Model reduction for controller design for infinite-dimensional systems

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Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
maandag 11 september 2006
om 16.15 uur

door

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geboren op 17 september 1979
te Amsterdam

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ISBN: 90-367-2699-9

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Chapter 1

Introduction

In this thesis we study systems that can formally be described by the equations

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ x(0) & = & w \\ y(t) & = & Cx(t) + Du(t), \end{array} \qquad (1.1)$$

or by the equations

$$\begin{array}{rcl} x_{n+1} & = & Ax_n + Bu_n \\ x_0 & = & w \\ y_n & = & Cx_n + Du_n. \end{array} \qquad (1.2)$$

Here A, B, C and D are linear operators on Hilbert spaces.

The main problem in the field of linear systems and control theory is, for a given system (1.1), designing a system

$$\begin{array}{rcl} \dot{x}_c(t) & = & A_c x_c(t) + B_c u_c(t) \\ x_c(0) & = & w_c \\ y_c(t) & = & C_c x_c(t) + D_c u_c(t), \end{array} \qquad (1.3)$$

such that if we put $u_c = y$ and $u = y_c$, then the resulting closed-loop system has some prespecified properties. The system (1.1) is usually referred to as the plant and the system (1.3) as the controller. This type of control is called feedback control since the output y of the system (1.1) is, after being processed by the system (1.3), fed back into the system (1.1) (see figure 1.1).

Which properties the closed-loop system is required to have depends very much on the particular application. We will focus on two properties that are almost always required: stabilization and low complexity of the controller. To

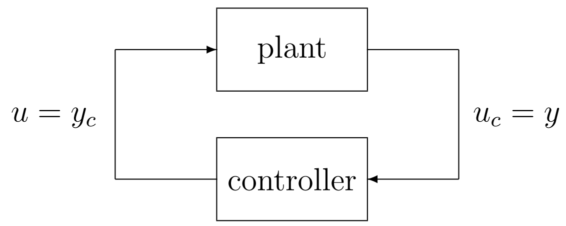

Figure 1.1: Feedback interconnection of plant and controller.

start with the second property, the measure of 'complexity' of the controller (1.3) we will use is the dimension of the Hilbert space in which the state x_c of the controller takes its values. In engineering applications it is often paramount that this dimension is small. Most standard controller design methods lead to a controller with the same state space as the plant. In the examples we are interested in the plant has an infinite-dimensional state space, so applying these standard controller design procedures to the plant is not an attractive option. There are two possible solutions to this problem. The first is to develop completely new controller design methods that do not have this disadvantage of producing in a controller with the same state space as the plant. The second is to use standard controller design procedures, but not for the plant, but for a low-dimensional approximation of it. One then has to prove that the controller designed based on the approximation, when interconnected with the plant, results in a closed-loop system that has the prespecified properties (other than having low complexity of the controller). We will focus on this second alternative. The other prespecified property we will look at is, as mentioned earlier, stabilization. There are several types of stability that one can demand the closed-loop system to have. We concentrate on obtaining so-called input-output stability of the closed-loop system, since under appropriate stabilizability conditions this is equivalent to the other (a priori stronger) types of stability.

Infinite-dimensional continuous-time systems

Examples we are interested in are systems described by partial differential equations. Continuous-time systems like (1.1) are an abstract representation of such systems described by partial differential equations. The operators A, B, C and D that result from writing the partial differential equation examples in this form are usually unbounded, which introduces severe technical difficulties. These difficulties already start with a correct notion of 'solution' of the equations (1.1). For an arbitrary quadruple of unbounded operators

A, B, C, D it is impossible to do even this, let alone developing a theory for control design. During the last decades much effort has gone into the question under which conditions on the quadruple of operators one can develop a theory of control design for stabilization. The present state-of-the-art class of systems for which there is a reasonably complete theory (well-posed linear systems, see Staffans [89]) does not include all systems described by partial differential equations that one would like to study. In part II of this thesis we present a new class of systems, larger than the class of well-posed linear systems, which also includes many interesting partial differential equations that are not well-posed. We were able to complete the program outlined above at this level of generality, showing that a reasonably complete control theory for this class of systems is certainly feasible. We note that the completion of the program outlined above is new even for well-posed linear systems (in fact, even for systems where all operators are bounded it is new). We were able to complete the program outlined above for our general class of systems by making the connection with discrete-time systems using a Cayley transform approach.

Infinite-dimensional discrete-time systems

In contrast to the infinite-dimensional continuous-time case, the infinite-dimensional discrete-time case does not provide problems due to unbounded operators. For all purposes one may assume that A, B, C and D are bounded. As mentioned before, our approach to carrying out the program outlined above is to first obtain the results in discrete-time and then to translate these results, using the Cayley transform, to continuous-time. It turns out that to be able to translate the results using the Cayley transform the results have to be 'optimal', i.e. one has to prove the desired theorems under the weakest possible assumptions. This meant that we had to rewrite large parts of infinite-dimensional discrete-time systems theory, since the existing results were proven under conditions that were too strong and therefore do not translate well under the Cayley transform. In particular, we had to prove theorems under weaker stability/stabilizability conditions than the standard power stability/stabilizability.

Outline of this thesis

This thesis consists of two parts. Part I is the longest and deals with discrete-time systems. Part II deals with continuous-time systems.

We now briefly outline the contents of Part I. In Chapter 2 some basic notions are defined. Chapter 3 deals with stability and Chapter 4 with stabilizability. The aspects of energy preserving systems that are needed in this thesis are collected in Chapter 5. The very important linear quadratic regulator problem is the subject of Chapter 6. In Chapter 7 coprime factorizations are studied. The existence result for (strongly) coprime factorizations proven in this chapter is probably one of the most important results presented in this thesis. Robustly stabilizing controllers are the topic of Chapter 8. The metric in which we measure the distance between systems, the gap metric, is the topic of Chapter 9. Part I concludes with Chapter 10 on balanced realizations. These balanced realizations are used to define the desired approximations of the plant. In this chapter we show that, under certain conditions, a robustly stabilizing controller based on an approximation of the plant stabilizes the original infinite-dimensional system.

Part II starts with Chapter 11 in which we introduce our new class of systems. In Chapter 12 we illustrate how systems described by partial differential equations fit into this abstract framework. The Cayley transform, which we use to translate results from discrete-time to continuous-time, is studied in Chapter 13. In Chapter 14 the continuous-time counterparts of the most important results obtained in discrete-time are presented. Chapter 15 illustrates the model reduction for controller design approach outlined in this thesis using an example of a system described by a partial differential equation (a beam).

In Chapter 16 the most important results obtained in the preceding chapters are collected.

There are two appendices; in the first one some basic results in Hardy space theory are recalled and in the second one some rather tedious algebraic calculations with algebraic Riccati equations are performed. At the end of the thesis one can find a short summary (both in English and in Dutch), a list of notations, a bibliography and an index.

Part I

Discrete-time systems

Chapter 2

Basic objects

In this chapter we introduce the main concept of part I of this thesis, that of a discrete-time system. We also introduce several objects associated with a discrete-time system that will be used throughout this thesis. Finally, we study several ways in which we can obtain a new discrete-time system from one or two known ones.

We first introduce the concept of a dynamical system. Note that $\mathbb{W}^{\mathbb{T}}$ denotes the set of functions from $\mathbb{T}$ to $\mathbb{W}$.

Definition 2.1. A **dynamical system** Σ is a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$ with $\mathbb{T}$ a set, called the time axis; $\mathbb{W}$ a set, called the signal space, and $\mathbb{B} \subset \mathbb{W}^{\mathbb{T}}$, the behavior of the system.

