to significantly reduce material consumption and waste, particularly for repetitive truss-like structures such as roof trusses, utility towers, and truss bridges.

Keywords: inventory-constrained design, component reuse, structural design, computational design, structural optimization

1. Introduction

Component reuse, and the use of minimally processed structural materials may provide significant life-cycle benefits for new structures. Milford [7] estimated that component reuse of steel in new structures could result in savings in CO_2-equivalent emissions of as high as 80-90% accounting for 10-20% member over-specification by mass. A 2012 survey found however, that in the UK only 7% of structural steel from deconstructed buildings was reused (Sansom and Avery [6]). Gorgolewski [4] identified a lack in flexibility in the design process as among the barriers to greater adoption of component reuse in steel design. The use of minimally processed round timber in construction presents a similar design challenge, with a potential savings of up to 66-72% CO_2-equivalent emissions per kilogram assuming one-to-one replacement of sawn timber with round timber by mass (Hammond and Jones [5]). This paper addresses basic theoretical challenges in inventory-constrained design and proposes some simple analytical and numerical optimization techniques for early-stage design with constrained inventories.

2. Structure-Inventory "Assignments"

In classical structural optimization, the design variables are typically structural geometry, topology, and element section. In inventory-constrained structural design, however, there is an additional set of design variables: the "assignment" of the available inventory elements to locations in the given structural design. An assignment can be informally defined as the set of instructions necessary to assemble a structure from the given set of inventory elements. The choice of assignment can have significant impact on the cost, material consumption, and life-cycle impact of a given structural design. For example, a poor assignment may result in large fractions of material being wasted as offcuts, elements being unnecessarily oversized, or large numbers of elements failing to be successfully placed and having to be built using conventionally sourced elements, with higher associated life-cycle impact.

This paper defines an assignment precisely for a specific inventory-constrained structural design problem: the design of a statically determinate pin-jointed truss using an inventory of bar elements (Figure 1). For this problem formulation, we only allow transverse cuts of elements, and attachment of elements to other elements at pin-jointed nodes. We do not allow longitudinal or transverse "gluing" or "welding" of elements together. The problem is formulated in Equations 1-7 as an integer program.

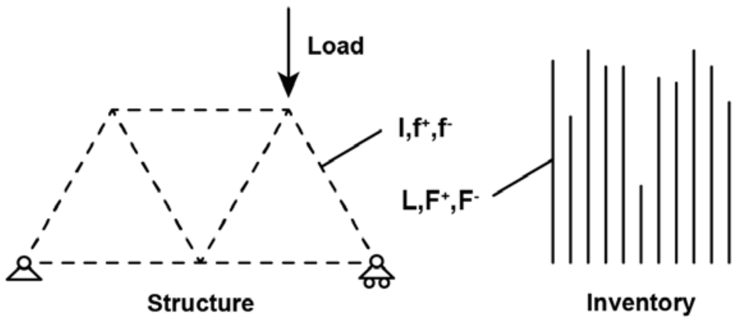


Figure 1: Problem: assignment of an inventory of bar elements to a statically determinate pin-jointed truss.

$$\begin{array}{llcll}
\min\limits_{x} & \text{Offcut Ratio} & & & (1)\\
\text{s.t.} & \sum_{i=1}^{m} x_{ij} & \leq & 1 \quad \forall j \in \{1,...,n\} & (2)\\
& \sum_{j=1}^{n} x_{ij} l_j & \leq & L_i \quad \forall i \in \{1,...,m\} & (3)\\
& x_{ij} f_j^+ & \leq & F_i^+ \quad \forall i \in \{1,...,m\}, \forall j \in \{1,...,n\} & (4)\\
& x_{ij} f_j^- & \leq & F_i^- \quad \forall i \in \{1,...,m\}, \forall j \in \{1,...,n\} & (5)\\
& x_{ij} & \in & \{0,1\} \quad \forall i \in \{1,...,m\}, \forall j \in \{1,...,n\} & (6)
\end{array}$$

$$x_{ij} = \begin{cases} 1, & \text{if structural element } j \text{ is to be fabricated from inventory element } i \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$$\begin{aligned}
m &= \text{number of inventory elements}\\
n &= \text{number of structure elements}\\
l_j &= \text{length of structure element } j\\
L_i &= \text{length of inventory element } i\\
f_j^+ &= \text{axial tensile force in structural element } j\\
f_j^- &= \text{axial compressive force in structural element } j\\
F_i^+ &= \text{axial tensile capacity of inventory element } i\\
F_i^- &= \text{axial compressive capacity of inventory element } i
\end{aligned}$$

