MODEL EQUATIONS FOR GRAVITY-CAPILLARY WAVES IN DEEP WATER

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Abstract.
The Euler equations for water waves in any depth have been shown to have solitary wave solutions when the effect of surface tension is included. This paper proposes three quadratic model equations for these types of waves in infinite depth with a two-dimensional fluid domain. One model is derived directly from the Euler equations. Two further simpler models are proposed, both having the full gravity-capillary dispersion relation, but preserving exactly either a quadratic energy or a momentum. Solitary wavepacket waves are calculated for each model. Each model supports the elevation and depression waves known to exist in the Euler equations. The stability of these waves is discussed, as is the dynamics resulting from instabilities and solitary wave collisions.

Key words. Water Wave, Solitary Wave, Nonlinear Schrödinger Equation, Gravity-Capillary Wave.

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1. Introduction. Gravity-capillary waves are surface waves in the regime where the restoring effects of both gravity and capillarity are similar in magnitude. For an air-water interface, this implies a free-surface length scale of approximately 1-cm. It is at this length scale, for example, that the initiation of the generation of wind ripples occur [30]. These waves can be detected by some forms of radar (SAR), and are used for remote sensing of the ocean surface, even finding oil spills [28]. In the nonlinear regime, it is important to understand the role that coherent structures play in the dynamics of these waves. The focus of this paper is to propose models for the weakly nonlinear evolution of these waves and study the dynamics of their solitary wave solutions. The dynamics of solitary waves on the free surface of a body of water is a classical problem in fluid dynamics. Shallow water gravity solitary waves have been studied and observed since Korteweg and de Vries (1895) and Russell (1834) [5][13]. More recently a new type of solitary wave has been discovered for waves where both surface tension and gravity play a role. In 1989, Longuet-Higgins provided physical motivation for why such solitary waves should exist [15]. Since then, traveling solitary waves have been found for the Euler equations in both two and three space dimensions [1][16][22] both in shallow and deep water. The waves have also been observed experimentally [17]. In deep water, the most relevant setting, very little is known about the dynamics of such waves. In this paper we aim to study the dynamic properties of these waves (their stability and interaction) in two dimensions. The three-dimensional case will be the subject of a forthcoming paper. The method used is to write a new quadratic model approximating the Euler equations that simplifies numerical computations while capturing the essential dynamic properties. In shallow water, there exist various quadratic models for gravity-capillary waves (the simplest of which is the 5th order Korteweg-de Vries equation), where the governing parameter is the non-dimensional Bond number, \( B = (\gamma gh_0^2) \). (Here, \( \gamma \) is the surface tension coefficient, \( g \) is the force due to gravity, and \( h_0 \) is the mean depth.) Solitary waves have been

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found for these types of models [3][9][18] - and their dynamics have been investigated numerically. However, as the gravity-capillary length scale is approximately 1-cm, the shallow water (or long wave) assumption of these models is extremely restrictive. Our work will not make this assumption - in fact we assume the more realistic case of infinite depth. Three model equations for gravity-capillary waves on infinite depth are proposed. One equation is derived directly as a quadratic approximation to the Euler equations. This equation conserves the leading order mass, but momentum and energy change at cubic order. The second equation is chosen to have the correct linear part, and a nonlinear term which conserves both the leading order mass and momentum. The third equation has the same linear part as the other two, but its nonlinearity conserves the leading order mass and energy. All models support wave-packet type solitary waves - solitary waves that bifurcate from an extremum of the phase speed at finite wave number. In the present setting, a minimum of the phase speed occurs when gravity and surface tension balance. One can easily show that at such a minimum the phase and group speed are equal and therefore wave-packets may behave as solitary waves. The model equations we derive are tools for approximating the dynamics of these wave-packet solitary wave solutions to Euler’s equations (2.1).

The Nonlinear Schrödinger Equation (NLS) is often used to study envelopes of small amplitude wave packets [5] and can also therefore be used to study these solitary waves. The coefficients of a NLS can be used to predict the existence of weakly nonlinear solitary wave-packets. The argument is intuitive: if the NLS is of the focusing type, it will have solitary wave solutions traveling at the group speed. If, in addition, the carrier wave lies at an extremum of the phase speed – where group and phase speed are equal – then, to the NLS order of approximation we have found a wave-packet solitary wave. The NLS equations has severe limitations however. Since the relative phase between the carrier and the envelope is arbitrary in the NLS approximation, it would predict a continuous family of solitary waves with arbitrary phase between the carrier and envelope peak. This does not occur in the Euler equations - where only symmetric wave-packet solitary waves are observed. The simplest of these are ones we denote “elevation waves” where the largest crest is at the center of the wave envelope and “depression waves” where the largest trough is at the center of the envelope. Our models capture these two principal solitary waves, and many other solitary wave-packet waves which have been found in the Euler equations [26]. These additional families of waves are composed of trains of a finite number of these simple waves and we do not discuss further these more complicated families in this paper.

For the simple wave-packet waves we find that elevation waves are unstable, whereas depression waves are stable - except for a finite amplitude instability in one model (as is the case in shallow water models and in Euler [9]). In time-dependent computations of the waves we show that elevation waves eventually evolve into depression waves and that the collision of depression waves can result in the coalescence of the waves or in inelastic collisions where both waves survive albeit as wavepackets. In all cases significant radiation is observed.