In Part I of this thesis we will be concerned with the following dynamical systems.

Let $\mathbb{T} = \mathbb{Z}^+$, the nonnegative integers, and $\mathbb{W} = \mathscr{U} \times \mathscr{X} \times \mathscr{Y}$, where $\mathscr{U}$, $\mathscr{X}$, $\mathscr{Y}$ are separable Hilbert spaces. Let $A \in \mathcal{L}(\mathscr{X})$, $B \in \mathcal{L}(\mathscr{U}, \mathscr{X})$, $C \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ and $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be bounded operators between the given spaces. Define the behavior by

$$\mathbb{B} := \left\{ \begin{bmatrix} u \\ x \\ y \end{bmatrix} \in \mathbb{W}^{\mathbb{T}} : \begin{bmatrix} x_{n+1} \\ y_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_n \\ u_n \end{bmatrix} \text{ for all } n \in \mathbb{Z}^+ \right\} \tag{2.1}$$

This type of dynamical system will be called a **discrete-time system** . The elements of the behavior are called **trajectories** of the system. It follows from (2.1) that for arbitrary $u : \mathbb{Z}^+ \to \mathscr{U}$ and $x_0 \in \mathscr{X}$ there exists a unique trajectory $[u; x; y] \in \mathbb{B}$. The sequence u is called the **input**, x_0 the **initial state**, x the **state** and y the **output**. The space $\mathscr{U}$ is called the **input space**, $\mathscr{X}$ is called the **state space** and $\mathscr{Y}$ the **output space**. Usually in control theory the goal is to choose for a given initial state an input such that the trajectory has some specified property.

Note that the operator

$$S := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.2}$$

is completely determined by the behavior in the following sense: if the behaviors corresponding to S_1 and S_2 are equal, then S_1 and S_2 are equal. This can be proven as follows. Let $x_0 \in \mathscr{X}$ and $u : \mathbb{Z}^+ \to \mathscr{U}$ be arbitrary. Since the behaviors corresponding to S_1 and S_2 are equal the trajectories corresponding to this initial state and input are equal. In particular the state at time one and the output at time zero are equal. Hence

$$S_1 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_0 \end{bmatrix} = S_2 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix},$$

and since u_0 and x_0 where arbitrary this shows that $S_1 = S_2$.

The above shows that the following definitions are unambiguous. The operators appearing in (2.1) have the following names: A is called the **state operator**, B the **input operator**, C the **output operator**, and D the **feedthrough operator** of the discrete-time system. The operator S is called the **system operator**.

We say that a sequence $h : \mathbb{Z} \to \mathscr{H}$ is **finitely nonzero** if only a finite number of elements in the sequence is nonzero.

A discrete-time system is called **approximately observable** if $[0; x; y] \in \mathbb{B}$ and $[0; w; y] \in \mathbb{B}$ implies $x = w$, i.e. if the output with zero input uniquely determines the state. A discrete-time system is called **approximately controllable** if the set

$$\{w \in \mathscr{X} : \text{ there exist } [u; x; y] \in \mathbb{B} \text{ with } u \text{ finitely nonzero, } N \in \mathbb{Z}^+ \text{ such that } x_0 = 0, x_N = w\}$$

is dense in $\mathscr{X}$. A discrete-time system is called **minimal** of it is both approximately controllable and approximately observable.

We define three maps on sequence spaces that will play an important role in this thesis. The **input map** of a discrete-time system is defined for finitely nonzero $u : \mathbb{Z}^- \to \mathscr{U}$ by (here $\mathbb{Z}^-$ is the set of negative integers)

$$\mathcal{B}u := \sum_{i=0}^{\infty} A^i B u_{-i-1},$$

the **output map** is defined for $x \in \mathscr{X}$ by

$$(\mathcal{C}x)_k := CA^k x \quad k \in \mathbb{Z}^+,$$

the **input-output map** is defined for finitely nonzero $u : \mathbb{Z} \to \mathscr{U}$ by

$$(\mathcal{D}u)_k := \sum_{i=0}^{\infty} CA^i B u_{k-i-1} + Du_k, \quad k \in \mathbb{Z}.$$

Remark 2.2. Let $J \subset \mathbb{Z}$. Denote by $l_c(J, \mathscr{H})$ the set of sequences $J \to \mathscr{H}$ with compact support and by $l(J, \mathscr{H})$ the set of all sequences $J \to \mathscr{H}$. As indicated above we consider $\mathcal{B} : l_c(\mathbb{Z}^-, \mathscr{U}) \to \mathscr{X}$, $\mathcal{C} : \mathscr{X} \to l(\mathbb{Z}^+, \mathscr{Y})$, $\mathcal{D} : l_c(\mathbb{Z}, \mathscr{U}) \to l(\mathbb{Z}, \mathscr{Y})$. In this thesis we will not need to consider topologies on $l_c(J, \mathscr{H})$ and $l(J, \mathscr{H})$. In connection with stability of the system in Chapter 3 and the subsequent chapters we sometimes consider the extension of $\mathcal{B}$ to a bounded operator on $l^2(\mathbb{Z}^-, \mathscr{U})$ (which when it exists is unique), $\mathcal{C}$ as a bounded operator into $l^2(\mathbb{Z}^+, \mathscr{Y})$ (which in some cases it may not be) and the extension of $\mathcal{D}$ as a bounded operator from $l^2(\mathbb{Z}, \mathscr{U})$ to $l^2(\mathbb{Z}, \mathscr{Y})$ (which when it exists is unique). It should be clear from the context on which spaces we consider the input, output and input-output map.

To further study the above maps we introduce the maps τ, π_- and π_+ on the space of sequences $\mathbb{Z} \to \mathscr{H}$ where $\mathscr{H}$ is a separable Hilbert space

$$(\tau h)_k := h_{k+1}, \quad (\pi_- h)_k := \begin{cases} h_k & k \in \mathbb{Z}^- \\ 0 & k \in \mathbb{Z}^+ \end{cases}, (\pi_+ h)_k := \begin{cases} 0 & k \in \mathbb{Z}^- \\ h_k & k \in \mathbb{Z}^+ \end{cases}.$$

The significance of the above maps is apparent from the following result. If $[u; x; y]$ is a trajectory with u finitely nonzero, then

$$\begin{aligned} x_n &= A^n x_0 + \mathcal{B}\pi_- \tau^n u \\ y &= \mathcal{C}x_0 + \mathcal{D}u. \end{aligned}$$

The above follows from an easy computation.

Proposition 2.3. *A discrete-time system is approximately observable if and only if its output map $\mathcal{C}$ is one-to-one and approximately controllable if and only if its input map $\mathcal{B}$ has dense range.*

Proof. This follows easily from the above characterization of a trajectory in terms of the main operator, the input map, the output map and the input-output map. □

The fourth map on sequence spaces that will play an important role in this thesis is the following map. The **Hankel map** is defined for finitely nonzero $u : \mathbb{Z}^- \to \mathscr{U}$ by

$$(\mathcal{H}u)_k := \sum_{i=0}^{\infty} CA^i B u_{k-i-1} \quad k \in \mathbb{Z}^+.$$

We state the following lemma on the Hankel map.

Lemma 2.4. *For the Hankel map of a discrete-time system we have*

$$\mathcal{C}\mathcal{B} = \mathcal{H} = \pi_+\mathcal{D}\pi_-,$$

where $\mathcal{B}$ is the input map, $\mathcal{C}$ the output map and $\mathcal{D}$ the input-output map of the discrete-time system.