The assignment in this formulation is defined as the binary matrix *x*, where each entry corresponds to whether a given structure element *j* is to be fabricated from a given inventory element *i*. The objective function in Equation 1, "Offcut Ratio", will be discussed in the subsequent section. The constraint in Equation 2 ensures that each structure element is created at most once. Equation 3 ensures that the total length of structure elements cut from a given inventory element does not exceed the length of that inventory element. Equations 4 and 5 ensure that the axial capacity of each newly created structure element is sufficient to resist the loads in that location in that structure as determined by a linear, first-order structural analysis.

3. New Objectives in Inventory-Constrained Structural Design: "Offcut Ratio"

Classical structural optimization generally treats structural mass or structural stiffness as objectives to guide designs towards ultimate design goals: the minimization of cost, material consumption, or life-cycle impact. While these objectives are appropriate in many contexts, for the case of inventory-constrained structural design with discrete elements, other metrics of structural performance may be better correlated with the life-cycle impact, material consumption, and cost of a structure.

One limitation of structural mass as a design objective is that it does not capture the actual amount of inventory material consumed in order to fabricate the elements used in a structure for designs using discrete elements (such as steel or timber frames). In particular, structural mass does not capture the amount of material wasted as offcuts during fabrication which are either discarded or diverted to lower-value product streams. It may be more appropriate in these cases therefore to treat inventory consumption and offcut waste as objectives in evaluating inventory-constrained designs.

While it may be practical in some cases to minimize for one or both of these objectives directly, it is not convenient to use these metrics for comparing assignments for different designs, because the absolute quantities of used and wasted material depend on the size of the structures being considered. This paper introduces a new optimization objective, "Offcut Ratio" – the ratio of offcut waste mass to the mass of inventory material consumed. Because it is normalized against a measure of the total size of a design, Offcut Ratio may be a helpful benchmarking tool for comparing the relative performance of assignments for different designs. Offcut Ratio is defined in Equation 8.

$$OR = \frac{M_{\text{offcuts}}}{M_{\text{used}}} \quad ; \quad 0 \leq OR < 1 \tag{8}$$

$$\begin{aligned} OR &= \text{Offcut Ratio} \\ M_{\text{offcuts}} &= \text{total mass of offcuts} \\ M_{\text{used}} &= \text{total mass of inventory elements used} \end{aligned}$$

4. Analytical Solutions for Offcut Ratio for Simple Design Cases

It is valuable to understand the effect of basic design variables on the Offcut Ratio for simple structures and inventories in an analytical way, so that any numerical solutions can be placed in context.

A simple example is considered: a simply-supported pin-jointed truss consisting of elements all of some length l_0 (Figure 2). The inventory in this example is infinite and consists of bars of some length *L*. For the purposes of analysis, the structure is rationalized as *n* identical repeating "monomers" each consisting of 4 bar elements, and a terminating group of 3 elements.

In this example, we assume that under whatever the given loading, every element in the inventory has sufficient axial capacity to meet ultimate and serviceability limit state requirements if placed at any location in the structure. For this example, we will investigate, from the designer's perspective, what effect changing l_0 with respect to *L* has on the Offcut Ratio metric. It is shown that the choice of scaling of the truss has a significant impact on the amount of material wasted during assembly. We present an easily-visualized approximation which can be used as a design aid for inventory-constrained design of truss-like structures.

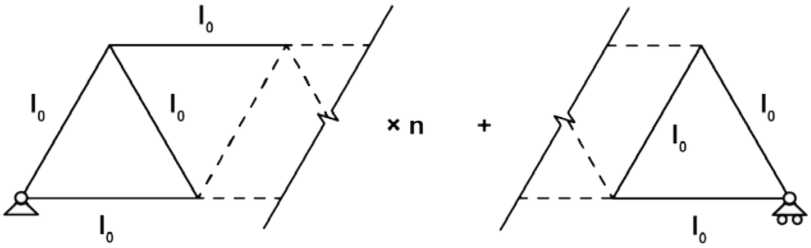


Figure 2: Simply-supported pin-jointed truss consisting of elements all of length l_0

Because all of the inventory elements are identical, and all of the structure elements are identical, it is clear that the optimal assignment with respect to Offcut Ratio is one in which structure elements are sequentially cut from inventory elements, using up the maximal amount from each inventory element until all the structure elements have been fabricated. This assignment results in two possible conditions for the inventory for any given l_0 and L (Figure 3).