The paper is organized as follows. The derivation of the model equation based on the full water wave problem is given in Section 2. The conservation based models are presented in Section 3. Computation of solitary waves and comparison with NLS equations is presented in Section 4. The stability analysis is presented in Section 5 and collision experiments are presented in Section 6.

2. Derivation. A weakly nonlinear one way equation for gravity-capillary waves on infinite depth is derived. This equation is derived as an approximation to the Euler
Gravity-capillary waves

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equations as in [18]. Here \( \eta(x, y, t) \) is the free surface displacement, \( \phi(x, y, z, t) \) is the velocity potential, and \( \hat{n} \) is the unit normal to the free surface.

\[
\Delta \phi + \phi_{zz} = 0, \quad -H_0 < z < \epsilon \eta, \quad (2.1a)
\]
\[
\phi_z = 0, \quad z = -H_0, \quad (2.1b)
\]
\[
\eta_t + \epsilon \nabla \eta \cdot \nabla \phi = \phi_z, \quad z = \epsilon \eta, \quad (2.1c)
\]
\[
\phi_t + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} (\phi_z)^2 + \eta - \frac{1}{\epsilon} \nabla \cdot \hat{n} = 0 \quad z = \epsilon \eta \quad (2.1d)
\]

These equations have been nondimensionalized using a characteristic wave height \( a_0 \), a lengthscale \( L = \gamma^{1/2} g^{-1/2} \), a timescale \( T = \gamma^{1/4} g^{-3/4} \), and a potential scale \( \Phi = a \gamma^{1/4} g^{1/4} \). The parameter \( \epsilon = a_0/L \) is assumed to be small. In cgs units, \( g = 981 \text{cm/sec}^2 \), \( \gamma = 73.50 \text{cm}^3/\text{sec}^2 \) [27].

Expanding the free surface conditions about \( z = 0 \) and combining (2.1b) and (2.1d) yields a single boundary condition to be evaluated at \( z = 0 \).

\[
\phi_{tt} + (1 - \Delta) \phi_z + \epsilon Q(\phi, \phi) + \epsilon^2 C(\phi, \phi, \phi) = O(\epsilon^3)
\]
(2.2)

with

\[
Q(\phi, \phi) = \frac{1}{2} (\nabla \phi^2)_t + \frac{1}{2} (\phi_z^2)_t + \nabla \cdot S (\nabla \phi \Phi^{-1} \phi_t) - (\phi_{zt} \Phi^{-1} \phi_t)_t,
\]

and

\[
C(\phi, \phi, \phi) = S \nabla \cdot (\nabla \phi \Phi^{-1} \left( \frac{1}{2} (\nabla \phi^2) + \frac{1}{2} (\phi_z^2 - \phi_{zt} \Phi^{-1} \phi_t) \right) + \frac{1}{2} \nabla \cdot (\nabla \phi \Phi^{-1} \phi_t)_t^3 + \left( \phi_{zt} \Phi^{-1} \phi_t - \frac{1}{2} (\nabla \phi^2) - \frac{1}{2} (\phi_z^2) \right)_t - \frac{1}{2} S \nabla \cdot (\Phi^{-1} \phi_t)_t^2 \nabla \phi_z + (\phi_{tt} \Phi^{-1} \phi_t)_t - \frac{1}{2} ((\nabla \phi_z^2 \Phi^{-1} \phi_t)_t - \frac{1}{2} (\phi_z^2 \Phi^{-1} \phi_t)_t
\]

The operator \( S = (1 - \Delta) \). The \( z \) derivatives are left explicit, so \( \nabla = (\partial_x, \partial_y) \) and \( \Delta = (\partial_x^2 + \partial_y^2) \). The \( z \) dependence of (2.2) for \( H_0 \rightarrow \infty \) is eliminated by solving Laplace’s Equation in the lower-half plane

\[
\phi(x, y, t, z) = \mathcal{F}^{-1} \left\{ \mathcal{F} \{ \Phi(x, y, t) \} e^{i |k| z} \right\}.
\]

Here \( \mathcal{F} \) is the Fourier transform in \( (x, y) \) with dual variable \( k \). Clearly \( \Phi = \phi(x, y, t, 0) \), and each derivative of \( \phi \) with respect to \( z \) evaluated at \( z = 0 \) can be replaced by \( (-\Delta)^{1/2} \) interpreted in Fourier space as a multiplication by \( |k| \). Carrying out this procedure for (2.2) results in a single equation for \( \Phi \), which, truncated at second order in \( \epsilon \), is

\[
\Phi_{tt} + \mathcal{L} \Phi + \epsilon Q(\Phi, \Phi_t) = 0 \quad (2.3)
\]

with

\[
Q(\Phi, \Phi_t) = \frac{1}{2} (\nabla \Phi_t^2) + \frac{1}{2} (\mathcal{L} \Phi_t^2) + S \nabla \cdot (\nabla \Phi \Phi^{-1} \Phi_t) - (\mathcal{L} \Phi_t \Phi^{-1} \Phi_t)_t
\]

\[
\mathcal{L} = (1 - \Delta)
\]
\[
S = (-\Delta)^{1/2}
\]

