

The use of virtual work for the formfinding of fabric, shell and gridshell structures

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Abstract

The use of the virtual work theorem enables one to derive the equations of static equilibrium of fabric, shell and gridshell structures from the compatibility equations linking the rate of deformation of a surface to variations in its velocity. If the structure is treated as a continuum there is no need to consider its micro-structure provided that the grid is fine compared to the overall geometry. Thus we can include fabrics, ribbed shells, corrugated shells and gridshells with a fine grid, such as the Mannheim Multihalle. The equilibrium equations are almost identical to those obtained by assuming that a shell is thin and of uniform thickness, but are more general in their application. Our formulation introduces the concept of geodesic bending moments which are relevant to gridshell structures with continuous laths.

The virtual work theorem is more general than the energy theorems, which it includes as a special case. Hence it can be applied to surfaces which admit some form of potential, including minimal surfaces and hanging fabrics. We can then use the calculus of variations for the minimization of a surface integral to define the form of a structure.

Many existing formfinding techniques can be rewritten in this way, but we concentrate on surfaces which minimize the surface integral of the mean curvature subject to a constraint on the enclosed volume, producing a surface of constant Gaussian curvature. This naturally leads to the more general study of conjugate stress and curvature directions, and hence to quadrilateral mesh gridshells with flat cladding panels and no bending moments in the structural members under own weight.

Key words: Virtual work, fabric structure, shell theory, gridshell, calculus of variations, conjugate directions

1 Introduction

The principal of virtual work was formalized by Johann Bernoulli and Joseph-Louis Lagrange in the 18th century (Capecchi, 2012) and today is taught to all civil and mechanical engineering students. It is the basis of the application of the finite element method to structural mechanics, although the formulation it produces is often identical to that obtained using the Galerkin method. Virtual work is closely related to the calculus of variations and the concept of strain energy, but it is more general in that it can be applied to non-elastic materials and to loads which do not admit a potential, such as wind loads.

Virtual work considers a virtual or imaginary infinitesimal increment of displacement and deformation of a structure which may be undergoing a very large deformation. It then calculates the increment of work done by the loads on the structure and uses the divergence theorem to relate this to internal stresses and strains. It is perhaps better to instead imagine a virtual velocity, in which case the increment of work is replaced by the rate of work, or power, and in French they use the term *puissances virtuelles*, literally virtual power.

Virtual work requires the geometric compatibility equations relating increment of displacement to increment of deformation or strain and uses the virtual nature of the increment of deformation to derive the equations of static equilibrium. Thus the method is purely geometric, with no concept of resolution of forces or moments, which should hopefully appeal to those with a background in geometry. The advantage of using velocity instead of increment of displacement is that the velocity is the derivative of position with respect to time, and we can therefore use all the properties of differentiation, rather than the more unwieldy process of letting the magnitude of the displacement tend to zero.

It should be emphasized that we have essentially 3 types of equation,

- the compatibility equations relating variations in velocity to rate of deformation, both stretching and bending,
- the virtual work equation and
- the equilibrium equations relating loads to internal forces and moments.

Any 2 of these imply the 3rd and we will use the compatibility equations and virtual work to obtain the equilibrium equations. One could use the compatibility equations and the equilibrium equations to prove the virtual work equation, and engineers often use the equilibrium equations and virtual work to solve a geometric problem involving the deformation of truss structures.

We shall assume that the structure is either a continuous shell or fabric structure, or has a fine grid so that it can be treated as a continuum, both from the geometrical and structural points of view.

In general we will follow the notation in Green and Zerna (1968) for both geometric

quantities and quantities such as force and stress. There are many books and papers on shell theory, for example Timoshenko and Woinowsk-Krieger (1959), Flügge (1973), Calladine (1982) and Axelrad and Emmerling (1988), but in our view the notation in Green and Zerna is to be preferred because it has the added advantage that it is essentially as that used by Einstein for the general theory of relativity (Dirac, 1975), where, of course, the stress-energy tensor causes the curvature of space-time so that stress is essentially a geometric concept with principal values equal to the density and the 3 principal stresses.

The references cited in the previous paragraph could be loosely described as ‘engineering’ texts. But there are numerous other relevant references in mathematics, architectural geometry and computer graphics including Kupferman et al. (2017), Vouga et al. (2012), Yang et al. (2011), Jiang et al. (2015) and Diamanti et al. (2014).

2 The application of virtual work to pin jointed space structures

Before considering shell structures let us examine a simpler case, that of pin jointed space structures. This enables us to understand the fundamental idea behind the application of virtual work. One could imagine a shell, or even a 3 dimensional continuum, as being made up of a very fine structure of pin ended members, and indeed this is done in the numerical methods, peridynamics (Silling and Lehoucq, 2010) and smoothed particle hydrodynamics (Monaghan, 2012). At the molecular level the assumptions of continuum mechanics break down, so that a pin jointed framework is theoretically just as realistic as a continuum.

The length L_i of the member with ends at nodes numbered a_i and b_i is given by Pythagoras’ theorem,

$$L_i^2 = (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \cdot (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \quad (1)$$

where \mathbf{x}_j is the position vector of node j .

Differentiating with respect to time we obtain

$$2L_i\dot{L}_i = 2(\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \cdot (\dot{\mathbf{x}}_{a_i} - \dot{\mathbf{x}}_{b_i})$$

so that

$$\dot{L}_i = \frac{(\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \cdot (\dot{\mathbf{x}}_{a_i} - \dot{\mathbf{x}}_{b_i})}{L_i} \quad (2)$$

The virtual work equation is

$$\sum_{j=0}^n (\mathbf{p}_j \cdot \dot{\mathbf{x}}_j) = \sum_{i=0}^m (T_i \dot{L}_i) \quad (3)$$

where the members are numbered from 0 to m and the nodes are numbered from 0 to n . \mathbf{p}_j is the load applied to node j and T_i is the tension in member i .

Therefore using (2),

$$\sum_{j=0}^n (\mathbf{p}_j \cdot \dot{\mathbf{x}}_j) = \sum_{i=0}^m \left(\frac{T_i}{L_i} (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \cdot (\dot{\mathbf{x}}_{a_i} - \dot{\mathbf{x}}_{b_i}) \right).$$

But this applies for *any virtual set of nodal velocities*. So if we imagine that all the nodes are stationary except for node j

$$\mathbf{p}_j \cdot \dot{\mathbf{x}}_j = \sum_{i=0}^m \left(\frac{T_i}{L_i} (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \cdot (\delta_{ja_i} \dot{\mathbf{x}}_j - \delta_{jb_i} \dot{\mathbf{x}}_j) \right)$$

in which

$$\begin{aligned} \delta_{ja_i} &= 0 & \text{if } j \neq a_i \\ &= 1 & \text{if } j = a_i. \end{aligned} \quad (4)$$

Therefore since the direction of $\dot{\mathbf{x}}_j$ is arbitrary,

$$\mathbf{p}_j = \sum_{i=0}^m \left(\frac{T_i}{L_i} (\delta_{ja_i} - \delta_{jb_i}) (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \right) \quad (5)$$

which are the equilibrium equations that we could have obtained by resolving forces at the nodes. We do not actually need the δ_{ja_i} and δ_{jb_i} in a numerical implementation since we sum over all the members adding forces to the nodes as appropriate.

Thus any 2 of equations (2), (3) or (5) imply the third.

Equation (3) looks like a statement of conservation of energy, rate of work being done by loads equal rate of work being absorbed by members, but there is no suggestion the the deformation or the forces are 'real', provided that $\dot{\mathbf{x}}_j$ and \dot{L}_i are geometrically compatible and \mathbf{p}_j and T_i are in equilibrium with each other.

Note there is absolutely no assumption that the displacements are small, although we do have to consider velocities or increments of displacement. L_i is the current length of a member, which might be stretched to many times its original length. Similarly \mathbf{x}_j are the current positions and the initial member lengths and positions do not appear at all. Indeed there may be no meaningful concept of initial lengths and positions since we may have to move nodes and stretch or compress members to fit the structure together, possibly deforming the members permanently.

There is also no assumption about material properties, the members do not have to be elastic and we have made no assumption about the relationship between T_i and L_i .