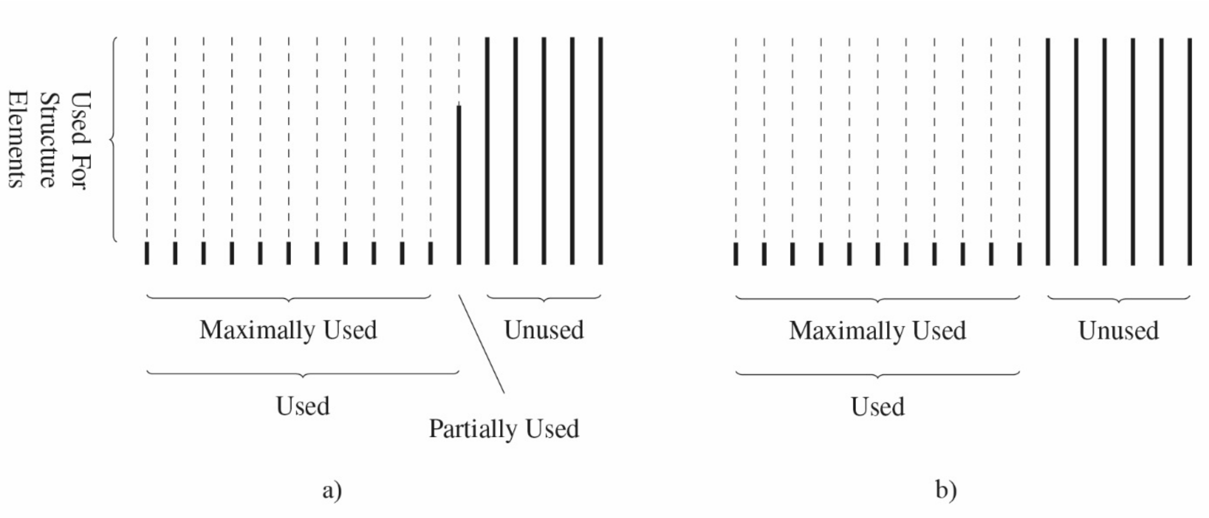


Figure 3: Inventory conditions after fitting: a) inventory contains a partially used element b) inventory contains only maximally used elements.

To compute the Offcut Ratio for this structure, inventory, and assignment, we need to find the total mass of all offcuts, and the total mass of the inventory elements which have been used. To do this we must find the number of structure elements and the number of maximally filled inventory elements. The number of partially filled inventory elements will always be either *1* or *0*. The number of structure elements is $4n + 3$, where $4n$ corresponds to the number of structure elements in the n repeating monomers of the truss, and *3* corresponds to the additional terminating group. The number of structure elements which can fit in each inventory element can be found by:

$$\left\lfloor \frac{L}{l_0} \right\rfloor \tag{9}$$

where $\lfloor\ \rfloor$ is the *floor* operator, indicating rounding down to the nearest integer. The number of maximally used inventory elements can be found by applying the *floor* operator to the quotient of the total number of inventory elements and the maximum number of structure elements per inventory element:

$$\left\lfloor \frac{4n+3}{\left\lfloor \frac{L}{l_0} \right\rfloor} \right\rfloor \tag{10}$$

The number of structure elements remaining after filling the maximally used inventory elements can be found by:

$$(4n+3) \bmod \left\lfloor \frac{L}{l_0} \right\rfloor \tag{11}$$

where *mod* is the modulo operator, indicating remainder. The offcut waste mass per maximally used inventory element can be found by:

$$A\rho(L \bmod l_0) \tag{12}$$

where A is the cross-sectional area of the inventory element, and ρ is the density of the inventory element material. The offcut waste mass in the partially used inventory element, if it exists, is:

$$A\rho\left(L - \left((4n+3) \bmod \left\lfloor \frac{L}{l_0} \right\rfloor\right) l_0\right) \tag{13}$$