Similar equations have been derived in [3], for shallow water and intermediate depth. Henceforth we shall consider these equations only for a two-dimensional fluid domain,
that is, a one dimensional free surface. The linear part of the equations has the
standard gravity-capillary dispersion relation
\[ \omega^2(k) = |k|(1 + k^2). \]

We are interested in wavepacket solitary waves to the gravity-capillary problem and
these types of waves are known to bifurcate from extrema in the dispersion curve.
The gravity-capillary dispersion curve has a minimum in phase speed \( c_p \equiv \omega/k \) at
\( k = 1 \). Furthermore, at an extremum, the phase speed is equal to the group speed
\( c_g = d\omega/dk \), since
\[ 0 = c'_p = \frac{c_g}{k} - \frac{c_p}{k}. \]

Because of this one expects that the carrier and envelope of a wavepacket will travel
at the same speed forming a solitary wave, if the envelope equation alone allows for
solitary waves. In two dimensions and for deep water this envelope equation is the
Nonlinear Schrödinger Equation (NLS) which, depending on its coefficients is denoted
as “focusing” or “defocusing”. The “focusing” NLS (also called the Benjamin-Feir
unstable case) has solitary wave solutions corresponding to approximate wave-packet
solitons in the full equation that we are seeking. Either through such an asymptotic
argument of from a direct numerical calculation, we have found that equation (2.3)
has wave packet type solitary wave solutions. For most time dependent computations,
however, (2.3) is impractical since the \( \Phi_{tt} \) term occurs in nonlinearity and nonlocally
and therefore requires the construction and inversion of a full matrix at each time
step. If one replaces \( \Phi_{tt} \) with \(-(1 - \Delta)\mathcal{L}\Phi \) in the nonlinearity, as is usual in formal
derivations of evolution equations, the resulting equation loses the wavepacket solitary
waves (the wavepacket analysis yields a defocusing NLS). Instead, we shall proceed
further in the derivation and split equation (2.3) into two one-way equations for left
and right moving waves.

Equation (2.3) can be factored into a system of first order equations as follows
\[
\begin{align*}
2u &\equiv \Phi_t + i\Omega\Phi \\
2v &\equiv \Phi_t - i\Omega\Phi \\
\Phi_t &\equiv (u + v) \\
\Phi &\equiv (i\Omega)^{-1}(u - v)
\end{align*}
\]
\[
\begin{align*}
u_t - i\Omega u + \frac{\epsilon}{2}Q(\Phi, \Phi_t) &= 0 \\
v_t + i\Omega v + \frac{\epsilon}{2}Q(\Phi, \Phi_t) &= 0
\end{align*}
\]

The operator \( \Omega \) has Fourier symbol
\[ \hat{\Omega} = \text{sign}(k)\sqrt{|k|(1 + k^2)}. \]

We take \( \Phi, u, v \) to have mean zero. Equation (2.3) is now recast as a system of two
equations
\[
\begin{align*}
u_t - i\Omega u + \frac{\epsilon}{2}Q((u + v), (i\Omega)^{-1}(u - v)) &= 0 \\
v_t + i\Omega v + \frac{\epsilon}{2}Q((u + v), (i\Omega)^{-1}(u - v)) &= 0
\end{align*}
\]
From \( u, v \), one can recover the variables of most interest, \( \eta \) and \( v \), the vertical velocity at \( z = 0 \), with
\[
\eta = -S^{-1}(u + v) + O(\epsilon) \quad \text{and} \quad v = \frac{L}{i\Omega}(u - v).
\]

If \( \Phi \) has small projection on waves in one direction (suppose that \( v(x, t) = O(\epsilon) \)), then these equations decouple for long times. Suppose that the initial data for \( v \) is small. The principal mechanism for energy transfer for weakly nonlinear waves occurs through triads and quartets of monochromatic waves. It is simple to show that, for gravity-capillary waves, all triads involve same-direction waves and that quartets coupling right- and left-traveling waves have at most one wave moving in the opposite direction as the others. Furthermore, when this situation occurs, the wave moving in the opposite direction is the one with smallest \( |k| \).

Resonant quartets satisfy the system of equations
\[
k_1 + k_2 = k_3 + k_4 \tag{2.4}
\]
\[
\omega_1 + \omega_2 = \omega_3 + \omega_4, \tag{2.5}
\]
where \( \omega_j = \omega(k_j) \). Consider, without loss of generality, a quartet with of three left-traveling waves, \( k_1, k_2, k_3 > 0 \) and one right-traveling wave, \( k_4 < 0 \). By definition, all of the frequencies are negative, \( \omega_1, \omega_2, \omega_3, \omega_4 < 0 \). Now consider solutions that travel leftward and are primarily supported about \( k = 1 \) in Fourier space (such as the solitary waves of interest here). There are two possible quartet configurations for energy transfer to the right: Case I where \( k_1 = 1 + \delta_1, k_2 = 1 + \delta_2 \), and Case II where \( k_1 = 1 + \delta_1, k_3 = 1 + \delta_3 \) where \( \delta_j \) are small. Since \( k_4 < 0 \) we have that, in Case I, \( k_3 \geq 2 + O(\delta) \). Thus equation (2.5) gives
\[
\sqrt{|k_4|(1 + k_4)} \leq \sqrt{8} - \sqrt{10} + O(\delta),
\]
which cannot be satisfied for \( k_4 \) and there is no resonant quartet with an opposite traveling wave. In Case II, \( 0 < k_3 = O(\delta) \) and there are possible quartets. However, from the quartet interaction equations (see [6]) this case is linearly stable with respect to perturbations of small amplitude waves with wavenumber \( k_2, k_4 \). Thus when \( u \) is in the gravity-capillary regime and \( v \) is initially small, \( v \) will remain small for long times.