Proof. The equality $\mathcal{H} = \pi_+\mathcal{D}\pi_-$ is immediate from the definitions. The equality $\mathcal{C}\mathcal{B} = \mathcal{H}$ is proven as follows. Let $u : \mathbb{Z}^- \to \mathscr{U}$ be finitely nonzero and $k \in \mathbb{Z}^+$, we then have

$$\begin{aligned}
(\mathcal{C}\mathcal{B}u)_k &= CA^k \textstyle\sum_{i=0}^{\infty} A^i B u_{-i-1} \\
&= \textstyle\sum_{i=0}^{\infty} CA^{k+i} B u_{-i-1} && \text{using that } u_{-i-1} = 0 \text{ for } i < 0 \\
&= \textstyle\sum_{i=-k}^{\infty} CA^{k+i} B u_{-i-1} && \text{substituting } j := k + i \\
&= \textstyle\sum_{j=0}^{\infty} CA^j B u_{k-j-1} \\
&= (\mathcal{H}u)_k.
\end{aligned}$$

This shows that $\mathcal{C}\mathcal{B} = \mathcal{H}$. □

We define four operator-valued holomorphic functions associated with a discrete-time system that will play an important role in this thesis. We define them through power series expansions. Note that, as in the scalar case, operator-valued power series have a radius of convergence and we consider the four operator-valued holomorphic functions as functions on an open disc with the radius of convergence as radius. The **state function** of a discrete-time system is defined by

$$\mathsf{A}(z) = \sum_{i=0}^{\infty} A^i z^i.$$

Note that this series converges for $|z| < 1/\|A\|$, since it forms a geometric series with common ratio Az. The radius of convergence of this series equals $1/r(A)$, where $r(A)$ denotes the spectral radius of A, and so $\mathsf{A} : \mathbb{D}_{1/r(A)} \to \mathcal{L}(\mathscr{X})$. We denote $1/r(A)$ by r_{A}. The **input function** of a discrete-time system is defined by

$$\mathsf{B}(z) = \sum_{i=0}^{\infty} A^i B z^{i+1}.$$

We have $\mathsf{B} : \mathbb{D}_{r_{\mathsf{B}}} \to \mathcal{L}(\mathscr{U}, \mathscr{X})$, where r_{B} is the radius of convergence of the power series. The **output function** $\mathsf{C} : \mathbb{D}_{r_{\mathsf{C}}} \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$ of a discrete-time system is defined by

$$\mathsf{C}(z) = \sum_{i=0}^{\infty} CA^i z^i$$

and the **transfer function** $\mathsf{D} : \mathbb{D}_{r_{\mathsf{D}}} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by

$$\mathsf{D}(z) = D + \sum_{i=0}^{\infty} CA^i B z^{i+1}.$$

Note that

$$A = \mathsf{A}'(0), \quad B = \mathsf{B}'(0), \quad C = \mathsf{C}(0), \quad D = \mathsf{D}(0). \tag{2.3}$$

Remark 2.5. We remark that we have the following inequalities

$$r_{\mathsf{A}} \leq r_{\mathsf{B}}, r_{\mathsf{C}} \leq r_{\mathsf{D}}$$

and the equalities

$$\begin{aligned} \mathsf{A}(z)Bz &= \mathsf{B}(z) \quad \forall z \in \mathbb{D}_{r_{\mathsf{A}}}, \\ C\mathsf{A}(z) &= \mathsf{C}(z) \quad \forall z \in \mathbb{D}_{r_{\mathsf{A}}}, \\ D + C\mathsf{B}(z) &= \mathsf{D}(z) \quad \forall z \in \mathbb{D}_{r_{\mathsf{B}}}, \\ D + \mathsf{C}(z)Bz &= \mathsf{D}(z) \quad \forall z \in \mathbb{D}_{r_{\mathsf{C}}}. \end{aligned}$$

A sequence $h : \mathbb{Z}^+ \to \mathscr{H}$ is called Z**-transformable** if the power series

$$\sum_{i=0}^{\infty} h_i z^i$$

has a positive radius of convergence. The Z-transform of a Z-transformable sequence h is denoted by $\hat{h}$.

Lemma 2.6. *If the input of a discrete-time system is Z-transformable, then the state and output are Z-transformable and they satisfy*

$$\begin{aligned} \hat{x}(z) &= \mathsf{A}(z)x_0 + \mathsf{B}(z)\hat{u}(z), \\ \hat{y}(z) &= \mathsf{C}(z)x_0 + \mathsf{D}(z)\hat{u}(z), \end{aligned}$$

for z such that $|z| < r_{\mathsf{A}}$ and $|z|$ smaller than the radius of convergence of the power series corresponding to the sequence u.

Proof. Due to the linearity of the system we can prove this in two steps: in the first we can take $u = 0$ and in the second $x_0 = 0$.

First step ($u = 0$). Since $x_{n+1} = Ax_n$, we obtain $x_i = A^i x_0$ and so the Z-transform of the state is

$$\sum_{i=0}^{\infty} x_i z^i = \sum_{i=0}^{\infty} A^i x_0 z^i = \mathsf{A}(z)x_0$$

and this power series converges for $|z| < r_{\mathsf{A}}$. Since $y_n = Cx_n$ we obtain that y is Z-transformable and

$$\hat{y}(z) = \mathsf{C}(z)x_0.$$

Second step ($x_0 = 0$). The state is now given by

$$x_n = \sum_{i=0}^{n-1} A^i B u_{n-i-1},$$

and so its Z-transform is

$$\sum_{n=0}^{\infty} x_n z^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} A^i B u_{n-i-1} z^n.$$

On the other hand, we have

$$\mathsf{B}(z)\hat{u}(z) = \sum_{j=0}^{\infty} A^j B z^{j+1} \sum_{k=0}^{\infty} u_k z^k = \sum_{n=0}^{\infty} \sum_{i=0}^{n} A^i B u_{n-i-1} z^n,$$

where the rearranging of terms is justified, since the series converge absolutely. Hence x is Z-transformable and satisfies $\hat{x}(z) = \mathsf{B}(z)\hat{u}(z)$. The proof that $\hat{y}(z) = \mathsf{D}(z)\hat{u}(z)$ follows along the same lines.

Combining steps 1 and 2 and using linearity proves the lemma. □

We now define four other operator-valued holomorphic functions that will also play a role in this thesis. They are defined on a set that we denote by $1/\rho(A)$ and that is defined as follows:

$$1/\rho(A) := \{z \in \mathbb{C} : 1/z \in \rho(A)\} \cup \{0\}.$$

Here $\rho(A)$ denotes the resolvent set of the operator A. The **resolvent** $\mathfrak{A} : 1/\rho(A) \to \mathcal{L}(\mathscr{X})$ of a discrete-time system is defined by

$$\mathfrak{A}(z) := (I - zA)^{-1},$$

the **incoming wave function** $\mathfrak{B} : 1/\rho(A) \to \mathcal{L}(\mathscr{U}, \mathscr{X})$ of a discrete-time system is defined by

$$\mathfrak{B}(z) := z(I - zA)^{-1}B,$$

the **outgoing wave function** $\mathfrak{C} : 1/\rho(A) \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$ of a discrete-time system is defined by

$$\mathfrak{C}(z) := C(I - zA)^{-1},$$

and the **characteristic function** $\mathfrak{D} : 1/\rho(A) \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ of a discrete-time system is defined by

$$\mathfrak{D}(z) := D + Cz(I - zA)^{-1}B.$$

It is easily seen that

$$\mathfrak{A}(z) = \mathsf{A}(z),\ \mathfrak{B}(z) = \mathsf{B}(z),\ \mathfrak{C}(z) = \mathsf{C}(z),\ \mathfrak{D}(z) = \mathsf{D}(z),\ \text{for } |z| < r_{\mathsf{A}}, \quad (2.4)$$

but the following examples show that these functions are not identical.