This results in two possible cases for the Offcut Ratio, depending on whether there is a partially used inventory element (Equation 14, corresponding to Figure 4a), or if there is not (Equation 15, corresponding to Figure 4b). Note that that the $A\rho$ terms have cancelled.

$$\frac{\left\lfloor \frac{4n+3}{\left\lfloor \frac{L}{l_0} \right\rfloor} \right\rfloor (L \bmod l_0) + \left(L - \left((4n+3) \bmod \left\lfloor \frac{L}{l_0} \right\rfloor\right) l_0\right)}{L\left(\left\lfloor \frac{4n+3}{\left\lfloor \frac{L}{l_0} \right\rfloor} \right\rfloor + 1\right)} \tag{14}$$

$$\frac{L \bmod l_0}{L} \tag{15}$$

It is clear that for large numbers of bays n, and for reasonably large l_0 / L the relative contribution of the partially used element to offcut waste is small. Furthermore, the larger the offcut from this partially used element, the more likely it is to be useful in a later design, and therefore not treated as waste. Therefore, it is reasonable to approximate the offcut ratio for most design situations with Equation 15, which disregards the Offcut Ratio contribution of the partially used inventory element. Equation 15 is a convenient approximation because, as long as n is high, n can be disregarded, meaning that designs of different sizes can be compared on the basis of simply a relationship between their geometry as defined by l_0, and their inventory, as defined by L. Equation 15 is shown plotted in Figure 4b as a function of the ratio l_0 / L. It is interesting to note that the peak values of Equation 15 coincide with l_0 / L.

From Figure 4, it is clear that the choice of l_0 with respect to L has significant impact on the Offcut Ratio. For this particular design example, the designer may use this information, along with other design metrics, to choose a depth of truss such that l_0 is equal or just less than L/i, where i is some positive integer which is unlikely to exceed *4* or *5* for most inventory-constrained design cases. Figure 4b may be a helpful visual aid for early stage inventory-constrained structural design, when a designer is considering the overall dimensions of a design with respect to the available inventory.

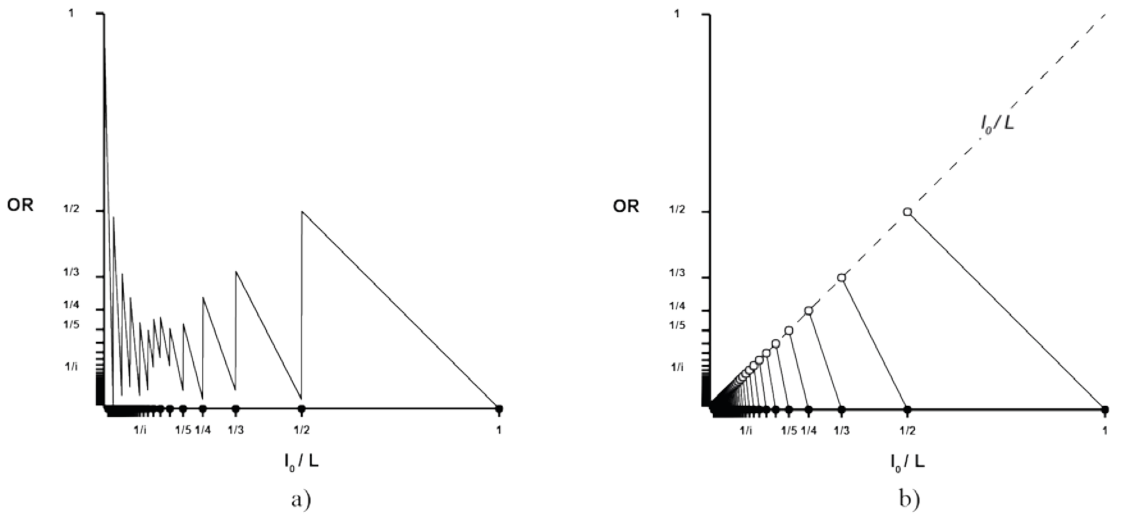


Figure 4: Offcut Ratio for the Single-Length Truss Problem: a) exact solution (Equations 14,15) for $n = 10$ b) approximate solution (Equation 15) for any *n*.