Setting \( v = 0 \) we now write an approximate single one-way equation for \( u(x, t) \), effectively diagonalizing and decoupling to this order the free-surface problem in infinite depth.
\[
u_t - i\Omega u + \epsilon \left( u_{xx} - \frac{1}{i\Omega} u_x + \frac{L}{i\Omega} u - \frac{3}{2} \left( S^{-1} u_x |\Omega| S^{-1} u_x \right)_x \right) = 0 \tag{2.6}
\]
\[
\hat{S} = (1 + k^2), \quad \hat{L} = |k|, \quad \hat{\Omega} = \text{sign}(k) \sqrt{|k|(1 + k^2)}, \quad |\hat{\Omega}| = \sqrt{|k|(1 + k^2)}
\]
Cubic terms have been neglected. The nonlinear term in parentheses will be referred sometimes as \( \frac{1}{2}Q_0(u) \). Equation (2.6) is a one-way quadratic model, derived directly from Euler’s equations (2.1). It will henceforth be referred to as the U-equation and governs left-traveling waves. The free surface, \( \eta \), can be recovered from \( u \) using the identity \( \eta = -S^{-1}(u) + O(\epsilon) \).
3. Conservation Properties. The Euler equations (2.1) conserve mass, momentum, and energy. A successful model equation should also conserve these quantities.

\[
\text{Mass} = \int \epsilon \eta \, dx \quad \text{Momentum} = \int \int_{-\infty}^{\infty} \epsilon (\phi_x, \phi_z) \, dz \, dx
\]

\[
\text{Energy} = \int \int_{0}^{\infty} z \, dz \, dx + \int \int_{-\infty}^{\infty} \frac{\epsilon_2}{2} (\phi_x^2 + \phi_z^2) \, dz \, dx + \int (\sqrt{1 + \epsilon_2^2 \eta^2} - 1) \, dx
\]

The integrals in \( x \) are either over a period of the periodic solution or over the whole real line if solutions decay to zero at infinity. The U-equation (2.6) is a quadratic approximation to (2.1). It conserves mass, momentum, and energy to quadratic order.

To see this, mass, momentum, and energy can be truncated to quadratic order, and then rewritten in terms of \( u \). In this section, we often pass a nonlocal Fourier symbol from one function to another under an integral sign (the analogue of integration by parts) and use, when one has two operators \( \mathcal{A}, \mathcal{B} \) with Fourier symbols \( \hat{\mathcal{A}}(k) \) and \( \hat{\mathcal{B}}(k) \) respectively, the Parseval's identity

\[
\int \mathcal{A}u(\mathcal{B}u)^* \, dx = \int \hat{\mathcal{A}}(k)\hat{\mathcal{B}}^*(k)|\hat{u}(k)|^2 \, dk. \tag{3.1}
\]

For real functions \( u \), when the product of Fourier symbols is odd, then this integral is zero.

Mass is exactly conserved in the Euler equations. If we truncate Euler’s equations, throwing away cubic terms, the mass changes only at cubic order. To see this, first replace \( \eta \) with \( \phi \) using the expanded equation (2.1d).

\[
\text{Mass} = \int -S^{-1} \left( \epsilon \Phi_t + \frac{\epsilon_2}{2} (\Phi_x^2 + (\mathcal{L}\Phi)^2) - \epsilon_2 \mathcal{L}\Phi_t S^{-1}\Phi_t \right) \, dx + \text{h.o.t.}
\]

Taking a time derivative yields

\[
\frac{d\text{Mass}}{dt} = \int -S^{-1} \left( \epsilon \Phi_{tt} + \frac{\epsilon_2}{2} (\Phi_x^2 + (\mathcal{L}\Phi)^2)_{tt} - \epsilon_2 (\mathcal{L}\Phi_t S^{-1}\Phi_t)_{tt} \right) \, dx + \text{h.o.t.}
\]

At this order, \( \Phi \) solves the truncated Euler equations (2.3), thus replacing \( \Phi_{tt} \) yields

\[
\frac{d\text{Mass}}{dt} = \int \epsilon \mathcal{L}\Phi + \epsilon_2 \partial_x (\partial_x \Phi S^{-1}\Phi_t) \, dx + O(\epsilon^3)
\]

The linear and quadratic part vanish and thus the truncation of the Euler equations conserves mass to cubic order. A similar argument applies to the mass conservation for the U-equation where the mass is proportional to the mean of \( u \). Note that the U-equation (2.6) both conserves the mean of \( u \) and is invariant to changes in mean and therefore, in terms of mass conservation, it is consistent with the Euler equations.

To find a truncated expression for momentum, recall that \( \phi(x, z, t) = \mathcal{F}^{-1}\{ \hat{\Phi}(k, t) e^{i[k|z]} \} \).