2.1 The stiffness matrix

We are not concerned with stiffness in this paper. However, since there is a great deal of confusion attached to the subject, let us differentiate (5) with respect to

time,

$$\dot{\mathbf{p}}_j = \sum_{i=0}^m \left(\frac{d}{dt} \left(\frac{T_i}{L_i} \right) (\delta_{ja_i} - \delta_{jb_i}) (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \right) + \sum_{i=0}^m \left(\frac{T_i}{L_i} (\delta_{ja_i} - \delta_{jb_i}) (\dot{\mathbf{x}}_{a_i} - \dot{\mathbf{x}}_{b_i}) \right). \quad (6)$$

If the members are elastic, or if they are plastic and we have sufficient knowledge of past deformation, we can write

$$\frac{d}{dt} \left(\frac{T_i}{L_i} \right) = s_i \frac{\dot{L}_i}{L_i} = s_i \frac{d}{dt} (\log L_i) \quad (7)$$

where the member stiffness s_i may itself be a function of the current L_i .

Then

$$\begin{aligned} \dot{\mathbf{p}}_j &= \sum_{i=0}^m \left(s_i \frac{\dot{L}_i}{L_i} (\delta_{ja_i} - \delta_{jb_i}) (\mathbf{x}_{a_i} - \mathbf{x}_{b_i}) \right) + \sum_{i=0}^m \left(\frac{T_i}{L_i} (\delta_{ja_i} - \delta_{jb_i}) (\dot{\mathbf{x}}_{a_i} - \dot{\mathbf{x}}_{b_i}) \right) \\ &= \sum_{i=0}^m \left((\delta_{ja_i} - \delta_{jb_i}) \left(s_i \frac{(\mathbf{x}_{a_i} - \mathbf{x}_{b_i})(\mathbf{x}_{a_i} - \mathbf{x}_{b_i})}{L_i^2} + \frac{T_i}{L_i} \mathbf{I} \right) \cdot (\dot{\mathbf{x}}_{a_i} - \dot{\mathbf{x}}_{b_i}) \right) \end{aligned} \quad (8)$$

in which \mathbf{I} is the unit tensor in 3 dimensional space.

The term containing s_i is the elastic stiffness and the term containing $\frac{T_i}{L_i}$ is the geometric stiffness. The geometric stiffness is so called because it only depends upon the geometry and the state of stress, not the elastic properties, except in so much as they may influence the state of stress.

3 The equations of static equilibrium of shell structures

The concept of virtual work is much more difficult to grasp for shell structures than for the pin jointed structures described in section 2 because the associated mathematics involving the differential geometry of a deforming surface is complicated. Therefore, rather than leave the main results until after they have been proved, we will quote them now as an incentive to follow their derivation.

The equation of equilibrium of forces is

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{p} = 0. \quad (9)$$

This is a vector equation corresponding to equilibrium in 3 directions, that is the 2 directions tangential to the surface and the normal direction. The vector \mathbf{p} is the load per unit area on the structure and $\boldsymbol{\sigma}$ is a second order tensor containing the membrane stresses and the normal shear forces associated with bending. The $\nabla \cdot$ is the divergence described in Section 4.3. In the case of a structure which is not static we can include inertia forces using D'Alembert's principle.

The form of eq. (9) is essentially the same as eq. (5) and indeed much of the theory of peridynamics and smoothed particle hydrodynamics is to establish link tensions which will have the same effect as stresses in a solid or a fluid.

Equation (9) is identical to that for a 3 dimensional continuum and in relativity theory $\nabla \cdot \boldsymbol{\sigma} = 0$ is the equation for the conservation of momentum and mass-energy.



Figure 1: Mannheim Multihalle, Germany 1974. Frei Otto, Carlfried Mutschler and Winfried Langner, Ove Arup and Partners

The equation of equilibrium of moments is

$$(\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c}) = (\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c})^T \quad (10)$$

where the superscript T means the transpose. Thus we are saying that the second order tensor $(\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c})$ is symmetric, which is equivalent to the resultant moment about 3 directions being zero, again 2 directions tangent to the surface and the surface normal. \mathbf{m} is the third order tensor containing the moments in the surface, both 'ordinary' bending and twisting moments about axes in the plane of the surface and 'geodesic' bending moments about the normal which are relevant to gridshell structures with continuous laths, like the Mannheim Multihalle gridshell (fig. 1) or a kitchen sieve made from a woven wire mesh. Both the Mannheim Multihalle and a sieve have a fine grid, making it appropriate to treat them as continua. The second order tensor \mathbf{c} is the loading couple per unit area, which is zero in almost all practical applications, and again we can include the effect of acceleration using D'Alembert's principle.

To our knowledge this is the first time that this concept of geodesic moments has been introduced.

4 Differential geometry of a stationary surface

4.1 The base vectors and the first fundamental form

We first need to define the geometric quantities of a stationary surface before we can examine how they vary as the surface deforms. The contents of this section will be familiar to some readers, but we need to define all our terms so that we can differentiate them with respect to time in Section 5 where we consider a deforming surface.

Imagine a surface defined by the curvilinear coordinates, θ^1 and θ^2 . The position vector of a typical point on the surface and its Cartesian coordinates are given by

$$\mathbf{r}(\theta^1, \theta^2) = x(\theta^1, \theta^2)\mathbf{i} + y(\theta^1, \theta^2)\mathbf{j} + z(\theta^1, \theta^2)\mathbf{k} \quad (11)$$

in which \mathbf{i} , \mathbf{j} and \mathbf{k} are unit base vectors in the direction of the Cartesian axes, x , y and z . We use the surface coordinates or parameters θ^1 and θ^2 with superscripts to replace the more usual u and v so that we can use the tensor notation, which is indispensable if one is to consider both geometry and structural concepts such as stress. Following the notation in Green and Zerna (1968), the covariant base vectors, \mathbf{a}_α , in which α is equal to 1 or 2, are given by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} = \mathbf{r}_{,\alpha} \quad (12)$$

A comma will be used to denote partial differentiation. \mathbf{a}_1 is tangential to a curve $\theta^2 = \text{constant}$ on the surface and \mathbf{a}_2 is tangential to a curve $\theta^1 = \text{constant}$. In general neither \mathbf{a}_1 nor \mathbf{a}_2 will be unit vectors, and they will not be perpendicular to each other.

The square of the distance between two adjacent points on the surface is equal to

$$\sum_{\alpha=1}^2 \sum_{\beta=1}^2 a_{\alpha\beta} d\theta^\alpha d\theta^\beta = a_{\alpha\beta} d\theta^\alpha d\theta^\beta \quad (13)$$

$$a_{\alpha\beta} = a_{\beta\alpha} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$$

in which we have used the Einstein summation convention for the implied summation for repeated subscripts and superscripts. $a_{\alpha\beta}$ are the covariant components of the metric tensor, also known as the coefficients of the first fundamental form. Eisenhart (1947) uses $g_{\alpha\beta}$ and Struik (1961) uses E , F and G .

The unit normal is

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \quad (14)$$

and here our notation differs from Green and Zerna who use \mathbf{a}_3 . We will also have

occasion to use the contravariant base vectors, \mathbf{a}^α , defined by

$$\begin{aligned}\mathbf{a}^\alpha \cdot \mathbf{a}_\beta &= \delta_\beta^\alpha = 0 \text{ if } \alpha \neq \beta \\ &= 1 \text{ if } \alpha = \beta \\ \mathbf{a}^\alpha \cdot \mathbf{n} &= 0.\end{aligned}\tag{15}$$

\mathbf{a}^1 lies in the tangent plane to the surface perpendicular to a curve $\theta^1 = \text{constant}$ and its magnitude is such that $\mathbf{a}^1 \cdot \mathbf{a}_1 = 1$, and similarly for \mathbf{a}^2 .

Using $a_{\alpha\beta}$ and

$$a^{\alpha\beta} = a^{\beta\alpha} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta\tag{16}$$

we can raise and lower indices using equations such as

$$\begin{aligned}\mathbf{a}^\alpha &= a^{\alpha\beta} \mathbf{a}_\beta \\ \mathbf{a}_\alpha &= a_{\alpha\beta} \mathbf{a}^\beta \\ q_{\alpha\beta} &= a_{\alpha\lambda} q^\lambda{}_\beta\end{aligned}$$

in which the dot in $q^\lambda{}_\beta$ is used maintain the order of indices. If the second order tensor \mathbf{q} is symmetric then $q_{\alpha\beta} = q_{\beta\alpha}$ and we can dispense with the dot and write $q^\lambda{}_\beta$.