Example 2.7. Let $\mathscr{U} = \mathscr{X} = \mathscr{Y} = \mathbb{C}$ and $A = -1$, $B = C = D = 0$. Then both the transfer function and the characteristic function are zero, but the transfer function has domain $\mathbb{C}$ while the characteristic function has domain $\mathbb{C}\backslash\{-1\}$. This shows that the transfer function and the characteristic function are not identical. Similar arguments apply to the other functions.

The above example is somewhat pathological, since on the intersection of their domains the functions are equal. This example identifies the only possible difference when the state space $\mathscr{X}$ is finite-dimensional. In the case that $\mathscr{X}$ is infinite-dimensional, the transfer function and the characteristic function need not even be equal on the intersection of their domains, as the following example shows.

Example 2.8. Let $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ and $\mathscr{X} = l^2(\mathbb{Z})$. Define the operators A, B, C by

$$(Ax)_k = x_{k-1}, \quad (Bu)_k = \begin{cases} u & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}, \quad Cx = x_{-1},$$

and $D = 0$. Then $CA^iB = 0$ for all $i \geq 0$ and so the transfer function is defined on the whole complex plane and equals zero. We calculate $\mathfrak{D}(2)$. We first note that the solution v of $B1 = (I - 2A)v$ has to satisfy

$$v_k - 2v_{k-1} = \begin{cases} 0 & \text{for } k \neq 0 \\ 1 & \text{for } k = 0. \end{cases}$$

The unique solution in $l^2(\mathbb{Z})$ is given by

$$v_k = \begin{cases} -2^k & \text{for } k < 0 \\ 0 & \text{for } k \geq 0. \end{cases}$$

So $v = (I - 2A)^{-1}B1$. It follows that $\mathfrak{D}(2)1 = C2(I - 2A)^{-1}B1 = 2v_{-1} = -1$. Hence $\mathfrak{D}(2) = -1$. We conclude that the transfer function and the characteristic function are both defined in 2, but that their values in this point are different. So the transfer function and the characteristic function are not equal on the intersection of their domains.

The following result shows on which domain we do have equality of the transfer function and the characteristic function.

Proposition 2.9. *For a discrete-time system we have the following equalities.*

$$\begin{aligned}
\mathsf{A}(z) &= \mathfrak{A}(z) \quad \forall z : |z| < r_{\mathsf{A}}, \\
\mathsf{B}(z) &= \mathfrak{B}(z) \quad \forall z : |z| < r_{\mathsf{A}}, \\
\mathsf{C}(z) &= \mathfrak{C}(z) \quad \forall z : |z| < r_{\mathsf{A}}, \\
\mathsf{D}(z) &= \mathfrak{D}(z) \quad \forall z : |z| < \max\{r_{\mathsf{B}}, r_{\mathsf{C}}\}, z \in 1/\rho(A).
\end{aligned}$$

Proof. The first three equalities were already mentioned in (2.4). We prove the fourth equality. From (2.4) we conclude that $\mathsf{C}(z)(I - zA) = C$ for all z with $|z| < r_{\mathsf{A}}$. Since both sides are holomorphic this equality extends to all z with $|z| < r_{\mathsf{C}}$. We now multiply both sides by $(I - zA)^{-1}zB$ which is well-defined on $1/\rho(A)$ and obtain $\mathsf{C}(z)zB = C(I - zA)^{-1}zB$ for all $z \in 1/\rho(A)$ with $|z| < r_{\mathsf{C}}$. This shows that $\mathsf{D}(z) = \mathfrak{D}(z)$ for these z and proves the fourth equality in the case that $r_{\mathsf{C}} \geq r_{\mathsf{B}}$. If $r_{\mathsf{B}} < r_{\mathsf{C}}$, then a similar argument with B instead of C proves the assertion. □

Example 2.10. We apply Proposition 2.9 to Example 2.8. It is easily seen that the spectral radius of A equals one. So we have $\mathsf{A} = \mathfrak{A}$, $\mathsf{B} = \mathfrak{B}$, $\mathsf{C} = \mathfrak{C}$, $\mathsf{D} = \mathfrak{D}$ on the open unit disc. It is not very difficult to show that $r_{\mathsf{B}} = r_{\mathsf{C}} = 1$, so that the in principle more precise condition for equality of the transfer function and the characteristic function from Proposition 2.9 in this case gives the same as the condition based on the spectral radius of A.

As we saw the transfer function of a discrete-time system is always holomorphic at zero. The following result shows that any function that is holomorphic in zero is the transfer function of some discrete-time system. We first give the relevant definition.

Definition 2.11. Let G be a $\mathcal{L}(\mathscr{U}, \mathscr{Y})$-valued function defined in a neigbourhood of zero. A discrete-time system Σ is called a **realization** of G if the transfer function of Σ coincides with G in a neighbourhood of zero.

Proposition 2.12. *Any $\mathcal{L}(\mathscr{U}, \mathscr{Y})$-valued function which is holomorphic at zero has a realization.*

Proof. In this proof we will use the Hardy spaces H^2 and H^∞ (see Appendix A). First assume that the given function G satisfies $\mathsf{G}(0) = 0$ and $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Define $\mathscr{X} := l^2(\mathbb{Z}^+, \mathscr{Y})$ and for $x \in \mathscr{X}$ the operator $A \in \mathcal{L}(\mathscr{X})$ by $(Ax)_n = x_{n+1}$ and the operator $C \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ by $Cx = x_0$.

For $u \in U$ define $\mathsf{F}(z) := \mathsf{G}(z)u$. Since $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we have $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{Y})$. It follows from Lemma A.2 that $\mathsf{F}(z) = \sum_{n=0}^\infty F_n z^n$, with the sequence $F_n \in l^2(\mathbb{Z}^+, \mathscr{Y})$. Define $B \in \mathcal{L}(\mathscr{U}, \mathscr{X})$ by $(Bu)_n = F_n$. This operator is bounded since

$$\|Bu\|_{\mathscr{X}} = \|(F_n)_{n\geq 0}\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|\mathsf{F}\|_{H^2(\mathbb{D},\mathscr{Y})}\| \leq \|\mathsf{G}\|_{H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U},\mathscr{Y}))} \ \|u\|_{\mathscr{U}},$$

where we have used Lemmas A.2 and A.4. It is easily seen that $CA^nB = F_n$ for all $n \in \mathbb{Z}^+$, from which it follows that G is the transfer function of the discrete-time system with system operator $[A, B; C, 0]$.

If $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, but $\mathsf{G}(0) \neq 0$, then by applying the above to $\mathsf{G}(z) - \mathsf{G}(0)$ we see that the discrete-time system with system operator $[A, B; C, \mathsf{G}(0)]$ has the transfer function G.

If G is holomorphic at zero, then there exists a $r > 0$ such that $\mathsf{G}_r(z) := \mathsf{G}(rz)$ is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Applying the above to G_r we obtain a realization $[A_r, B_r; C_r, D_r]$ of G_r. Define $[A, B; C, D] := [A_r/r, B_r; C_r/r, D_r]$, then it is easily seen that this is a realization of G. □

Remark 2.13. The realization constructed in the proof of Proposition 2.12 for a function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is called the **backward shift realization**. If we define

$$\mathscr{X}_{\min} := \overline{\operatorname{span}}\{A^n Bu : u \in \mathscr{U}, n \geq 0\},$$

then obviously the system operator (see (2.2)) restricts to a bounded operator from $[\mathscr{X}_{\min}; \mathscr{U}]$ to $[\mathscr{X}_{\min}; \mathscr{Y}]$ and the resulting discrete-time system is approximately controllable and approximately observable and has the same transfer function as the backward shift realization. This realization is called the **restricted backward shift realization**. We will denote its system operator by S^{rs} and we will denote its components similarly.