Then, the \( z \)-dependence of the momentum integral can be evaluated

\[
\text{Momentum} = \int \epsilon \left( \mathcal{F}^{-1}\left\{ \frac{k}{|k|} \hat{\Phi}(k, t) e^{i[k|\eta(x, t)]} \right\}, \mathcal{F}^{-1}\{ \hat{\Phi}(k, t) e^{i[k|\eta(x, t)]} \} \right) \, dx.
\]
Recal that the potential $\Phi(x,t)$ is taken to have $\Phi(0,t) = 0$ (mean zero) which is consistent since equation (2.3) is invariant to changes in the mean of $\Phi$. Truncating the exponentials at second order gives

$$\text{Momentum} = \int \epsilon \left( F^{-1}\left\{ i \frac{k}{|k|} \hat{\Phi}(k,t)(1 + \epsilon |k| \eta(x,t)) \right\}, F^{-1}\{ \hat{\Phi}(k,t)(1 + \epsilon |k| \eta(x,t)) \} \right) \ dx.$$ 

The linear terms vanish yielding

$$\text{Momentum} = \int \epsilon^2 (\eta \Phi_x, \eta \mathcal{L} \Phi) \ dx + \text{h.o.t.}$$

For the one dimensional waves in the U-equation (2.6), we have the relation $(1+k^2)\hat{\eta} = -i\Omega(k)\hat{\Phi} + O(\epsilon)$. The momentum can be rewritten in terms of $\eta$

$$\text{Momentum}(\eta) = \int \epsilon^2 \left( -\eta |\Omega(\eta, i\Omega(\eta)) \right) \ dx + \text{h.o.t.}$$

The vertical component of this integral vanishes. Henceforth, the horizontal component will be referred to as the momentum. The truncated momentum can be written in terms of the variable $u$ using $u = \Phi_t + O(\epsilon) = -S\eta + O(\epsilon)$.

$$\text{Momentum}(\eta) = -\epsilon^2 \int \eta |\Omega(\eta) \ dx \quad \text{Momentum}(u) = -\epsilon^2 \int S^{-1} u |\Omega| S^{-1} u \ dx$$

The quadratic momentum changes at cubic order in $u$:

$$\frac{\partial \text{Momentum}(u)}{\partial t} = \epsilon^3 \int S^{-1} u |\Omega| S^{-1} Q_0(u) \ dx$$

It does not appear that a higher order truncation of momentum improves on conservation properties, although, in numerical calculations, the changes in this quadratic momentum appear to be of much higher order than this formal calculation implies.

To find the quadratic approximation to the energy, first use $(\phi_x^2 + \phi_z^2) = (\partial_x, \partial_z) \cdot (\phi \phi_x, \phi \phi_z) - \phi (\phi_{xx} + \phi_{zz})$. Since the $\phi$ is harmonic, $(\phi_x^2 + \phi_z^2) = (\partial_x, \partial_z) \cdot (\phi \phi_x, \phi \phi_z)$. Then,

$$\text{Energy} = \int \int_{-\infty}^{\epsilon \eta} \int_{0}^{\epsilon \eta} z \ dz \ dx + \frac{1}{2} \int \int_{-\infty}^{\epsilon \eta} \epsilon^2 (\partial_x, \partial_z) \cdot (\phi \phi_x, \phi \phi_z) \ dz \ dx + \int \left( \sqrt{1 + \epsilon^2 \eta^2_x} - 1 \right) \ dx$$

The second integral is rewritten using the divergence theorem.

$$\text{Energy} = \int \int_{0}^{\epsilon \eta} z \ dz \ dx + \frac{1}{2} \int \epsilon^2 \phi \frac{\partial \phi}{\partial n} |_{z=\epsilon \eta} \ dx + \int \left( \sqrt{1 + \epsilon^2 \eta^2_x} - 1 \right) \ dx$$

The second integral is an integral in $x$, evaluated at $z = \epsilon \eta$, where $n$ is the unit normal vector to the free surface $z = \epsilon \eta$. Taylor expanding about $z = 0$ yields

$$\text{Energy} = \frac{1}{2} \int \epsilon^2 (\phi \phi_x + \eta_x^2 + \eta^2) \ dx + \text{h.o.t.}$$

This energy agrees with the calculation by Lighthill [14]. The truncated energy can be rewritten in terms of $u$ or $\eta$

$$\text{Energy}(u) = \epsilon^2 \int u S^{-1} u \ dx, \quad \text{Energy}(\eta) = \epsilon^2 \int \eta S \eta \ dx = \epsilon^2 \int \eta^2 + \nu^2 \ dx = ||\eta||_H^2.$$
Like the momentum, time variations in energy are at cubic order:

$$\frac{\partial \text{Energy}(u)}{\partial t} = -\epsilon^3 \int Q_0(u)S^{-1}u \, dx$$

In the previous section the nonlinear part of the U-equation (2.6) is derived directly from the Euler equations (2.1). In the U-equation, mass, momentum and energy change at cubic order in $u$. Of course, the mass, momentum and energy are constant for traveling waves. One can also write a quadratic model equation with the infinite depth gravity-capillary dispersion relation and ask which nonlinearity will conserve our truncated mass, momentum and energy exactly. A model which exactly conserves the truncated mass and momentum is

$$\eta_t - i\Omega \eta - \frac{1}{2}|\Omega|^{-1}(\eta^2)_x = 0 \quad (3.2)$$

Henceforth equation (3.2) will be referred to as the P-equation, as it conserves momentum. This equation conserves energy to cubic order as the U-equation. A model equation which exactly conserves the truncated mass and energy is

$$\eta_t - i\Omega \eta - \frac{1}{\sqrt{2}}S^{-1}(\eta^2)_x = 0 \quad (3.3)$$

Equation (3.3) will be called the E-equation. This equation conserves momentum to cubic order as the U-equation.