The components of the permutation tensor,

$$\begin{aligned}\varepsilon_{\lambda\mu} &= -\varepsilon_{\mu\lambda} \\ \varepsilon_{11} &= 0, \varepsilon_{12} = -\varepsilon_{21} = \sqrt{a}, \varepsilon_{22} = 0\end{aligned}\tag{17}$$

in which

$$a = |\mathbf{a}_1 \times \mathbf{a}_2|^2 = a_{11}a_{22} - (a_{12})^2\tag{18}$$

are used to perform the vector products,

$$\begin{aligned}\mathbf{a}_\alpha \times \mathbf{a}_\beta &= \varepsilon_{\alpha\beta} \mathbf{n} \\ \mathbf{n} \times \mathbf{a}^\alpha &= \varepsilon^{\alpha\beta} \mathbf{a}_\beta.\end{aligned}\tag{19}$$

a is not a scalar since it is a property of the coordinate system.

4.2 The second fundamental form and the Christoffel symbols

The components of the normal curvature tensor, or coefficients of the second fundamental form, are

$$b_{\alpha\beta} = b_{\beta\alpha} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n} = \frac{\partial^2 \mathbf{r}}{\partial \theta^\alpha \partial \theta^\beta} \cdot \mathbf{n} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}.\tag{20}$$

Eisenhart (1947) uses $d_{\alpha\beta}$ and Struik (1961) uses e , f and g . $b_{\alpha\beta}$, together with $a_{\alpha\beta}$, contain all the information about the normal curvature and twist of the surface, including the principal curvatures and their directions. The Gaussian curvature,

$$K = \frac{b_{11}b_{22} - (b_{12})^2}{a_{11}a_{22} - (a_{12})^2}\tag{21}$$

is equal to the product of the principal curvatures and the mean curvature,

$$H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = \frac{1}{2} b_{\alpha}^{\alpha} \quad (22)$$

is the average of the principal curvatures.

$b_{\alpha\beta}$ give the component of $\mathbf{a}_{\alpha,\beta}$ normal to the surface and the Christoffel symbols,

$$\Gamma_{\alpha\beta}^{\chi} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^{\chi} = \frac{1}{2} a^{\chi\eta} (a_{\eta\alpha,\beta} + a_{\beta\eta,\alpha} - a_{\alpha\beta,\eta}) \quad (23)$$

give the components of $\mathbf{a}_{\alpha,\beta}$ tangent to the surface. Note that the Christoffel symbols are not the components of a tensor because they represent properties of the coordinate system rather than the surface itself.

The fundamental theorem of surface theory states that the tensor components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ define the shape of a surface, but not its position and orientation in space. $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are not independent since they come from differentiating 3 Cartesian coordinates with respect to the surface coordinates. Writing

$$\mathbf{a}_{\alpha,\beta\chi} = \left(b_{\alpha\beta} \mathbf{n} + \Gamma_{\alpha\beta}^{\lambda} \mathbf{a}_{\lambda} \right)_{,\chi} = \mathbf{a}_{\alpha,\chi\beta} = \left(b_{\alpha\chi} \mathbf{n} + \Gamma_{\alpha\chi}^{\lambda} \mathbf{a}_{\lambda} \right)_{,\beta} \quad (24)$$

gives the 3 conditions that the surface 'fits together'. These are known as Gauss's *theorem egregium* and the Codazzi-Mainardi equations,

$$\nabla_{\lambda} b_{\alpha\beta} = \nabla_{\alpha} b_{\lambda\beta} \quad (25)$$

in which the covariant derivative,

$$\nabla_{\lambda} b_{\alpha\beta} = b_{\alpha\beta,\lambda} - b_{\eta\beta} \Gamma_{\alpha\lambda}^{\eta} - b_{\alpha\eta} \Gamma_{\beta\lambda}^{\eta} \quad (26)$$

4.3 Component free notation and the gradient of a tensor

It is rather unsatisfactory to only be able to talk about the components of a tensor, rather than the tensor itself, and we can write the second order normal curvature tensor, \mathbf{b} , as

$$\mathbf{b} = b_{\alpha\beta} \mathbf{a}^{\alpha} \mathbf{a}^{\beta} = b_{\alpha}^{\beta} \mathbf{a}^{\alpha} \mathbf{a}_{\beta} = b^{\alpha\beta} \mathbf{a}_{\alpha} \mathbf{a}_{\beta} \quad (27)$$

in which the product $\mathbf{a}^{\alpha} \mathbf{a}^{\beta}$, without a dot or a cross is the tensor product, or outer product, sometimes written with a \otimes , defined by

$$\begin{aligned} (\mathbf{de}) \cdot \mathbf{g} &= \mathbf{d}(\mathbf{e} \cdot \mathbf{g}) \\ \mathbf{g} \cdot (\mathbf{de}) &= (\mathbf{g} \cdot \mathbf{d}) \mathbf{e} \end{aligned} \quad (28)$$

where \mathbf{d} , \mathbf{e} and \mathbf{g} are any vectors. We shall also use the double dot notation,

$$\begin{aligned} (\mathbf{de}) \cdot \cdot (\mathbf{gh}) &= (\mathbf{d} \cdot \mathbf{g})(\mathbf{e} \cdot \mathbf{h}) \\ (\mathbf{cde}) \cdot \cdot \cdot (\mathbf{ghpq}) &= (\mathbf{c} \cdot \mathbf{g})(\mathbf{d} \cdot \mathbf{h})(\mathbf{e} \cdot \mathbf{p}) \mathbf{q} \end{aligned} \quad (29)$$

which can be extended to any number of dots.

Let us now imagine that we have a vector field on a surface, that is a first order tensor field,

$$\mathbf{v} = v^\alpha \mathbf{a}_\alpha + v\mathbf{n} \quad (30)$$

with components tangential to the surface, v^α , and normal to the surface, v . We will define the gradient of this vector field as the second order tensor,

$$\begin{aligned} \nabla \mathbf{v} &= \mathbf{a}^\beta \mathbf{v}_{,\beta} = \mathbf{a}^\beta (v^\alpha \mathbf{a}_\alpha + v\mathbf{n})_{,\beta} \\ &= \mathbf{a}^\beta \left((\nabla_\beta v^\alpha - v b_\beta^\alpha) \mathbf{a}_\alpha + (v^\alpha b_{\alpha\beta} + v_{,\beta}) \mathbf{n} \right) \end{aligned} \quad (31)$$

in which the covariant derivative,

$$\nabla_\beta v^\alpha = v^\alpha_{,\beta} + v^\eta \Gamma_{\eta\beta}^\alpha. \quad (32)$$

The covariant derivative of the components of the metric tensor and of the permutation tensor are all zero and

$$\nabla \mathbf{n} = -\mathbf{b}. \quad (33)$$

The divergence of the vector \mathbf{v} ,

$$\nabla \cdot \mathbf{v} = \mathbf{a}^\alpha \cdot \mathbf{v}_{,\alpha} = \nabla_\alpha v^\alpha - v b_\alpha^\alpha \quad (34)$$

in which

$$\nabla_\alpha v^\alpha = v^\alpha_{,\alpha} + v^\lambda \Gamma_{\lambda\alpha}^\alpha = \frac{(v^\alpha \sqrt{a})_{,\alpha}}{\sqrt{a}}. \quad (35)$$

4.4 The divergence theorem on a surface

This last result enables us to prove the divergence theorem on a surface for a vector with no normal component, $\mathbf{v} = v^\alpha \mathbf{a}_\alpha$,

$$\begin{aligned} \int_A \nabla \cdot \mathbf{v} dA &= \int_A \nabla_\alpha v^\alpha \sqrt{a} d\theta^1 d\theta^2 = \int_A (v^\alpha \sqrt{a})_{,\alpha} d\theta^1 d\theta^2 \\ &= \int_{\partial A} v^\alpha \varepsilon_{\alpha\beta} d\theta^\beta = \int_{\partial A} (d\mathbf{r} \times \mathbf{n}) \cdot \mathbf{v} \end{aligned} \quad (36)$$

in which ∂A is the boundary of the surface A . This result is central to the application of virtual work to surface structures.

5 Deformation of a surface

5.1 Velocity

Now let us imagine a moving and deforming surface defined by the curvilinear coordinates, θ^1 and θ^2 , and time t . The position vector of a typical point on the surface is now given by

$$\mathbf{r}(\theta^1, \theta^2, t) = x(\theta^1, \theta^2, t) \mathbf{i} + y(\theta^1, \theta^2, t) \mathbf{j} + z(\theta^1, \theta^2, t) \mathbf{k}. \quad (37)$$

The velocity vector is

$$\mathbf{u} = u_\mu \mathbf{a}^\mu + u \mathbf{n} = \frac{\partial \mathbf{r}}{\partial t}. \quad (38)$$

We imagine that the coordinates move with the surface, like the laths of a gridshell.