Remark 2.14. In general, a function has infinitely many realizations. If $[A, B; C, D]$ is a realization of G and $S \in \mathcal{L}(\mathscr{X})$ has a bounded inverse, then $[SAS^{-1}, SB; CS^{-1}, D]$ is also a realization of G. In fact there are many more realizations. For a reasonably complete discussion of realization theory we refer to Staffans [89, Chapter 9].

Since in operator theory the adjoint (also known as conjugate or dual) of an operator plays an essential role, it might not come as a surprise that the following dual system plays an important role in systems theory.

Definition 2.15. The **dual system** of a discrete-time system with system operator S is the discrete-time system with system operator S^*.

Note that the state space of the dual system equals the state space of the system itself, but that the input and output spaces are interchanged.

The following results show how the holomorphic functions of a system are related to those associated with its dual system. To formulate this we need the following notation: let $f : \Lambda \subset \mathbb{C} \to \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable Hilbert spaces, then $f^\dagger : \overline{\Lambda} \to \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ is defined by $f^\dagger(s) := f(\bar{s})^*$.

Proposition 2.16. *The resolvent, the wave functions and the characteristic function of the dual system satisfy*

$$\left[\begin{array}{c|c} \mathfrak{A}_{\text{dual}} & \mathfrak{B}_{\text{dual}} \\ \hline \mathfrak{C}_{\text{dual}} & \mathfrak{D}_{\text{dual}} \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{A}^\dagger & \mathfrak{C}^\dagger \\ \hline \mathfrak{B}^\dagger & \mathfrak{D}^\dagger \end{array}\right].$$

Proof. This follows easily from the definitions. □

Proposition 2.17. *The state function, input function, output function and transfer function of the dual system are given by*

$$\left[\begin{array}{c|c} \mathsf{A}_{\text{dual}} & \mathsf{B}_{\text{dual}} \\ \hline \mathsf{C}_{\text{dual}} & \mathsf{D}_{\text{dual}} \end{array}\right] = \left[\begin{array}{c|c} \mathsf{A}^\dagger & \mathsf{C}^\dagger \\ \hline \mathsf{B}^\dagger & \mathsf{D}^\dagger \end{array}\right].$$

Proof. This follows easily from the definitions. □

Definition 2.18. The **series interconnection** of the system Σ_1 and the system Σ_2 is defined when $\mathscr{Y}_1 = \mathscr{U}_2$ by its system operator

$$S_{\text{series}} = \left[\begin{array}{c|c} A_{\text{series}} & B_{\text{series}} \\ \hline C_{\text{series}} & D_{\text{series}} \end{array}\right] := \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right].$$

Proposition 2.19. *The resolvent, the wave functions and the characteristic function of the series interconnection satisfy*

$$\left[\begin{array}{c|c} \mathfrak{A}_{\text{series}} & \mathfrak{B}_{\text{series}} \\ \hline \mathfrak{C}_{\text{series}} & \mathfrak{D}_{\text{series}} \end{array}\right] = \left[\begin{array}{cc|c} \mathfrak{A}_1 & 0 & \mathfrak{B}_1 \\ \mathfrak{B}_2\mathfrak{C}_1 & \mathfrak{A}_2 & \mathfrak{B}_2\mathfrak{D}_1 \\ \hline \mathfrak{D}_2\mathfrak{C}_1 & \mathfrak{C}_2 & \mathfrak{D}_2\mathfrak{D}_1 \end{array}\right].$$

Proof. This is an easy calculation. □

Proposition 2.20. *The state function, input function, output function and transfer function of the series interconnection are given by*

$$\left[\begin{array}{c|c} \mathsf{A}_{\text{series}} & \mathsf{B}_{\text{series}} \\ \hline \mathsf{C}_{\text{series}} & \mathsf{D}_{\text{series}} \end{array}\right] = \left[\begin{array}{cc|c} \mathsf{A}_1 & 0 & \mathsf{B}_1 \\ \mathsf{B}_2\mathsf{C}_1 & \mathsf{A}_2 & \mathsf{B}_2\mathsf{D}_1 \\ \hline \mathsf{D}_2\mathsf{C}_1 & \mathsf{C}_2 & \mathsf{D}_2\mathsf{D}_1 \end{array}\right].$$

Proof. This is an easy calculation. □

Lemma 2.21. *A realization of the transfer function of the series interconnection of Σ_1 and Σ_2 is given by the system operator*

$$\left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ A_2 + B_2C_1 - A_1 & A_2 & B_2D_1 - B_1 \\ \hline D_2C_1 + C_2 & C_2 & D_2D_1 \end{array}\right].$$

Proof. This follows from applying the state space transformation

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

to the system operator of the series interconnection given in Definition 2.18. □

Proposition 2.22. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{H})$ be holomorphic with $0 \in D(\mathsf{G})$. Let $[A, B; C, D]$ be a realization of G and assume that D is boundedly invertible. Then $\mathsf{G}(z)$ is invertible in a neighbourhood of zero and the inverse of G has a realization $[A - BD^{-1}C, BD^{-1}; -D^{-1}C, D^{-1}]$.*

Proof. This follows from writing down realizations of the transfer functions of the series interconnection of $[A, B; C, D]$ and $[A-BD^{-1}C, BD^{-1}; -D^{-1}C, D^{-1}]$ in both orders using Lemma 2.21. Since these are both equal to the identity the result follows. □

Proposition 2.23. *Let $\check{\Sigma}$ be a discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Denote its transfer function by $[\check{\mathsf{D}}_1; \check{\mathsf{D}}_2]$. Assume that $\check{D}_1$ has a bounded inverse. Define the discrete-time system Σ by its system operator:*

$$S = \left[\begin{array}{c|c} \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 & \check{B}\check{D}_1^{-1} \\ \hline \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1 & \check{D}_2\check{D}_1^{-1} \end{array}\right]. \tag{2.5}$$

Then the transfer function D of Σ satisfies $\mathsf{D}(z) = \check{\mathsf{D}}_2(z)\check{\mathsf{D}}_1(z)^{-1}$.

Proof. The operator $[\check{A}, \check{B}; \check{C}_2, \check{D}_2])$ is a realization of $\check{\mathsf{D}}_2$. A realization of $\check{\mathsf{D}}_1^{-1}$ can be obtained from Proposition 2.22. A realization of the transfer function of their series interconnection is provided by Lemma 2.21 as

$$\left[\begin{array}{cc|c} \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 & 0 & \check{B}\check{D}_1^{-1} \\ 0 & \check{A} & 0 \\ \hline \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1 & \check{C}_2 & \check{D}_2\check{D}_1^{-1} \end{array}\right].$$

It follows from Proposition 2.20 that this system is a realization of $\check{\mathsf{D}}_2\check{\mathsf{D}}_1^{-1}$. It is easily seen that the transfer function of this system equals that of Σ. □

Notes

The concept of dynamical system as defined in Definition 2.1 is taken from Polderman and Willems [76]. Discrete-time systems have been studied for quite some time. The first account in book form of the infinite-dimensional case seems to be Fuhrmann [31]. Chapter 12 of Staffans [89] contains some more recent developments. Example 2.8 is adapted from Curtain and Zwart [18, Example 4.3.8]. The backward shift realization mentioned in Remark 2.13 is due to Fuhrmann [30] and Helton [37].