The P and E-equations are not the only equations that conserve momentum or energy - one can construct whole families of such equations - however, we have not found an equation conserving both. For example, a family of equations conserving momentum is

$$\eta_t - i\Omega \eta - C|\Omega|^{\alpha - 1}(|\Omega|^{\alpha} \eta)_x^2 = 0, \quad (3.4)$$

where $C$ and $\alpha$ are arbitrary constants. A similar family can be written which conserves energy. The E and P-equations we show are arbitrary choices from these families, with $\alpha$ set to zero and $C$ chosen so that the coefficients in their respective Nonlinear Schrödinger Equation (4.2) analysis are approximately equal for all three model equations.

4. Solitary Waves. One of the aims of this research is to study unsteady gravity-capillary motion and particularly the dynamics of wave packet type solitary waves like those known to exist in the Euler equations. The U-equation (2.6), P-equation (3.2), and E-equation (3.3) are quadratic equations which capture features of the Euler equations (2.1). Here, we compute and compare solitary waves for the three equations, and for the full Euler problem.

The simplest approximation for studying the small amplitude nonlinear wave packets is to write a Nonlinear Schrödinger Equation (NLS) that governs the wave’s envelope. The NLS equation is found via the ansatz, in, for example, the U-equation

$$u = A(\epsilon(x - ct), \epsilon^2 t)e^{ik_0x - i\omega t} + c.c. + \epsilon A_2(x, t) + \epsilon^2 A_3(x, t) + ... \quad (4.1)$$

At $O(\epsilon^2)$ the solvability condition for the leading order envelope amplitude $A$ is the Nonlinear Schrödinger Equation.

$$iA_x + \lambda A_{xx} = \chi |A|^2A \quad (4.2)$$
In equation (4.2), $\tau = \epsilon^2 t$, and $X = \epsilon(x - c_g t)$. The coefficients of dispersion $\lambda = \frac{1}{2}k'(k_0)$ and nonlinearity $\chi$, depend on the equation we are approximating and the carrier wave frequency $k_0$. We shall choose $k_0 = 1$, corresponding to the minimum of the phase speed. Here, $c_g = c_p = \sqrt{2}$. Then, for the three equations $\lambda = \frac{1}{2\sqrt{2}}$ and

$$\chi_U = -\frac{3(17 - 7\sqrt{5})}{20(\sqrt{10} - \sqrt{8})}, \quad \chi_P = -\frac{1}{2\sqrt{5}(\sqrt{10} - \sqrt{8})}, \quad \chi_E = -\frac{1}{5(\sqrt{10} - \sqrt{8})}$$

The product $\lambda \chi < 0$ in all cases, corresponding to a Benjamin-Feir unstable, or focusing, NLS equation, which admits solitary wave type solutions [6].

$$A(X, \tau) = a[2\lambda/\chi]^{1/2}\text{sech}(a(X - 2b\lambda \tau))e^{ibX - i\lambda(b^2 - a^2)\tau}, \quad (4.3)$$

where $a$ and $b$ are both free real parameters. The solitary wave solutions to NLS (4.3) are approximate nonlinear wave packets of the model equations. At first order, the envelope, $A$, moves at the group velocity, and the carrier wave moves at the phase speed. At a carrier wave frequency where the phase speed is equal to the group speed the NLS solitary wave solution corresponds to an approximate wave packet solitary wave to times $t \sim O(1/\epsilon)$. For the proper choice of parameters, $b = -\frac{1}{2}\epsilon a^2$, the NLS solitary wave is a wavepacket solitary wave to times $t \sim O(1/\epsilon^3)$. This wave has speed $c = c_g - \lambda(\epsilon a)^2$. This corrected speed depends on amplitude and we shall compare it to the speed of computed solitary waves in our three model equations.

To compute wavepacket solitary waves for our model equations directly, traveling solutions are expanded in Fourier components. For example, for the U-equation

$$u(x, t) = \sum_{n=-M/2}^{M/2} a_n e^{ik_n(x-ct)},$$

We take $a_0 = 0$, as the mean level has no meaning for a fluid of infinite depth. We project the equation onto $M$ Fourier modes, yielding a nonlinear system of $M$ algebraic equations, for $M + 1$ unknowns: the remaining $M$ Fourier amplitudes and the speed $c$. We add to this an equation fixing a measure of the solution amplitude (either a
norm or the height at the central crest). Newton’s method is used to find solutions to this system, with the NLS solitary wavepacket as an initial guess. Solutions for different amplitudes are computed by continuation. Two branches of solitary waves were computed and are referred to as depression and elevation waves, based on the height at the waves’ center (see Figure 4.1). (We have found that our equations, as the Euler equations [4], have many families of wavepacket solitary waves. However, we compute and study the two simplest ones.) Solutions to the full Euler equations are also computed for comparison. These are calculated using a boundary integral method for Laplace’s equation on a discretized free surface. The resulting nonlinear system is also solved using Newton’s method. The method is similar to that used in [26]. Figure 4.2 compares the speed-amplitude relation of depression wavepacket solitary waves from the Euler equation, the U-equation, the P-equation, and the E-equation to that of an NLS. The amplitude is measured either as the maximum free surface amplitude or as the energy. As expected these waves travel at speeds lower than all linear wave speeds in the problem and, generally, their speed decreases with amplitude. For the E-equation, there is a range of energies for which there are multiple possible waves with the same energy. We shall return to this in Section 5.