5.2 Strain rate and angular velocity

The gradient of the velocity is

$$\begin{aligned} \nabla \mathbf{u} &= \mathbf{a}^\lambda \mathbf{u}_{,\lambda} = (\nabla_\lambda u_\mu - b_{\lambda\mu} u) \mathbf{a}^\lambda \mathbf{a}^\mu + (u_\mu b_\lambda^\mu + \nabla_\lambda u) \mathbf{a}^\lambda \mathbf{n} \\ &= (\gamma_{\lambda\mu} + \omega \varepsilon_{\lambda\mu}) \mathbf{a}^\lambda \mathbf{a}^\mu + \varepsilon_{\lambda\mu} \omega^\mu \mathbf{a}^\lambda \mathbf{n} \end{aligned} \quad (39)$$

in which the symmetric strain rate tensor,

$$\begin{aligned} \gamma &= \gamma^T = \frac{1}{2} \left((\nabla \mathbf{u} - \nabla \mathbf{u} \cdot \mathbf{nn}) + (\nabla \mathbf{u} - \nabla \mathbf{u} \cdot \mathbf{nn})^T \right) \\ &= \gamma_{\lambda\mu} \mathbf{a}^\lambda \mathbf{a}^\mu \end{aligned} \quad (40)$$

and the anti-symmetric angular velocity tensor, or vorticity tensor,

$$\begin{aligned} \omega &= -\omega^T = \frac{1}{2} \left((\nabla \mathbf{u} - \mathbf{n} \nabla \mathbf{u} \cdot \mathbf{n}) - (\nabla \mathbf{u} - \mathbf{n} \nabla \mathbf{u} \cdot \mathbf{n})^T \right) \\ &= \omega \varepsilon_{\lambda\mu} \mathbf{a}^\lambda \mathbf{a}^\mu + \varepsilon_{\lambda\mu} \omega^\mu \left(\mathbf{a}^\lambda \mathbf{n} - \mathbf{n} \mathbf{a}^\lambda \right). \end{aligned} \quad (41)$$

ω is defined by only 3 quantities, ω , ω^1 and ω^2 , which could be considered to be the components of a vector.

We can write

$$\nabla \mathbf{u} = \gamma + \omega - \mathbf{nn} \cdot \omega \quad (42)$$

$$\mathbf{u}_{,\lambda} = \mathbf{a}_\lambda \cdot (\gamma + \omega) \quad (43)$$

$$\frac{\partial \mathbf{n}}{\partial t} = \mathbf{n} \cdot (\gamma + \omega) = \mathbf{n} \cdot \omega \quad (44)$$

and we have the results

$$\frac{\partial a_{\alpha\beta}}{\partial t} = \mathbf{u}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \mathbf{u}_{,\beta} = \mathbf{a}_\alpha \cdot \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \cdot \mathbf{a}_\beta = 2\gamma_{\alpha\beta} \quad (45)$$

$$\frac{\partial a^{\alpha\beta}}{\partial t} = -2\gamma^{\alpha\beta} \quad (46)$$

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial t} = a^{\alpha\beta} \gamma_{\alpha\beta}. \quad (47)$$

5.3 Rate of bending, or rate of change of normal and geodesic curvature

In order to find how the coefficients of the second fundamental form and the Christoffel symbols vary with time we need

$$\begin{aligned}
 \frac{\partial \mathbf{a}_{\alpha,\beta}}{\partial t} &= \frac{\partial}{\partial t} \left(b_{\alpha\beta} \mathbf{n} + \Gamma_{\alpha\beta}^{\lambda} \mathbf{a}_{\lambda} \right) \\
 &= \frac{\partial b_{\alpha\beta}}{\partial t} \mathbf{n} + \frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t} \mathbf{a}_{\lambda} + b_{\alpha\beta} \frac{\partial \mathbf{n}}{\partial t} + \Gamma_{\alpha\beta}^{\lambda} \mathbf{u}_{,\lambda} \\
 &= \mathbf{u}_{,\alpha\beta} = \mathbf{a}_{\alpha} \cdot \left(\mathbf{a}^{\lambda} \mathbf{u}_{,\lambda} \right)_{,\beta} - \mathbf{a}_{\alpha} \cdot \mathbf{a}^{\lambda}_{,\beta} \mathbf{u}_{,\lambda} = \mathbf{a}_{\alpha} \cdot \left(\mathbf{a}^{\lambda} \mathbf{u}_{,\lambda} \right)_{,\beta} + \Gamma_{\alpha\beta}^{\lambda} \mathbf{u}_{,\lambda} \\
 &= (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla \nabla \mathbf{u} + \Gamma_{\alpha\beta}^{\lambda} \mathbf{u}_{,\lambda}
 \end{aligned} \tag{48}$$

and

$$\begin{aligned}
 (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla \nabla \mathbf{u} &= (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla (\gamma + \omega - \mathbf{nn} \cdot \omega) \\
 &= (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla (\gamma + \omega) + b_{\alpha\beta} \mathbf{n} \cdot \omega \\
 &= (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla (\gamma + \omega) + b_{\alpha\beta} \frac{\partial \mathbf{n}}{\partial t}
 \end{aligned} \tag{49}$$

which mean that

$$\frac{\partial b_{\alpha\beta}}{\partial t} \mathbf{n} + \frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t} \mathbf{a}_{\lambda} = (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla \nabla \mathbf{u} - b_{\alpha\beta} \frac{\partial \mathbf{n}}{\partial t} = (\mathbf{a}_{\beta} \mathbf{a}_{\alpha}) \cdot \nabla (\gamma + \omega) \tag{50}$$

$$\frac{\partial b_{\alpha\beta}}{\partial t} = (\mathbf{a}_{\beta} \mathbf{a}_{\alpha} \mathbf{n}) \cdot \nabla (\gamma + \omega) \tag{51}$$

$$\frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t} = (\mathbf{a}_{\beta} \mathbf{a}_{\alpha} \mathbf{a}^{\lambda}) \cdot \nabla (\gamma + \omega) . \tag{52}$$

Equation (51) tells us about the rate of change of normal curvature, while eq. (52) tells us about the rate of change of geodesic curvature. Equation (52) leads to the

somewhat surprising conclusion that $\frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t}$ are the components of a tensor, even though $\Gamma_{\alpha\beta}^{\lambda}$ are not, although in the general theory of relativity the derivatives of the Christoffel symbols lead to the components of the Riemann–Christoffel tensor.

Differentiating eq. (23) with respect to time,

$$\begin{aligned}
 \frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t} &= \frac{1}{2} \frac{\partial a^{\alpha\beta}}{\partial t} (a_{\eta\alpha,\beta} + a_{\beta\eta,\alpha} - a_{\alpha\beta,\eta}) + \frac{1}{2} a^{\lambda\eta} \left(\frac{\partial a_{\beta\eta,\alpha}}{\partial t} + \frac{\partial a_{\eta\alpha,\beta}}{\partial t} - \frac{\partial a_{\alpha\beta,\eta}}{\partial t} \right) \\
 &= -2\gamma^{\lambda\eta} a_{\mu\eta} \Gamma_{\alpha\beta}^{\mu} + a^{\lambda\eta} \left(\begin{array}{l} \nabla_{\beta} \gamma_{\eta\alpha} + \gamma_{\chi\alpha} \Gamma_{\eta\beta}^{\chi} + \gamma_{\eta\chi} \Gamma_{\alpha\beta}^{\chi} \\ + \nabla_{\alpha} \gamma_{\beta\eta} + \gamma_{\chi\eta} \Gamma_{\alpha\beta}^{\chi} + \gamma_{\beta\chi} \Gamma_{\eta\alpha}^{\chi} \\ - \nabla_{\eta} \gamma_{\alpha\beta} - \gamma_{\chi\beta} \Gamma_{\alpha\eta}^{\chi} - \gamma_{\alpha\chi} \Gamma_{\beta\eta}^{\chi} \end{array} \right) \\
 &= a^{\lambda\eta} (\nabla_{\beta} \gamma_{\eta\alpha} + \nabla_{\alpha} \gamma_{\beta\eta} - \nabla_{\eta} \gamma_{\alpha\beta})
 \end{aligned} \tag{53}$$

which confirms that $\frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t}$ are indeed the components of a tensor. But note that the rate of change of the Christoffel symbols of the first kind, $\frac{\partial \Gamma_{\alpha\beta\lambda}}{\partial t}$, are *not* the components of a tensor.