Chapter 3

Stability

The concept of stability plays a key role in systems theory. In this chapter we study different notions of stability for discrete-time systems.

Definition 3.1. A discrete-time system is called

- **exponentially stable** if for all sequences x with $[0; x; y] \in \mathbb{B}$ we have $x \in l^2(\mathbb{Z}^+, \mathscr{X})$.
- **strongly stable** if for all sequences x with $[0; x; y] \in \mathbb{B}$ we have $\|x_n\| \to 0$ as $n \to \infty$.
- **output stable** if for all sequences y with $[0; x; y] \in \mathbb{B}$ we have $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$.
- **input stable** if the dual system is output stable.
- **input-output stable** if $u \in l^2(\mathbb{Z}^+, \mathscr{U})$, $x_0 = 0$ and $[u; x; y] \in \mathbb{B}$ implies $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$.

We remark that exponential stability is often referred to as **power stability** in the literature.

We will first give alternative characterizations of the concepts just introduced. Then we will show that exponential stability implies all the other types of stability (Proposition 3.28).

The Hardy spaces H^2 and H^∞ play a role in this chapter. The reader is referred to Appendix A for the relevant background.

Proposition 3.2. *The following are equivalent.*

1. *The discrete-time system is output stable.*
2. *The output map $\mathcal{C}$ is an element of $\mathcal{L}(\mathscr{X}, l^2(\mathbb{Z}^+, \mathscr{Y}))$.*

3. *There exists a nonnegative self-adjoint operator $L \in \mathcal{L}(\mathscr{X})$ such that $A^*LA - L + C^*C = 0$.*

4. *We have $r_{\mathsf{C}} \geq 1$ and for all $x \in \mathscr{X}$ the restriction of $\mathsf{C}(\cdot)x$ to the open unit disc is in $H^2(\mathbb{D}, \mathscr{Y})$.*

Proof. We show that output stability is equivalent to 2, that 2 is equivalent to 3 and that 4 is equivalent to output stability.

(i) 2 implies output stability. The output for initial state x_0 and zero input is given by $\mathcal{C}x_0$ and since $\mathcal{C}$ maps into $l^2(\mathbb{Z}^+, \mathscr{Y})$, by assumption, we obtain the desired result.

(ii) output stability implies 2. Output stability shows that the range of the output map is contained in $l^2(\mathbb{Z}^+, \mathscr{Y})$. We show that $\mathcal{C}$ is closed. Assume that $x^n \to x$ in $\mathscr{X}$ and $\mathcal{C}x^n \to y$ in $l^2(\mathbb{Z}^+, \mathscr{Y})$ as $n \to \infty$. We have to show that $y = \mathcal{C}x$. Since $\mathcal{C}x^n \to y$ we have for all $k \in \mathbb{Z}^+$ that $(\mathcal{C}x^n)_k \to y_k$. Using the definition of output map we see that this is equivalent to $CA^kx^n \to y_k$. Since C and A are bounded operators and $x^n \to x$ we also have for all $k \in \mathbb{Z}^+$ that $CA^kx^n \to CA^kx$. This shows that for all $k \in \mathbb{Z}^+$ the following holds $(\mathcal{C}x)_k = y_k$, in other words $\mathcal{C}x = y$. This proves that $\mathcal{C}$ is closed. Since it is everywhere defined, it follows from the closed graph theorem that it is bounded.

(iii) 2 implies 3. Since $\mathcal{C}$ is bounded $L := \mathcal{C}^*\mathcal{C}$ is a nonnegative self-adjoint element of $\mathcal{L}(\mathscr{X})$. We show that it satisfies the given equation. We have for $x \in \mathscr{X}$

$$\langle LAx, Ax\rangle - \langle Lx, x\rangle + \langle Cx, Cx\rangle = \langle \mathcal{C}Ax, \mathcal{C}Ax\rangle - \langle \mathcal{C}x, \mathcal{C}x\rangle + \langle Cx, Cx\rangle =$$

$$\sum_{k=0}^{\infty} \|(\mathcal{C}Ax)_k\|^2 - \sum_{k=0}^{\infty} \|(\mathcal{C}x)_k\|^2 + \|Cx\|^2 = \sum_{k=0}^{\infty} \|(\mathcal{C}x)_{k+1}\|^2 - \|(\mathcal{C}x)_k\|^2 + \|Cx\|^2.$$

The reordering of terms is permitted, since the series involved converge absolutely. Noting that the last series above telescopes, we obtain that the above expression equals zero. Since this is true for all $x \in \mathscr{X}$, we see that L satisfies the above mentioned equation.

(iv) 3 implies 2. Multiply the given equation from the left with A^{*k} and from the right with A^k and sum from $k = 0$ to n to obtain

$$\begin{aligned} \sum_{k=0}^{n} A^{*k}C^*CA^k &= \sum_{k=0}^{n} A^{*k}LA^k - \sum_{k=0}^{n} A^{*k+1}LA^{k+1} \\ &= L - A^{*n+1}LA^{n+1} \leq L. \end{aligned}$$

From this we obtain for all $x \in \mathscr{X}$ and $n \in \mathbb{Z}^+$

$$\sum_{k=0}^{n} \|CA^k x\|^2 \leq \langle Lx, x \rangle. \tag{3.1}$$

Letting $n \to \infty$ shows that $\mathcal{C}$ is a bounded map.

(v) 4 implies output stability. This follows from the fact that the Z-transform maps $l^2(\mathbb{Z}^+, \mathscr{Y})$ one-to-one onto $H^2(\mathbb{D}, \mathscr{Y})$. Since by assumption, the Z-transform of the output with initial condition x_0 and zero input restricts to a function in $H^2(\mathbb{D}, \mathscr{Y})$ it follows that this output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$.

(vi) Output stability implies 4. Using the fact that the Z-transform maps $l^2(\mathbb{Z}^+, \mathscr{Y})$ one-to-one onto $H^2(\mathbb{D}, \mathscr{Y})$, we obtain that the Z-transform of the output with initial condition x_0 and zero input restricts to a function in $H^2(\mathbb{D}, \mathscr{Y})$. This is equivalent to $\mathsf{C}x_0$ restricting to a function in $H^2(\mathbb{D}, \mathscr{Y})$. This shows that C is defined on the open unit disc and since an operator-valued function is holomorphic in the strong topology if and only if it is holomorphic in the uniform topology, it follows that C is holomorphic on the open unit disc, and this implies that we have $r_{\mathsf{C}} \geq 1$. □

Example 3.3. The backward shift realization and the restricted backward shift realization from Remark 2.13 are output stable. It is easily seen that the identity is a solution of the equation mentioned in part 3 of Proposition 3.2.

We formulate a corollary about input stability.

Corollary 3.4. *The following are equivalent.*

1. *The discrete-time system is input stable.*
2. *The input map $\mathcal{B}$ extends uniquely to an element of $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), \mathscr{X})$.*
3. *There exists a nonnegative self-adjoint operator $L \in \mathcal{L}(\mathscr{X})$ such that $ALA^* - L + BB^* = 0$.*
4. *We have $r_{\mathsf{B}} \geq 1$ and for all $x \in \mathscr{X}$ the restriction of $\mathsf{B}^\dagger(\cdot)x$ is in $H^2(\mathbb{D}, \mathscr{U})$.*

Proof. This follows by applying Proposition 3.2 to the dual system. □

From the above results on input and output stability we obtain the following result on boundedness of the Hankel map.