We note that we have not found any asymmetric solitary waves for any of our models. In fact this is one of the differences between the NLS approximation to wavepacket solitary waves and the solitary waves of the underlying equations: in NLS the relative phase between the crest of the envelope and the crest of the carrier is arbitrary whereas in exact calculations it is always 0 or π always yielding symmetric solutions. For other types of equations this symmetry has been studied using exponential asymptotics [29].

5. Stability of Solitary Waves. For Euler’s equations (and the 5th order KDV equation), elevation wavepacket solitary waves are linearly unstable [9],[10]. In this section, we show that this is also the case for the model equations in this study. Each model equation was linearized about elevation waves, and the spectrum of the linearized equation was computed. The eigenvalues were computed in a frame traveling with the wave by substituting $\tilde{u} = u_0 + \delta v(x - ct)e^{\lambda t}$, with $u_0$ an exact solution, into

![Image of Figure 4.2](image-url)
Gravity-capillary waves

one of the model equations and linearizing:

\[ \lambda v - cv_x - i\Omega v - N'(u)v = 0. \]  \hspace{1cm} (5.1)

The Fourier transform of (5.1), taking the form of an eigenvalue problem \( A\hat{v} = \lambda\hat{v} \), was approximated numerically with an appropriate truncation of \( A \). The resolved single real positive eigenvalue of this problem for each model equation is plotted in the left panel of Figure 5.1 as a function of the solitary wave amplitude. The criterion used for keeping eigenfunction-eigenvalue pairs as non-spurious was that the eigenfunction was resolved on the grid. Depression waves for the P and U-equation are considered stable since the same numerical procedure yields eigenvalues with real parts of the order of machine accuracy.

In addition to computing the spectrum of the linearized equations, the nonlinear evolution of this instability was computed. To observe the dynamics, both unstable elevation and stable depression solitary waves are used as initial data for the model equations. Time evolutions were computed using a Fourier pseudo-spectral method for the spatial dependence and fourth order Runge-Kutta for the time stepping.

The unstable elevation waves eventually break up, generically resulting in a depression solitary wave and small amplitude radiation. An example of the evolution of an elevation wave into a depression wave for the U-equation is shown in Figure 5.2 where numerical error was sufficient to incite the instability. This evolution of an elevation wave into a depression wave has been observed in the (shallow water) 5th order KDV equation [9]. All of our model equations have this feature. For the U-equation and P-equation all depression waves evolve indefinitely with no appreciable change.

In the E-equation, there is a band in the depression branch where waves are linearly unstable. The single real positive eigenvalue of the E-equation linearized about depression waves is plotted in the right panel of Figure 5.1. When evolved numerically this type of wave dissolves, resulting in a time periodic travelling wave, oscillating about a stable depression wave. The onset of this instability occurs where the waves’ energy as a function of speed has an extremum (see Figure 4.2 right panel, top curve). The unstable band corresponds to where the energy of the waves is increasing with speed. This phenomenon of exchange of stability at an extrema of the energy-speed
Fig. 5.2. Elevation waves are unstable, and evolve into depression waves. The elevation wave on the left, from the U-equation (2.6) evolves into the depression wave on the right. The center picture is this evolution in the x-t plane in a reference frame moving to the left at the speed of the original elevation wave. The resulting depression wave is slower and of larger amplitude than the elevation wave.

Fig. 5.3. Left: The energy of waves in the E-equation (3.3) plotted as a function of the depth, $A$, of the central trough. When the slope is positive, these waves are unstable. Right: A phase portrait of the evolution of an unstable depression wave - denoted by a star - from the E-equation, which oscillates around a stable depression wave. Since the E-equation conserves energy, three solitary waves are possible in this evolution - denoted as a diamond, star and circle in both figures. The trajectory of the unstable wave is marked by the circles.

Curve is generic in hamiltonian systems (see, for example, [24]). In this unstable range of energies there are three waves of different speed (or, equivalently, amplitude) for a fixed value of the energy - two unstable and one stable. An approximate phase portrait capturing the time evolution of an unstable wave is shown in Figure 5.3.

6. Collisions of solitary waves. The second dynamic problem addressed in this paper is the collision of two solitary waves. In head on collisions, nonlinear interactions take place over a short time scale. A head on collision of two solitary waves in the quadratic truncation of the Euler equations (2.3) is in Figure 6.1. According to our computations for equation (2.3), these collisions are essentially the linear superposition of the two waves, in that the waves pass through each other, unchanged by the interaction. For significant nonlinear interaction, we look at overtaking collisions of solitary waves. Overtaking collisions were simulated numerically in all model equations. No theory exists to predict the collision behavior of wave packet type solitary waves. Vanden-Broeck and Malomed have studied numerically collisions of solitary waves to the fifth-order KDV equation [21]. As in [21], we do not attempt to classify
all types of solitary wave interactions. Instead, we present the two principal types of overtaking collisions we observe for small amplitude solitary waves.