For future use we will combine the components of $\frac{\partial b_{\alpha\beta}}{\partial t}$ and $\frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t}$ into the components of one third order tensor, β ,

$$\begin{aligned}\beta &= \mathbf{a}^{\alpha} \mathbf{a}^{\beta} \left(\beta_{\alpha\beta} \mathbf{n} + \beta_{\alpha\beta}^{\cdot\lambda} \mathbf{a}_{\lambda} \right) = \mathbf{a}^{\alpha} \mathbf{a}^{\beta} \left(\frac{\partial b_{\alpha\beta}}{\partial t} \mathbf{n} + \frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial t} \mathbf{a}_{\lambda} \right) \\ &= \mathbf{a}^{\alpha} \mathbf{a}^{\beta} (\mathbf{a}_{\alpha} \mathbf{a}_{\beta}) \cdot \nabla (\gamma + \omega)\end{aligned}\quad (54)$$

which we might call the rate of bending tensor. It includes both the rate of change of the normal curvature as well as the rate of change of the geodesic curvature of lines on the surface.

6 Puissances virtuelles - rate of virtual work being done on a surface structure

6.1 The rate of virtual work

The rate of virtual work being done on a surface A with boundary ∂A as it moves and deforms is

$$W = \int_{\partial A} (d\mathbf{r} \times \mathbf{n}) \cdot (\boldsymbol{\sigma} \cdot \mathbf{u} + \mathbf{m} \cdot \nabla (\gamma + \omega)) + \int_A (\mathbf{p} \cdot \mathbf{u} + \mathbf{c} \cdot \nabla (\gamma + \omega)) dA. \quad (55)$$

We do not have to justify this equation in any way, except to say that we assume that W only depends upon \mathbf{u} and $(\gamma + \omega)$ and the loading on the surface itself and its boundary. Equation (55) is the definition of the vector \mathbf{p} , the second order tensors \mathbf{c} and $\boldsymbol{\sigma}$ and the third order tensor \mathbf{m} , whose physical interpretation were given in Section 3.

We could add other terms, particularly those to include shear deformation as in a Timoshenko (1921) or Cosserat (1909) beam or shell. Such deformation was important in the design of the Mannheim Multihalle gridshell because of the flexible connection between the upper and lower parallel members. However, while not difficult to add such deformation, it introduces further complexity which is not relevant to this paper.

We can stipulate that

$$\mathbf{n} \cdot \boldsymbol{\sigma} = 0 \quad (56)$$

$$\mathbf{n} \cdot \mathbf{m} = 0 \quad (57)$$

because $(d\mathbf{r} \times \mathbf{n})$ lies in the plane of the surface. We will also stipulate that

$$(\mathbf{a}_\alpha \mathbf{n}) \cdot \mathbf{m} = 0 \quad (58)$$

$$\mathbf{n} \cdot \mathbf{c} = 0 \quad (59)$$

and include the work done by that part of $\boldsymbol{\omega}$ containing \mathbf{n} from the 'right hand part' of the third order tensor \mathbf{m} . Thus we can write the components

$$\boldsymbol{\sigma} = \mathbf{a}_\alpha \left(\sigma^{\alpha\beta} \mathbf{a}_\beta + \sigma^\alpha \mathbf{n} \right) \quad (60)$$

$$\mathbf{m} = \mathbf{a}_\alpha \mathbf{a}_\beta \left(m^{\alpha\beta} \mathbf{n} + m^{\alpha\beta}{}_{\cdot\chi} \mathbf{a}^\chi \right). \quad (61)$$

$\sigma^{\alpha\beta}$ are the components of membrane stress and σ^α are the components of normal shear force. $m^{\alpha\beta}$ are the components of normal bending and twisting moment and $m^{\alpha\beta}{}_{\cdot\chi}$ are the components of the geodesic bending moments.

We can now use the divergence theorem, eq. (36), to change the boundary integral in eq. (55) to a surface integral,

$$W = \int_A \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u} + \mathbf{m} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega})) dA + \int_A (\mathbf{p} \cdot \mathbf{u} + \mathbf{c} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega})) dA \quad (62)$$

in which

$$\begin{aligned} \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u} + \mathbf{m} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega})) &= \mathbf{a}^\alpha \cdot (\boldsymbol{\sigma} \cdot \mathbf{u} + \mathbf{m} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega}))_{,\alpha} \\ &= \nabla \cdot \boldsymbol{\sigma} \cdot \mathbf{u} + \nabla \cdot \mathbf{m} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega}) \\ &\quad + \boldsymbol{\sigma} \cdot \cdot (\mathbf{a}^\alpha \mathbf{a}_\alpha \cdot \nabla \mathbf{u}) + \mathbf{m} \cdot \cdot (\mathbf{a}^\alpha \mathbf{a}_\alpha \cdot \nabla (\boldsymbol{\gamma} + \boldsymbol{\omega})) \\ &= \nabla \cdot \boldsymbol{\sigma} \cdot \mathbf{u} + \nabla \cdot \mathbf{m} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega}) \\ &\quad + \boldsymbol{\sigma} \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega}) + \mathbf{m} \cdot \cdot \boldsymbol{\beta} \end{aligned} \quad (63)$$

so that

$$W = \int_A ((\nabla \cdot \boldsymbol{\sigma} + \mathbf{p}) \cdot \mathbf{u} + (\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c}) \cdot \cdot (\boldsymbol{\gamma} + \boldsymbol{\omega}) + \mathbf{m} \cdot \cdot \boldsymbol{\beta}) dA. \quad (64)$$

6.2 The equilibrium equations

If \mathbf{u} and $\boldsymbol{\omega}$ are such that we have a rigid body motion, then $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are both zero so that

$$W = \int_A ((\nabla \cdot \boldsymbol{\sigma} + \mathbf{p}) \cdot \mathbf{u} + (\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c}) \cdot \cdot \boldsymbol{\omega}) dA. \quad (65)$$

We now postulate that no net work can be done in *any* rigid body motion of a structure in equilibrium, so that $W = 0$, even if we change the location the boundary

relative to the surface, adding or removing parts of the surface to just include some arbitrary part of the structure. This leads to the equation of equilibrium of forces,

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{p} = 0 \quad (66)$$

and of moments

$$(\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c}) = (\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c})^T \quad (67)$$

because $\boldsymbol{\omega}$ is anti-symmetric.

6.3 The rate of work being absorbed

Finally we have the rate of work being absorbed by the structure,

$$\begin{aligned} W &= \int_A ((\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c}) \cdot \boldsymbol{\gamma} + \mathbf{m} \cdot \boldsymbol{\omega}) dA \\ &= \int_A ((\boldsymbol{\sigma} + \mathbf{c}) \cdot \boldsymbol{\gamma} + \mathbf{m} \cdot \nabla \boldsymbol{\omega}) dA + \int_{\partial A} (d\mathbf{r} \times \mathbf{n}) \cdot \mathbf{m} \cdot \boldsymbol{\gamma}. \end{aligned} \quad (68)$$

The term $\mathbf{c} \cdot \boldsymbol{\gamma}$ is there because the laths of a gridshell like that of the Mannheim Multihalle can undergo different angular velocities about an axis normal to the surface, like a pair of scissors, and the loading couples \mathbf{c} can therefore do work.

It is important to realize that $\frac{\partial b_{\alpha\beta}}{\partial t}$ may be non zero even when $\nabla \boldsymbol{\omega}$ is zero. For example when a spherical shell undergoes a uniform expansion there are no rotations but $b_{\alpha\beta}$ change because of the change in the magnitude of \mathbf{a}_α .