5. Offcut Ratio for Designs with Grouped Structure and Inventory Elements

Equation 15 and Figure 4b provide insight to the designer about efficient design choices for a simple case – the Single-Length Truss Problem. For the more general case of designs involving elements of multiple lengths, determining the optimal Offcut Ratio as a function of design geometry may be significantly more challenging due to the difficulty of finding optimal assignments (See Section 6). One strategy to simplify the analytical estimation of Offcut Ratio for such designs is to establish a pairing between batches of inventory elements and structure elements, so as to, in effect, break the larger problem into sub-problems identical to the Single-Length Truss Problem. The Offcut Ratio contributions of each sub-problem can then be combined, accounting for the cross-sectional areas, material densities, and structure and elements lengths of each group, to provide the global Offcut Ratio plot. The approximation in Equation 15 can be used to make the calculation simpler, while still being realistic. Local minima in this plot (see Figure 5c for an example) represent good potential choices of structural dimensions, given the available inventory. This problem setup is realistic, as structural detailing constraints often dictate where elements of certain sections can be used in the structure.

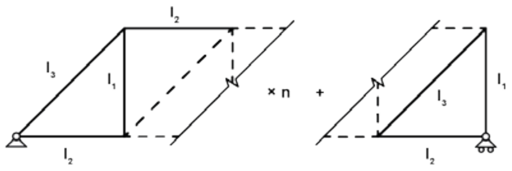
For a design with *G* individual groups of structure elements and inventory elements, to generate an approximate offcut ratio plot normalized in the same way as in Figure 4b, where the entire plotted region represents viable solutions, we must first find a datum group for which adjusting the length of the structure element will first yield $l > L$, and thus will define the feasible range for l_{datum}. This can be found by finding the structure/inventory element group corresponding to:

$$\min\left(\frac{L_i}{l_i}\right), \forall i \in \{1, \ldots, G\} \tag{16}$$

Now, we find the global Offcut Ratio by dividing the total offcut waste contributions of each group by the total inventory element mass used using Equation 17.

$$\frac{\sum_{i=1}^{G} N_i (L_i \bmod (\lambda_i l_{\text{datum}})) A_i \rho_i}{\sum_{i=1}^{G} N_i L_i A_i \rho_i} \tag{17}$$

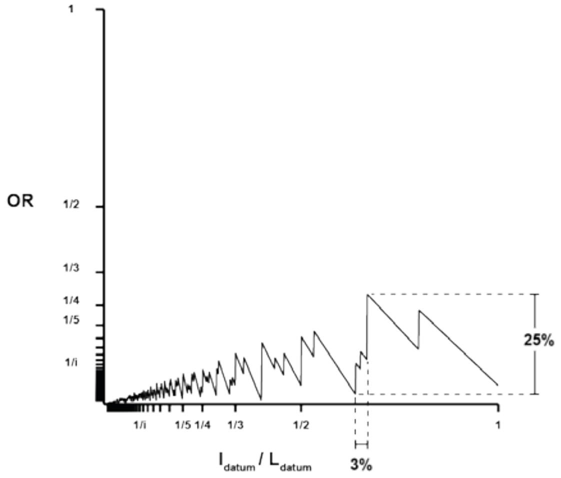
where $\lambda_i = l_i / l_{datum}$ and N_i is the number of structure elements in group *i*.



a)

Group	l	L	A	ρ	N	L/l
Verticals *(datum)*	1.0	1.3	1.0	1.0	1	1.3
Chords	1.0	2.0	1.75	1.0	2	2.0
Diagonals	$\sqrt{2}$	4.5	1.25	1.0	1	3.19

b)



c)

Figure 5: Example of global Offcut Ratio plot for a grouped design: a) design indicating lengths of structure elements b) table of relevant values for each group c) resulting approximate global Offcut Ratio plot.

An example of the global approximate Offcut Ratio plot for a grouped design is shown in Figure 5. Note that a change of around 3% in l_{datum} in one instance produces a difference in Offcut Ratio of about 25%. We believe that this analysis method may be useful for the optimization of designs for roof trusses in steel or round-timber. As we have demonstrated (Figures 4a, 4b, 5c), small changes in the geometry of such structures can result in significant savings in material consumption and waste.