There are two qualitatively different overtaking interactions observed in the model equations depending on amplitude. We denote these “weak” and “strong” interactions. Weak interactions occur when both solitary waves are of small amplitude. In this type of interaction, the two waves approach each other, interact through their slowly decaying oscillatory tails without ever colliding, and the resulting evolution is that of one solitary wave and one wave packet (although it is often difficult to discern whether the solitary wave is not a wavepacket with very close phase and group speeds). The small perturbations from the interaction seem to break the solitary character of the smaller wave but not its localization. Figure 6.2 shows the evolutions
Fig. 6.3. Snapshots of the overtaking collision of two waves in the $U$-equation (2.6). This is the same computation as in Figure 6.2. The waves are shown in a frame moving with the original speed of the wave on the left.

of two small amplitude solitary waves and their interaction. This interaction is typical of the smaller amplitude waves in the $U$-equation and the $P$-equation. We did not observe this type of interaction in the $E$-equation, where a band of depression waves of finite but small amplitude are unstable (see Section 5).

Strong interactions occur when at least one of the waves is of larger amplitude. The two waves collide, and after the collision only one solitary wave remains, along with some radiation. The strength of the interaction is due to a combination of two effects: larger amplitude waves are more localized (and therefore the interaction through the tails is much weaker) and the difference in wave speeds can be much larger allowing for the centers of the waves to approach each other before significant interaction changes their dynamics. In this collision, the wave that remains is larger than either of the two waves that collided to form it and slows down further. This type of collision was observed in all model equations. Figure 6.2 shows the one such collision viewed in the x-t plane for the $U$-equation. Snapshots of this collision are in Figure 6.3.

For the collision experiments, a simple absorbing boundary condition is used.
This boundary condition is presented in [11]. The idea is to add absorbing zones to the periphery of the computational domain, where the waves are relaxed to an analytical solution of the equation, in our case zero. At each time step the solution $u(x, t)$ is relaxed to zero at both boundaries by multiplying by a coefficient $c(x)$ which is one in the interior of the domain and decays at the boundaries:

$$U(x, t) \rightarrow U(x, t)c_R(x - x_R)c_L(x - x_L),$$

with

$$c_R(x) = \frac{1}{2} + \frac{1}{2}\tanh \left(2\pi \frac{x - \frac{1}{2}}{L/2}\right), \quad c_L(x) = \frac{1}{2} - \frac{1}{2}\tanh \left(2\pi \frac{x - \frac{1}{2}}{L/2}\right).$$

The parameters $x_R$ and $x_L$ are the right and left boundaries of the computational domain respectively; $L$ parameterizes the width of the absorbing zone, typically $4\pi$. This boundary condition was not used when evolving unstable waves, as they are very sensitive to perturbation. For collisions, the absorbing zone is used to prevent high frequency waves from wrapping many times around the domain. The type and character of collisions is not sensitive to whether or not this absorbing boundary condition is used.

7. Conclusion. Three weakly nonlinear model equations have been presented for deep water gravity-capillary surface waves. One model is derived as a quadratic truncation of the Euler equations for waves moving in one direction. The quadratic truncations of the physical quantities mass, momentum and energy, governed by the $U$-equation (2.6), change at cubic order. A second model, the $P$-equation (3.2), is proposed with same linear part as the $U$-equation (2.6), and a nonlinearity which conserves exactly the truncated mass and momentum (with the physical truncated energy changing at cubic order). The third model, the $E$-equation (3.3), again has the same linear part, but a nonlinearity which conserves exactly the truncated mass and energy (with the physical truncated momentum changing at cubic order). All three models support wave packet solitary wave solutions, of the same type that exist in Euler’s equations (2.1). Two branches of solitary waves - elevation and depression waves - are computed in each model equation. The $U$ and $P$-equations capture key features of the Euler equations (2.1), and all are simple to evolve numerically. From the perspective of modeling realistic surface tension flows, the $E$-equation should probably be discarded since it has a secondary instability of depression waves that, to our knowledge, does not appear in the Euler equations. The three model equations are used to study the dynamics of solitary waves.

Two types of dynamic experiments were presented. The first was the evolution of an linearly unstable elevation wave into a depression wave together with some radiation. The second dynamic experiment was the overtaking collision of two solitary waves. Two qualitatively different collisions were observed which we denote “weak” or “strong”. In weak collisions two wavepackets remain whereas in strong collisions only one solitary wave emerges.

Even though the Nonlinear Schrödinger Equation can be used to predict the existence of wave packet type solitary waves, it cannot be used to answer questions about the stability of these waves, or their unsteady evolution. For example, the NLS cannot discern the difference between elevation and depression waves. In the primitive equations, elevation and depression waves have very different dynamics, as observed in this study. Thus, our model equations serve as an intermediate step, simpler than
Euler, to study this regime. Our work focused on a one dimensional free surface. The two dimensional free surface problem is the subject of a forthcoming paper.

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**REFERENCES**