6.4 The equilibrium equations and rate of work being absorbed in terms of components

We ought, perhaps, to write the equilibrium equations, eq. (66) and eq. (67), in terms of components when they lose their essential simplicity,

$$\nabla_\alpha \sigma^{\alpha\beta} - \sigma^\alpha b_\alpha^\beta + p^\beta = 0 \quad (69)$$

$$\sigma^{\alpha\beta} b_{\alpha\beta} + \nabla_\alpha \sigma^\alpha + p = 0 \quad (70)$$

$$\varepsilon_{\alpha\beta} \left(\sigma^{\alpha\beta} - m^{\eta\alpha} b_\eta^\beta + \nabla_\eta m^{\eta\alpha\beta} + c^{\alpha\beta} \right) = 0 \quad (71)$$

$$\sigma^\beta + \nabla_\alpha m^{\alpha\beta} + \left(m^{\lambda\mu\beta} + m^{\lambda\beta\mu} \right) b_{\lambda\mu} + c^\beta = 0. \quad (72)$$

These equations are identical to equations (10.4.4) to (10.4.7) of Green and Zerna (1968), if one makes the following changes to the notation, $n^{\alpha\beta} = \sigma^{\alpha\beta}$, $q^\alpha = \sigma^\alpha$, $\tilde{p}^\beta = -c^\beta$, the sign of $m^{\alpha\beta}$ are reversed, $|_\alpha$ is used instead of ∇_α for the covariant derivative and the components $m^{\eta\alpha\beta} = 0$ and $c^{\alpha\beta} = 0$. Green and Zerna derive their equations from the 3 dimensional equations of equilibrium and integrating through

the thickness of a thin shell, but this is rather unsatisfactory for gridshells or ribbed shells.

The rate of work being absorbed, eq. (68), becomes

$$W = \int_A \left(\left(\sigma^{\alpha\beta} - m^{\eta\alpha} b_{\eta}^{\beta} + \nabla_{\eta} m^{\eta\alpha\beta} + c^{\alpha\beta} \right) \gamma_{\alpha\beta} + m^{\alpha\beta} \beta_{\alpha\beta} + m^{\alpha\beta}{}_{;\chi} \beta_{\alpha\beta}{}^{;\chi} \right) dA. \quad (73)$$

7 Elastic surfaces

We define an elastic surface as any surface whose strain energy per unit mass is a function of $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\Gamma_{\alpha\beta}^{\lambda}$. Such structures include soap films, fabric structures, shells and gridshells, as well as surfaces which minimize the Willmore energy, that is the surface integral of $H^2 - K$ (Velimirović et al., 2011; Williams, 1987).

For an elastic surface the virtual work formulation becomes identical to the minimization of the sum of the strain energy of the structure and the potential energy of the loads, assuming that they admit a potential. The same results could be obtained using the calculus of variations, which would need exactly the same geometric relationships and how they change under an increment of displacement, or better a velocity allowing us to differentiate with respect to time.

It is preferable to use the strain energy per unit mass than the strain energy per unit area because mass is conserved, whereas area changes. If Q is the strain energy per unit mass and ρ is the mass per unit area, we can write

$$\begin{aligned} \rho \frac{\partial Q}{\partial t} &= (\boldsymbol{\sigma} + \nabla \cdot \mathbf{m} + \mathbf{c}) \cdot \boldsymbol{\gamma} + \mathbf{m} \cdots \boldsymbol{\beta} \\ &= \left(\sigma^{\alpha\beta} - m^{\eta\alpha} b_{\eta}^{\beta} + \nabla_{\eta} m^{\eta\alpha\beta} + c^{\alpha\beta} \right) \frac{1}{2} \frac{\partial a_{\alpha\beta}}{\partial t} \\ &\quad + m^{\alpha\beta} \frac{\partial b_{\alpha\beta}}{\partial t} + m^{\alpha\beta}{}_{;\chi} \frac{\partial \Gamma_{\alpha\beta}^{\chi}}{\partial t} \end{aligned} \quad (74)$$

in which

$$\frac{\partial}{\partial t} (\rho \sqrt{a}) = 0 \quad (75)$$

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} = -\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial t} = -a^{\alpha\beta} \gamma_{\alpha\beta}. \quad (76)$$

7.1 Surfaces with constant mean curvature, including minimal surfaces

The strain energy per unit mass of a surface with a constant isotropic surface tension T is

$$Q = \frac{T}{\rho}. \quad (77)$$

Thus Q does not depend upon $b_{\alpha\beta}$ or $\Gamma_{\alpha\beta}^\lambda$ and so the moments are zero giving

$$\boldsymbol{\sigma} = \sigma^{\alpha\beta} \mathbf{a}_\alpha \mathbf{a}_\beta = \sigma^{\beta\alpha} \mathbf{a}_\alpha \mathbf{a}_\beta \quad (78)$$

and eq. (74) becomes

$$\frac{\partial}{\partial t} \left(\frac{T}{\rho} \right) = - \frac{T}{\rho^2} \frac{\partial \rho}{\partial t} = \frac{T}{\rho} a^{\alpha\beta} \gamma_{\alpha\beta} = \frac{1}{\rho} \sigma^{\alpha\beta} \gamma_{\alpha\beta} . \quad (79)$$

Thus the membrane stress

$$\begin{aligned} \sigma^{\alpha\beta} &= T a^{\alpha\beta} \\ \boldsymbol{\sigma} &= T \mathbf{J} \end{aligned} \quad (80)$$

corresponding to the uniform surface tension.

$$\mathbf{J} = a^{\alpha\beta} \mathbf{a}_\alpha \mathbf{a}_\beta \quad (81)$$

is the unit tensor on the surface.

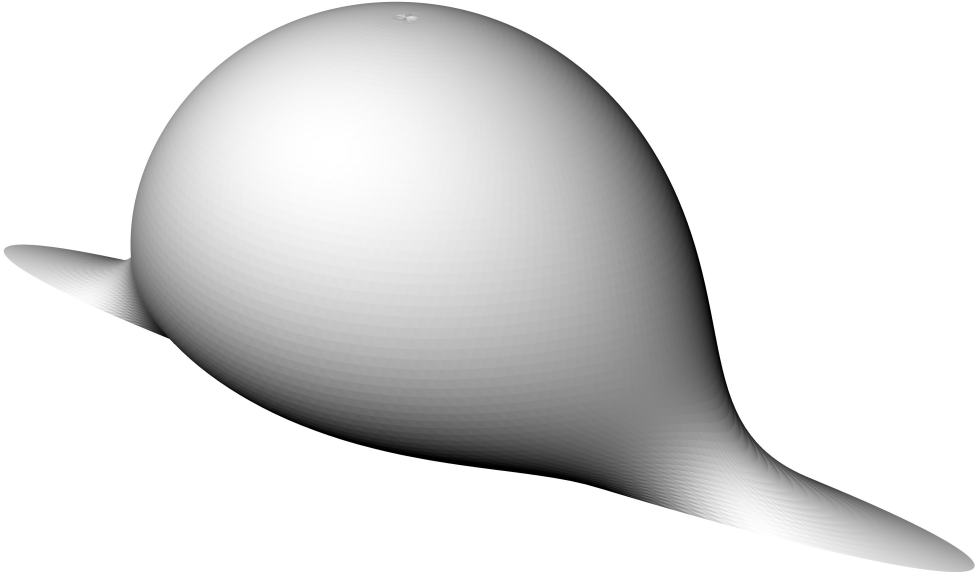


Figure 2: Surface with constant mean curvature on a plane elliptical boundary

The surface in fig. 2 was found using dynamic relaxation (Day, 1965). The boundary is a plane ellipse and the surface has the minimum surface area for a given enclosed volume, like an inflated soap film. Dynamic relaxation was also used to find the pressure necessary to enclose a fixed volume. The pressure cannot be kept constant for a surface such as this because the pressure *decreases* with increasing volume, once it is inflated beyond a certain point. It was found necessary to damp the pressure change with both the rates of change of pressure and volume.