Proposition 3.5. *The Hankel map of an input and output stable discrete-time system has a unique extension to an element of $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), l^2(\mathbb{Z}^+, \mathscr{Y}))$.*

Proof. Proposition 3.2 shows that the output map is bounded and Corollary 3.4 that the input map is bounded. Lemma 2.4 shows that the Hankel map is the product of these two operators. Hence the Hankel map is bounded. □

The operator $\mathcal{C}^*\mathcal{C}$ that we encountered in the proof of Proposition 3.2 plays an important role.

Definition 3.6. The **observability gramian** L_C of an output stable system is defined as $L_C := \mathcal{C}^*\mathcal{C}$. Here $\mathcal{C}^*$ is the adjoint of $\mathcal{C}$ considered as an operator in $\mathcal{L}(\mathscr{X}, l^2(\mathbb{Z}^+, \mathscr{Y}))$.

The proof of proposition 3.2 shows that the observability gramian is a solution of the **observation Lyapunov equation**

$$A^*LA - L + C^*C = 0. \tag{3.2}$$

This equation may have several other bounded nonnegative self-adjoint solutions. The following result gives two additional properties that the observability gramian has, each of which identifies it uniquely in the set of bounded nonnegative self-adjoint solutions of the observation Lyapunov equation. This will be of use to us later to show that a certain bounded nonnegative self-adjoint operator is the observability gramian of the system.

Lemma 3.7. *The set of bounded nonnegative self-adjoint solutions of the observation Lyapunov equation of an output stable discrete-time system has a unique element $L_{\min}$ such that $L_{\min} \leq L$ for all other bounded nonnegative self-adjoint solutions L. This unique element $L_{\min}$ is the observability gramian.*

This set also has a unique element L such that $L^{1/2}A^n x \to 0$ for all $x \in \mathscr{X}$ as $n \to \infty$. This unique element is the observability gramian.

Proof. From (3.1) we obtain by letting $k \to \infty$ that $L_C \leq L$ for all bounded nonnegative self-adjoint solutions L of the observation Lyapunov equation. Obviously the smallest element is unique.

We have

$$\|L_C^{1/2}A^n x\|^2 = \langle L_C A^n x, A^n x\rangle = \|\mathcal{C}A^n x\|^2 = \sum_{k=0}^{\infty} \|CA^k A^n x\|^2 = \sum_{i=n}^{\infty} \|CA^i x\|^2.$$

For $n \to \infty$ this converges to zero, since $\mathcal{C}x \in l^2(\mathbb{Z}^+, \mathscr{Y})$. This shows that the observability gramian indeed satisfies the given convergence condition. Let L be a bounded nonnegative self-adjoint solution of the observation Lyapunov equation with the above mentioned convergence property. Multiply the Lyapunov equation from the left with A^{*k} and from the right with A^k and sum from $k = 0$ to n to obtain

$$\begin{aligned}\sum_{k=0}^{n} A^{*k}C^*CA^k &= \sum_{k=0}^{n} A^{*k}LA^k - \sum_{k=0}^{n} A^{*k+1}LA^{k+1} \\ &= L - A^{*n+1}LA^{n+1}\end{aligned}$$

From this we obtain for all $x \in \mathscr{X}$

$$\sum_{k=0}^{n} \|CA^k x\|^2 = \langle Lx, x\rangle - \|L^{1/2}A^{n+1}x\|^2.$$

Letting $n \to \infty$ the left-hand side converges to $\|\mathcal{C}x\|^2 = \langle L_C x, x\rangle$ while, since $\|L^{1/2}A^{n+1}x\|^2 \to 0$ by assumption, the right-hand side converges to $\langle Lx, x\rangle$. Hence we obtain $\langle L_C x, x\rangle = \langle Lx, x\rangle$ for all $x \in \mathscr{X}$, which implies $L = L_C$. □

Lemma 3.8. *Let Σ be output stable and strongly stable. Then the observability gramian is the unique nonnegative self-adjoint solution of the observation Lyapunov equation.*

Proof. According to Lemma 3.7 the observability gramian is a nonnegative self-adjoint solution of the observation Lyapunov equation, so we only have to show that it is the unique nonnegative self-adjoint solution. Let L be a nonnegative self-adjoint solution of the observation Lyapunov equation. Then, as in the proof of Lemma 3.7, we have for all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} A^{*n}C^*CA^n = \sum_{n=0}^{N} A^{*n}LA^n - \sum_{n=0}^{N} A^{*n+1}LA^{n+1} = L - A^{*N+1}LA^{N+1}.$$

We then have for all $x, y \in X$

$$\left\langle \sum_{n=0}^{N} A^{*n}C^*CA^n x, y \right\rangle = \langle Lx, y\rangle - \langle LA^{N+1}x, A^{N+1}y\rangle.$$

Letting $N \to \infty$ and using the fact that A is strongly stable, we have for all $x, y \in X$

$$\langle L_C x, y\rangle = \langle Lx, y\rangle.$$

This implies that $L = L_C$. Since L was an arbitrary nonnegative self-adjoint solution, this implies that L_C is the unique nonnegative self-adjoint solution of the observation Lyapunov equation. □

Example 3.9. The backward shift realization and the restricted backward shift realization from Remark 2.13 have the identity as observability gramian. From Example 3.9 we obtain that the identity is a solution of the observation Lyapunov equation. Since the systems are strongly stable it follows from Proposition 3.8 that the identity is the observability gramian of both systems.

Proposition 3.10. *Consider an output stable discrete-time system with observability Gramian L_C. If $[u; x; y] \in \mathbb{B}$ with u finitely nonzero, then $L_C^{1/2} x_n \to 0$ as $n \to \infty$.*

Proof. Let N be such that $u_n = 0$ for $n \geq N$. We have $L_C^{1/2} A^n x_N \to 0$ by Lemma 3.7. Since for $k \geq N$ we have $x_{k+N} = A^k x_N$ this implies that $L_C^{1/2} x_n \to 0$. □

Proposition 3.11. *An output stable discrete-time system is approximately observable if and only if $L_C > 0$.*

Proof. This follows since $\langle L_C x, x\rangle = \|\mathcal{C}x\|^2$ and a discrete-time system is approximately observable if and only if $\mathcal{C}$ is one-to-one by Proposition 2.3. □

The following Lyapunov equation that we already encountered in Corollary 3.4 is called the **control Lyapunov equation**

$$ALA^* - L + BB^* = 0. \tag{3.3}$$

Definition 3.12. The **controllability gramian** L_B of an input stable system is defined as $L_B := \mathcal{B}\mathcal{B}^*$. Here $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ considered as an operator in $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), \mathscr{X})$.

By duality we obtain similar results for the control Lyapunov equation as we obtained for the observability Lyapunov equation.

Lemma 3.13. *The set of bounded nonnegative self-adjoint solutions of the control Lyapunov equation of an input stable discrete-time system has a unique element $L_{\min}$ such that $L_{\min} \leq L$ for all other bounded nonnegative self-adjoint solutions L. This unique element $L_{\min}$ is the controllability gramian.*

This set also has a unique element L such that $L^{1/2} A^{*n} x \to 0$ for all $x \in \mathscr{X}$ as $n \to \infty$. This unique element is the controllability gramian.

Proof. This follows from applying Lemma 3.7 to the dual system. □

Lemma 3.14. *Let Σ be input stable with a strongly stable dual system. Then the controllability gramian is the unique nonnegative self-adjoint solution of the control Lyapunov equation.*

Proof. This follows from applying Lemma 3.8 to the dual system. □

Proposition 3.15. *An input stable discrete-time system is approximately controllable if and only if $L_B > 0$.*

Proof. Assume the system is approximately controllable. Then the input map has dense range. It follows that its adjoint is injective. From this we obtain that $L_B = \mathcal{B}\mathcal{B}^*$ is a positive operator.