6. Generating Approximately Optimal Assignments with Polynomial-Time Heuristics

For more complex structural designs, it may not be possible to predetermine groupings of inventory and structure elements. Also, these groups may contain elements of various lengths. This generalized problem is shown in the integer program in Equations 1-7. For problems of this type, the discovery of optimal assignments becomes NP-Hard (this problem is identical to the bin-packing problem (Coffman, Garey, and Johnson [2]), except in the choice of objective function). This means that not only do no analytical solutions exist to describe the optimal assignment as a function of the problem parameters, but no algorithms can exist to find optimal assignments in polynomial time, unless P = NP (Cooke [3]).

Bukauskas et al. [1] addressed this challenge with polynomial-time heuristics which are able to generate *approximately* optimal assignments for arbitrary statically determinate pin-jointed truss structures and arbitrary inventories of bar elements in real-time. These heuristics can be used as part of an iterative designer-led, form-finding process for inventory-constrained structural design. Inventory consumption, offcut waste, and Offcut Ratio can be computed and displayed to the designer in real-time. Figure 6 shows how a designer might use these tools to explore the geometric design space for a parametrically defined roof truss with respect to a given constrained inventory. The heuristic in this example is a greedy "First-Fit" algorithm typically used for bin-packing problems with an additional pre-sorting step of the inventory and structure elements by length. This work demonstrates the generalizability of structural assignments for arbitrarily diverse and complex statically determinate pin-jointed truss designs.

7. Conclusions and Future Work

Inventory-constrained structural design is an emerging practice and research area, with significant potential to reduce the life-cycle impact, material consumption, and cost of new structures. This paper has introduced a precise definition for the concept of an "assignment" for inventory-constrained design with discrete elements. We have also identified the need for new optimization objectives which better quantify the performance of inventory-constrained designs. We have proposed inventory consumption, offcut waste, and the Offcut Ratio metric for this purpose. Future work will involve the development of other metrics which quantify the relative required use of material from outside the given inventory (i.e. fit failure), and relative over-specification of elements. We have also demonstrated simple analytical techniques which, subject to some approximations, can provide valuable and readily visualizable insights into choices of efficient design geometries. We suggest that these techniques may be particularly useful for reducing the material consumption and waste of repetitive truss structures, such as roof trusses, utility towers, and truss bridges. Finally, we suggest how polynomial-time fitting heuristics can be used to generate and evaluate assignments for the general case of statically-determinate pin-jointed trusses in real-time, a critical component in iterative, designer-led form-finding processes which are key in early-stage structural design.

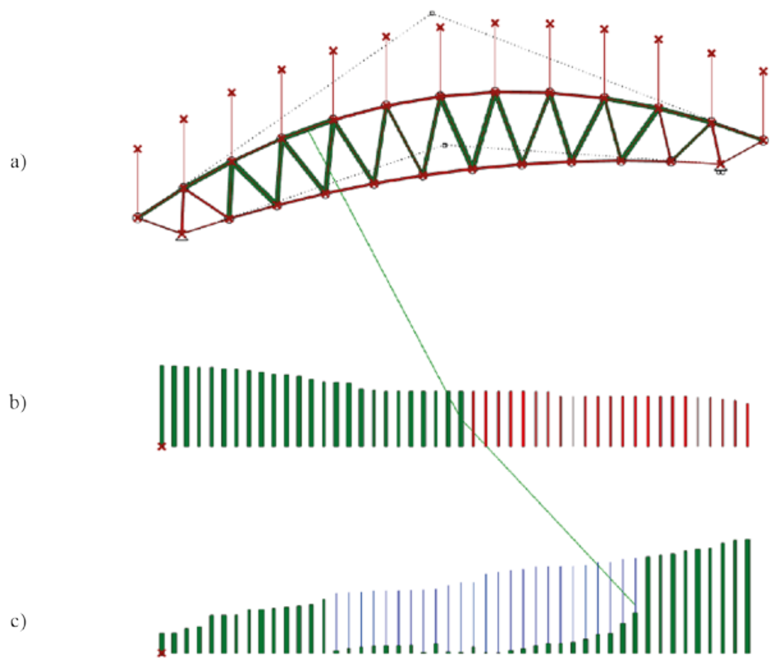


Figure 6: Iterative form-finding process for inventory-constrained design with real-time assignment-generation heuristics: a) parametrically defined structure with loads and support conditions b) structure elements pre-sorted in descending order by length c) inventory elements pre-sorted in ascending order by length.

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