7.2 Surfaces which minimize the integral of the mean curvature subject to a constant volume

If we assume that the strain energy per unit mass is equal to the mean curvature divided by the density, then

$$Q = \frac{H}{\rho} = \frac{a^{\alpha\beta} b_{\alpha\beta}}{2\rho} \quad (82)$$

and

$$\begin{aligned} \rho \frac{\partial Q}{\partial t} &= \frac{1}{2} \frac{\partial a^{\alpha\beta}}{\partial t} b_{\alpha\beta} + \frac{1}{2} a^{\alpha\beta} \frac{\partial b_{\alpha\beta}}{\partial t} - \frac{H}{\rho} \frac{\partial \rho}{\partial t} \\ &= \gamma^{\alpha\beta} (H a_{\alpha\beta} - b_{\alpha\beta}) + \frac{1}{2} a^{\alpha\beta} \frac{\partial b_{\alpha\beta}}{\partial t} \\ &= (H a^{\alpha\beta} - b^{\alpha\beta}) \frac{1}{2} \frac{\partial a^{\alpha\beta}}{\partial t} + \frac{1}{2} a^{\alpha\beta} \frac{\partial b_{\alpha\beta}}{\partial t}. \end{aligned} \quad (83)$$

Then comparison with eq. (74) shows that

$$m^{\alpha\beta} = \frac{1}{2} a^{\alpha\beta}. \quad (84)$$

Assuming that the loading couples, \mathbf{c} are zero, the equations of equilibrium of moments show that

$$\sigma^{\alpha\beta} = \sigma^{\beta\alpha} \quad (85)$$

$$\sigma^\alpha = 0 \quad (86)$$

so that

$$\sigma^{\alpha\beta} - m^{\eta\alpha} b_\eta^\beta = H a^{\alpha\beta} - b^{\alpha\beta} \quad (87)$$

and therefore

$$\begin{aligned} \sigma^{\alpha\beta} &= H a^{\alpha\beta} - b^{\alpha\beta} + \frac{1}{2} a^{\eta\alpha} b_\eta^\beta \\ &= \frac{1}{2} (b_\eta^\eta a^{\alpha\beta} - b^{\alpha\beta}). \end{aligned} \quad (88)$$

or

$$\sigma = H \mathbf{J} - \frac{\mathbf{b}}{2} = \frac{\boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon}}{2}. \quad (89)$$

We therefore have

$$\nabla_\alpha \sigma^{\alpha\beta} = \frac{1}{2} (\nabla_\alpha b_\eta^\eta a^{\alpha\beta} - \nabla_\alpha b^{\alpha\beta}) = \frac{1}{2} (\nabla_\alpha b_\eta^\eta - \nabla_\eta b_\alpha^\eta) a^{\alpha\beta} = 0 \quad (90)$$

from the Codazzi-Mainardi equations, eq. (25), and

$$\begin{aligned} \sigma^{\alpha\beta} b_{\alpha\beta} &= \frac{1}{2} (b_\eta^\eta a^{\alpha\beta} - b^{\alpha\beta}) b_{\alpha\beta} \\ &= \frac{1}{2} (b_\alpha^\alpha b_\beta^\beta - b_\beta^\alpha b_\alpha^\beta) \\ &= K. \end{aligned} \quad (91)$$

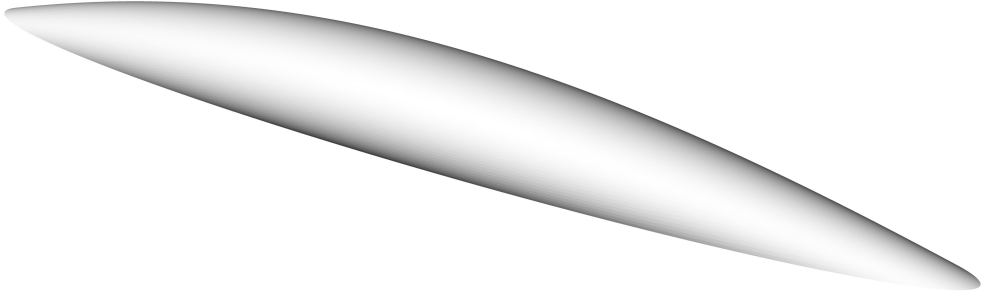


Figure 3: Surface with constant positive Gaussian curvature on a plane elliptical boundary

Thus the tangential components of load

$$p^\alpha = 0 \quad (92)$$

are zero and the normal component of load,

$$p = -K \quad (93)$$

where K is the Gaussian curvature.

Thus if we minimize, or possibly maximize, the surface integral of the mean curvature subject to the enclosed volume being constant we find that we need an internal, or possibly external pressure, which must be a constant and therefore the Gaussian curvature is also constant. One would imagine that this simple fact must have been known before.

The moments in the surface do not affect equilibrium, and can therefore be dispensed with and are purely a phantom, we only need the membrane stresses for equilibrium.

The mean membrane stress,

$$\frac{1}{2}a_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{2}b_\eta^\eta = H \quad (94)$$

and therefore for a sphere with an outwards pointing normal the mean stress is compressive, and we need an external pressure. On the other hand if we choose to have the normal pointing inwards we have a tensile mean stress and an internal pressure.

Figure 3 show a surface of constant positive Gaussian curvature on the same plane elliptic boundary as that in fig. 2. The numerical procedure uses flat triangular facets and the stress in the surface is represented by forces in each fold proportional to the angle of the fold from flat. Those familiar with the Airy stress function (Timoshenko, 1934) will realize that this is equivalent to the shell being its own stress function, but with no projection onto the plane. The equilibrium shape was again found using dynamic relaxation. Note that this procedure only controls the

shape of the surface, not the position of the nodes upon it and therefore some other constraint is required. In this case the folds were given an additional constant force density, but the normal component of the extra resultant force was removed before moving the nodes. This technique is commonly used to find geodesics on a surface for fabric structures (Williams, 1980).

8 Simultaneous conjugate directions for membrane stress and curvature

8.1 Non-orthogonal directions

In the stress state in eq. (89) the principal stress directions and principal curvature directions coincide. That means we could construct a gridshell structure with an orthogonal quadrilateral mesh with no bending moments and flat panels, subject to a pressure loading. In this section we shall relax the pressure loading requirement, because we are only really interested in the state of stress and the curvature. We can also relax the condition on the mesh being orthogonal.

We can write any state of membrane stress as

$$\begin{aligned}\boldsymbol{\sigma} &= A\mathbf{x}\mathbf{x} + B\mathbf{y}\mathbf{y} \\ \mathbf{x} \cdot \mathbf{x} &= 1 \\ \mathbf{y} \cdot \mathbf{y} &= 1 \\ \sigma^{\alpha\beta} &= Ax^\alpha x^\beta + By^\alpha y^\beta\end{aligned}\tag{95}$$

corresponding to two monoaxial stresses of magnitude A and B in the directions of the unit vectors \mathbf{x} and \mathbf{y} tangent to the surface. There are 3 values of the components σ^{11} , $\sigma^{12} = \sigma^{21}$ and σ^{22} , but 4 quantities A , B and the directions \mathbf{x} and \mathbf{y} , see fig. 4. We therefore need 1 further condition, often taken as $\mathbf{x} \cdot \mathbf{y} = 0$, which leads to the principal stresses and their directions, but we will not make that assumption at this juncture.

Now let us postulate that we can write the normal curvature tensor \mathbf{b} as

$$\begin{aligned}\mathbf{b} &= C(\boldsymbol{\varepsilon} \cdot \mathbf{x})(\boldsymbol{\varepsilon} \cdot \mathbf{x}) + D(\boldsymbol{\varepsilon} \cdot \mathbf{y})(\boldsymbol{\varepsilon} \cdot \mathbf{y}) \\ b_{\alpha\beta} &= \varepsilon_{\alpha\lambda} \varepsilon_{\beta\mu} (Cx^\lambda x^\mu + Dy^\lambda y^\mu)\end{aligned}\tag{96}$$

with the same unit vectors \mathbf{x} and \mathbf{y} . The reason for the $\boldsymbol{\varepsilon}$ in eq. (96) is that we want the stresses to coincide with the directions of 'folding', rather than the directions of curvature.

