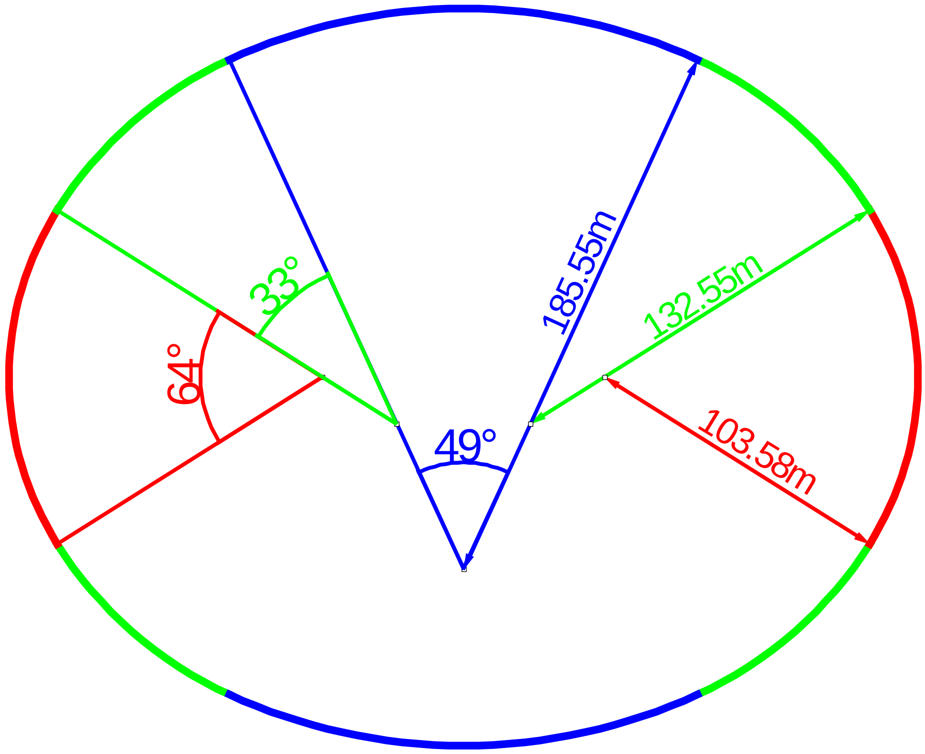
What's the Equation of a Stadium?

The Background

The main stadium for the London 2012 Olympics is instantly recognisable, but did you know it is not a circle? Although it has two lines of symmetry, the steel roof measures 150m along its long axis and only 122m along the short axis.

And whilst it may look like an ellipse, it is not. The shape has been carefully constructed from three different circular arcs, as shown below:



These circular segments allow for repetition in the design of the structure of the roof and the seating underneath, making the overall design easier and more efficient to build. It also makes it easier for the designers to calculate the circumference of the roof, and therefore how much steel will be required to build it.

The shape of the roof could have been described mathematically by an ellipse of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

…where a and b are the major and minor radii (half the long and short diameters).

However, this would have made the calculation of the exact circumference much more difficult. The exact equation for the circumference of an ellipse involves some nasty "elliptical integrals" which are very tricky things. In 1914, mathematical genius Ramanujan derived a pretty good approximation:

$$C \approx \pi \left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right]$$

The Problem

Use trigonometry to calculate the width and height of the stadium using only the information given in the sketch above. Check it by drawing to scale using a compass and ruler (or Autograph?).

Derive the circumference of the stadium roof by summing the different arc-lengths. Check it against your scale drawing using a piece of string.

Find what Ramanujan's approximation to the circumference would be if it were an ellipse of the same diameters.

