Derivation of Gaussian Distribution from Binomial

The number of paths that take k steps to the right amongst n total steps is:

$$\frac{n!}{k!(n-k)!},$$

and since each path has probability $1/2^n$, the total probability of paths with k right steps are:

$$p = \frac{n!}{k!(n-k)!}2^{-n}.$$

Now, consider the probability for $m/2$ more steps to the right than to the left, resulting in a position $x = m\Delta x$. Thus setting $k = \frac{n}{2} + \frac{m}{2}$.

$$p(m,n) = \frac{n!}{\left(\frac{n}{2}+\frac{m}{2}\right)!\left(\frac{n}{2}-\frac{m}{2}\right)!}2^{-n}.$$

Using Stirling's formula

$$z! \sim \sqrt{2\pi} z^z e^{-z}\sqrt{z}, \qquad z \to \infty,$$

we have,

$$\begin{aligned} p &\sim \frac{1}{\sqrt{2\pi}} \frac{n^n\sqrt{n}}{\left(\frac{n}{2}+\frac{m}{2}\right)^{n/2}\left(\frac{n}{2}-\frac{m}{2}\right)^{n/2}} \frac{2^{-n}}{\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{1/2}} \left(\frac{\frac{n}{2}-\frac{m}{2}}{\frac{n}{2}+\frac{m}{2}}\right)^{m/2} \\ &\sim \frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\left(1-\frac{m^2}{n^2}\right)^{-1/2}\left(1-\frac{m^2}{n^2}\right)^{-n/2}\left(1+2\frac{m}{n}\right)^{-m/2}. \end{aligned}$$

Taking the logarithm, and keeping the leading behavior of each term

$$\begin{aligned} \ln p &\sim \ln\frac{2}{\sqrt{2\pi}} - \frac{1}{2}\ln n - \frac{1}{2}\ln\left(1-\frac{m^2}{n^2}\right) - \frac{n}{2}\ln\left(1-\frac{m^2}{n^2}\right) - \frac{m}{2}\ln\left(1+2\frac{m}{n}\right) \\ &\sim \ln\frac{2}{\sqrt{2\pi}} - \frac{1}{2}\ln n + \frac{m^2}{2n^2} + \frac{m^2}{2n} - \frac{m^2}{n}. \end{aligned}$$

Now, the third term is smaller than the last two, so we can drop it and then exponentiate to get

$$p \sim \frac{1}{\sqrt{2\pi}}\frac{2}{\sqrt{n}}e^{-\frac{m^2}{2n}}.$$

Now, substituting $n = t/\Delta t$, $m = x/\Delta x$, and using the definition $D = \frac{1}{2}\frac{\Delta x^2}{\Delta t}$, and considering that the probability above is for values of x between x and $x+2\Delta x$ (since fixing n and changing k by one changes x by $2\Delta x$). Thus $p(x,t)2\Delta x = p(m,n)$. Combining these facts we get that

$$p(x,t) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2Dt}}\exp\left(-\frac{x^2}{4Dt}\right).$$