Then if $\boldsymbol{\sigma}$ and \mathbf{b} are known we have 6 equations in the unknowns A , B , C , D and the directions of \mathbf{x} and \mathbf{y} . These equations can be solved by introducing the orthogonal

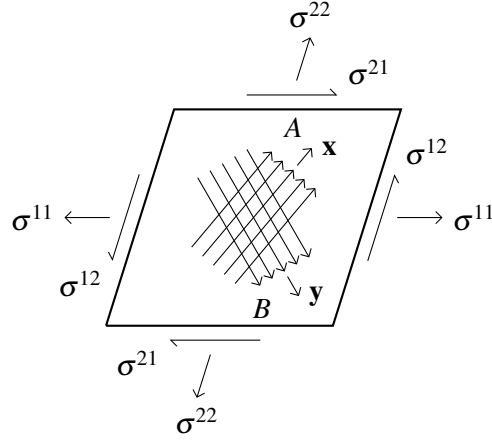


Figure 4: Monoaxial stress state of magnitude A and B acting in the directions of unit vectors \mathbf{x} and \mathbf{y} tangent to the surface on a small element of shell

unit vectors \mathbf{X} and \mathbf{Y} such the

$$\begin{aligned}\mathbf{x} &= \mathbf{X} \cos \frac{\theta}{2} + \mathbf{Y} \sin \frac{\theta}{2} \\ \mathbf{y} &= \mathbf{X} \sin \frac{\theta}{2} + \mathbf{Y} \cos \frac{\theta}{2}\end{aligned}\tag{97}$$

$$\begin{aligned}\mathbf{X} \cdot \mathbf{X} &= 1 \\ \mathbf{X} \cdot \mathbf{Y} &= 0 \\ \mathbf{Y} \cdot \mathbf{Y} &= 1\end{aligned}$$

and substituting into eqs. (95) and (96) to give

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{(A+B)}{2} (\mathbf{X}\mathbf{X} + \mathbf{Y}\mathbf{Y} + (\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{X}) \sin \theta) \\ &+ \frac{(A-B)}{2} (\mathbf{X}\mathbf{X} - \mathbf{Y}\mathbf{Y}) \cos \theta\end{aligned}\tag{98}$$

$$\begin{aligned}-\boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} &= \frac{(C+D)}{2} (\mathbf{X}\mathbf{X} + \mathbf{Y}\mathbf{Y} + (\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{X}) \sin \theta) \\ &+ \frac{(C-D)}{2} (\mathbf{X}\mathbf{X} - \mathbf{Y}\mathbf{Y}) \cos \theta.\end{aligned}\tag{99}$$

Hence

$$H\boldsymbol{\sigma} + S\boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} = \left(H \frac{(A-B)}{2} - S \frac{(C-D)}{2} \right) \cos \theta (\mathbf{X}\mathbf{X} - \mathbf{Y}\mathbf{Y})\tag{100}$$

where S is the mean stress. We can find \mathbf{X} and \mathbf{Y} by observing that $(\mathbf{X} + \mathbf{Y})$ and $(\mathbf{X} - \mathbf{Y})$ are the eigenvectors of $(H\boldsymbol{\sigma} + S\boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon})$. Having found \mathbf{X} and \mathbf{Y} we can find all the other unknowns and further study shows that $|\cos \theta| \leq 1$ and $|\sin \theta| \leq 1$. Thus it would appear that there is always a solution, except for the case when H and S are both zero.

8.2 Orthogonal directions

A special case of the previous section is when the curvature and membrane stress are such that they share the same principal directions, so that \mathbf{x} and \mathbf{y} are orthogonal.

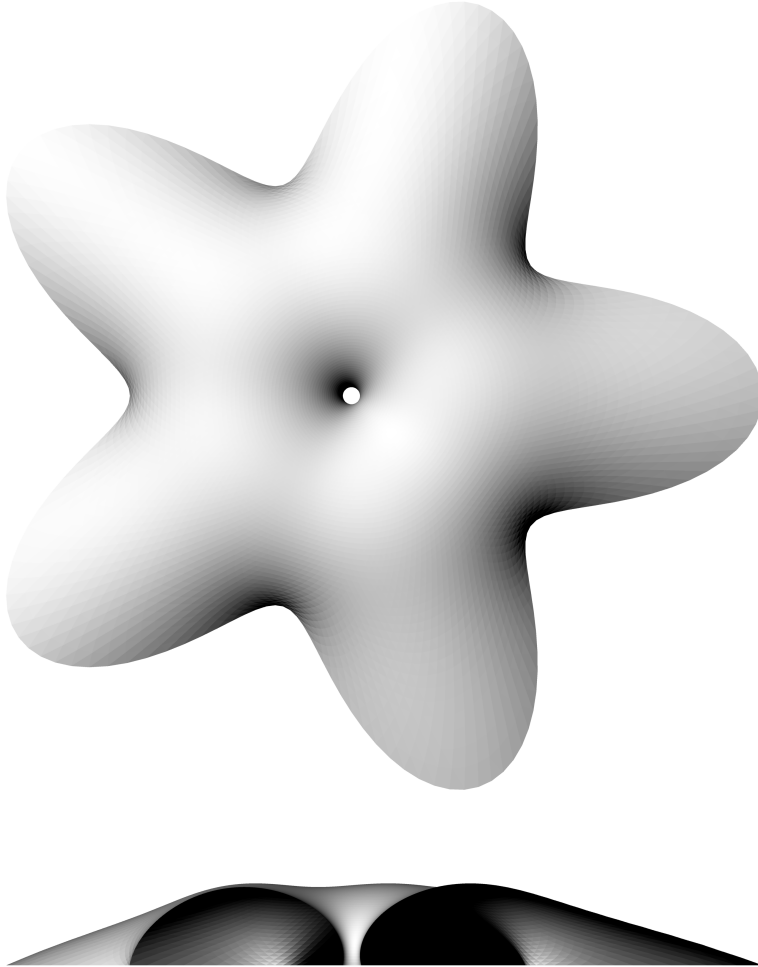


Figure 5: Plan and cross-section of shell with coincident principal membrane stress and principal curvature directions

Then we can generalize the state of stress in eq. (89) to

$$\begin{aligned}\boldsymbol{\sigma} &= \phi \mathbf{J} + \psi \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} \\ \sigma^{\alpha\beta} &= \phi a^{\alpha\beta} - \psi \varepsilon^{\alpha\lambda} \varepsilon^{\beta\mu} b_{\lambda\mu}\end{aligned}\tag{101}$$

where ϕ and ψ are scalar fields.

Using the Codazzi-Mainardi equations, eq. (25), the equilibrium equations become

$$(\nabla \cdot \boldsymbol{\sigma} + \mathbf{p}) \cdot \mathbf{J} = \nabla \phi + \nabla \psi \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} + \mathbf{p} \cdot \mathbf{J} = 0\tag{102}$$

in the plane of the surface and

$$(\nabla \cdot \boldsymbol{\sigma} + \mathbf{p}) \cdot \mathbf{n} = 2\phi H + 2\psi K + p = 0 \quad (103)$$

normal to the surface.

Let us assume that we have a load corresponding to a known weight per unit area w , then

$$\mathbf{p} = p^\beta \mathbf{a}_\beta + p\mathbf{n} = -w\mathbf{k} \quad (104)$$

where \mathbf{k} is a unit vector in the z direction. The equilibrium equations are then

$$\nabla\phi + \nabla\psi \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} = w\nabla z \quad (105)$$

$$2\phi H + 2\psi K = w\mathbf{k} \cdot \mathbf{n} . \quad (106)$$

The equilibrium equations in the plane of the surface, eq. (105), have one possible very simple solution.

$$\psi = \text{constant} \quad (107)$$

$$\phi = \phi(z) \quad (108)$$

$$w = w(z) = \frac{d\phi}{dz} \quad (109)$$

where $\phi(z)$ is a function we can choose. Having done this the equilibrium equation normal to the surface eq. (106) can be used to find the geometry of the surface, in exactly the same way as in sections 7.1 and 7.2. Aish et al. (2015) consider the special case $\psi = 0$.

If we assume that

$$\frac{\phi}{w} = -L = \text{constant} \quad (110)$$

so that the isotropic part of the membrane stress is proportional to the weight per unit area, then the isotropic stress and weight per unit area reduce with height,

$$\phi = \phi_0 e^{-(z-z_0)/L} . \quad (111)$$

Figure 5 shows a plan and cross-section of a shell corresponding to this isotropic stress and loading state plus the stress associated with a constant ψ . It can be seen that there is a concentration of vertical stress at the centre support which could not have been obtained with the isotropic stress state on its own. The high vertical stress is associated with ψ and the large curvature in the horizontal plane.

Having defined the surface we need to construct the orthogonal grid of principal curvature directions, which coincide with the directions of the principal membrane stresses. This is not a trivial task, particularly in ensuring that the variation in spacing of the lines is satisfactory (Sun et al., 2016).

9 Conclusion

We have derived the shell equilibrium equations using virtual work, which enables many formfinding methods to be reformulated as a minimization using the calculus of variations. This method of deriving the equilibrium equations naturally introduces the concept of geodesic bending moments for the analysis of gridshells and kitchen sieves via the Christoffel symbols, which become the components of a tensor upon differentiation with respect to time.

We have also demonstrated that minimizing the surface integral of the mean curvature subject to a constraint on enclosed volume gives a surface of constant Gaussian curvature, although one would imagine that this simple fact must have been known before.

These studies lead us to examine the conditions under which principal stress and principal curvature coincide and how this can be incorporated into a formfinding process.

Some of the ideas introduced in this paper could lead to further numerical studies and practical application.

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