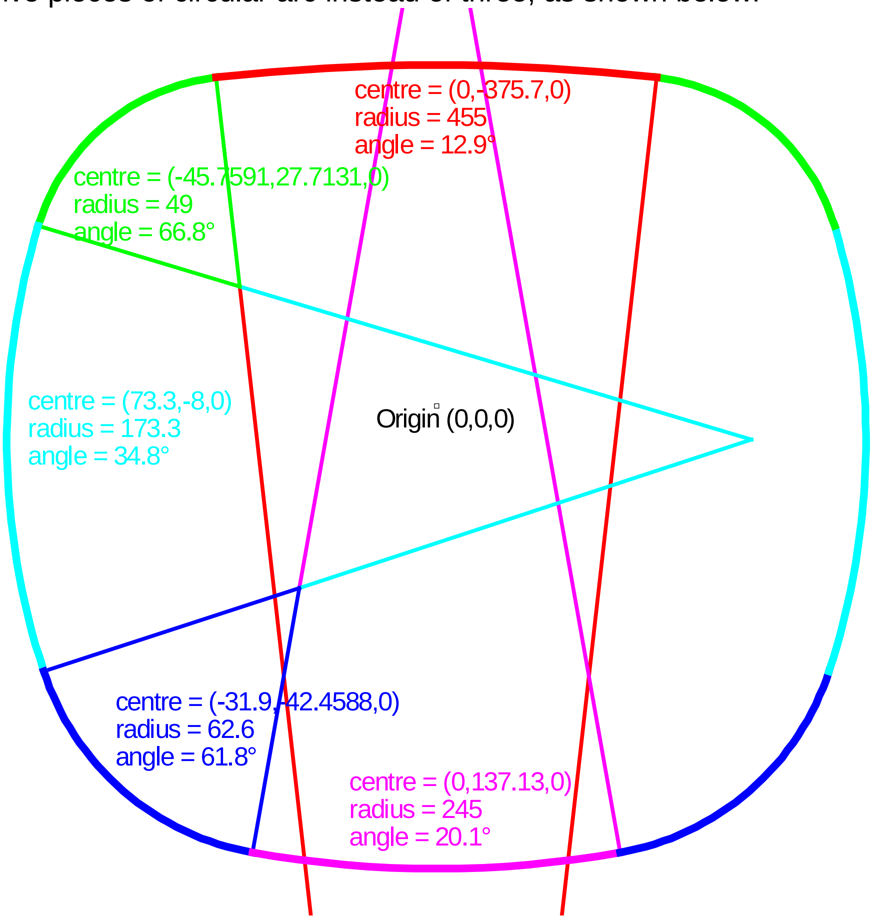
So do you think the Olympic roof is more "circular" than an ellipse, or more squashed?

Why not have a go at designing your own stadium using parts of circles meeting at a tangent?
What will be its circumference?
What will be its floor-area?

Teachers' Notes

By defining large, complicated buildings using mathematical equations, it allows the computer to take over most of the shape generation and drawing work. This means that changes to the design can be incorporated very quickly, because the designer simply changes the input (arc radii or angles) and the computer can instantly regenerate a new shape.

The shape of the new "Aviva Stadium" at Lansdowne Road, Dublin (Ireland's equivalent to Wembley) was designed entirely using equations and the relationships between them. This is a process called Parametric Modelling, because mathematical "parameters" (such as radius or length) are used to define a shape instead of coordinates. The shape where the Aviva stadium meets the ground was generated in just the same way as described above, but using five pieces of circular arc instead of three, as shown below:



References

[1] An interesting comparison of the various approximations to the circumference of an ellipse can be found here: http://dx.doi.org/10.3247/SL1Math05.004

[2] Ramanujan is himself worthy of a discussion http://en.wikipedia.org/wiki/Srinivasa_Ramanujan. His life is remarkable and he also created some highly accessible maths - see in particular his "Taxi Numbers" sum of two cubes (http://www.durangobill.com/Ramanujan.html) and his approximation for Pi (http://planetmath.org/encyclopedia/RamanujansFormulaForPi.html).

[3] A couple of my own research papers describe how Parametric Modelling is used in modern stadium design.
http://people.bath.ac.uk/ps281/research/publications/ijac_preprint1.pdf
http://people.bath.ac.uk/ps281/research/publications/ijac_preprint2.pdf