Assume $L_B > 0$. Then $\mathcal{B}^*$ is injective, from which it follows that $\mathcal{B}$ has dense range. This implies that the system is approximately controllable. □

The following lemma and its corollary will be used throughout the thesis.

Lemma 3.16. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, $Z \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$, $T \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ and $\lambda \neq 0$. Then $\lambda \in \sigma(ZT)$ if and only if $\lambda \in \sigma(TZ)$.*

Proof. Suppose that $\lambda \in \rho(ZT)$. Then we have

$$\frac{1}{\lambda}(I + T(\lambda I - ZT)^{-1}Z)(\lambda I - TZ) = I$$

and

$$(\lambda I - TZ)\frac{1}{\lambda}(I + T(\lambda I - ZT)^{-1}Z) = I.$$

This implies $\lambda \in \rho(TZ)$. The converse follows from interchanging the role of Z and T. Since the spectrum is the complement of the resolvent set the result follows. □

Note that $\lambda \neq 0$ is essential for Lemma 3.16 to hold: the left and right shift on $l^2(\mathbb{Z}^+)$ offers a counterexample for the case $\lambda = 0$. Lemma 3.16 has the following obvious corollary on the spectral radius of a product.

Corollary 3.17. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, $Z \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$, $T \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$. Then $r(ZT) = r(TZ)$.*

Proof. This follows from Lemma 3.16. □

We use Corollary 3.17 to prove the following.

Lemma 3.18. *Let Σ be an input and output stable discrete-time system. Let L_C and L_B be its observability and controllabilty gramian, respectively, and $\mathcal{H}$ its Hankel map. Then $\|\mathcal{H}\| = \sqrt{r(L_C L_B)}$.*

Proof. We use Corollary 3.17 and Lemma 2.4 to obtain the following.

$$r(L_C L_B) = r(\mathcal{C}^*\mathcal{C}\mathcal{B}\mathcal{B}^*) = r(\mathcal{B}^*\mathcal{C}^*\mathcal{C}\mathcal{B}) = r(\mathcal{H}^*\mathcal{H}).$$

Since $\mathcal{H}^*\mathcal{H}$ is a self-adjoint operator its spectral radius equals its norm. Since the norm of a self-adjoint operator T can be computed as

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

we obtain

$$\|\mathcal{H}^*\mathcal{H}\| = \sup_{\|x\|=1} |\langle \mathcal{H}^*\mathcal{H}x, x\rangle| = \sup_{\|x\|=1} |\langle \mathcal{H}x, \mathcal{H}x\rangle| = \|\mathcal{H}\|^2.$$

Combing the above we obtain

$$\|\mathcal{H}\|^2 = \|\mathcal{H}^*\mathcal{H}\| = r(\mathcal{H}^*\mathcal{H}) = r(L_C L_B),$$

as desired. □

Lemma 3.19. *Let Σ be an input and output stable discrete-time system. Let L_C and L_B be its observability and controllabilty gramian, respectively, and let L_c and L_b be arbitrary nonnegative self-adjoint solutions of its observation and control Lyapunov equations, respectively. Then $r(L_C L_B) \le r(L_c L_b)$.*

Proof. Lemma 3.17 implies that

$$r(L_C L_B) = r(L_C^{1/2} L_B L_C^{1/2}).$$

By Lemma 3.7 we have $L_C \le L_c$ and by Lemma 3.13 we have $L_B \le L_b$. From $L_B \le L_b$ we conclude that $L_C^{1/2} L_B L_C^{1/2} \le L_C^{1/2} L_b L_C^{1/2}$. This implies that

$$r(L_C^{1/2} L_B L_C^{1/2}) \le r(L_C^{1/2} L_b L_C^{1/2}).$$

Using Lemma 3.17 we obtain

$$r(L_C^{1/2} L_b L_C^{1/2}) = r(L_b^{1/2} L_C L_b^{1/2}).$$

Since $L_C \le L_c$ we obtain $L_b^{1/2} L_C L_b^{1/2} \le L_b^{1/2} L_c L_b^{1/2}$, which implies that

$$r(L_b^{1/2} L_C L_b^{1/2}) \le r(L_b^{1/2} L_c L_b^{1/2}).$$

Using Lemma 3.17 again we obtain

$$r(L_b^{1/2} L_c L_b^{1/2}) = r(L_c L_b).$$

Combing the above obtained inequalities we arrive at $r(L_C L_B) \le r(L_c L_b)$. □

Output stability tells us the following about the transfer function.

Proposition 3.20. *For an output stable discrete time system we have the following:*

$$\begin{aligned} \mathsf{D}(z) &= D + \mathsf{C}(z)zB \quad \forall z \in \mathbb{D}, \\ \mathsf{D}(z) &= \mathfrak{D}(z) \qquad\qquad \forall z \in \rho(A) \cap \mathbb{D}, \end{aligned}$$

and for all $u \in \mathscr{U}$ we have that $\mathsf{D}(\cdot)u$ restricts to a function in $H^2(\mathbb{D}, \mathscr{Y})$.

Proof. Proposition 3.2 part 4 shows that $r_{\mathsf{C}} \geq 1$. Remark 2.5 and Proposition 2.9 now give the indicated equalities. The first of these equalities together with Proposition 3.2 part 4 (with $x = Bu$) shows the H^2 property. □

The dual result reads as follows.

Proposition 3.21. *For an input stable discrete time system we have the following:*

$$\begin{aligned} \mathsf{D}(z) &= D + C\mathsf{B}(z) \quad \forall z \in \mathbb{D}, \\ \mathsf{D}(z) &= \mathfrak{D}(z) \qquad\quad \forall z \in \rho(A) \cap \mathbb{D}, \end{aligned}$$

and for all $y \in \mathscr{Y}$ we have that $\mathsf{D}^\dagger(\cdot)y$ restricts to a function in $H^2(\mathbb{D}, \mathscr{U})$.

Proof. This follows along similar lines as Proposition 3.20. □

We now give a necessary and sufficient condition for input-output stability.

Proposition 3.22. *A system is input-output stable if and only if $r_{\mathsf{D}} \geq 1$ and D restricts to a function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.*

Proof. We first prove the if part. Let $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and denote the output corresponding to this input and initial condition zero by y. By Lemma 2.6 the Z-transform of the output is given by $\mathsf{D}(z)\hat{u}(z)$. From Lemmas A.2 and A.4 we obtain that $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. Hence the system is input-output stable.

We now prove the only if part. We first show that the map from the input to the output (with initial condition zero) is closed from $l^2(\mathbb{Z}^+, \mathscr{U})$ to $l^2(\mathbb{Z}^+, \mathscr{Y})$. So assume that $u^n \to u$ in $l^2(\mathbb{Z}^+, \mathscr{U})$ and the corresponding outputs $y^n \to y$ in $l^2(\mathbb{Z}^+, \mathscr{Y})$. We have to show that y is the output for input u. For y^n we have

$$y_k^n = \sum_{i=0}^{k-1} CA^i Bu_{k-i-1}^n + Du_k^n.$$

Since A, B, C and D are continuous and u_j^n converges to u_j, we have

$$y_k^n \to \sum_{i=0}^{k-1} CA^i B u_{k-i-1} + Du_k.$$

On the other hand, since $y^n \to y$, we have $y_k^n \to y_k$. This shows that

$$y_k = \sum_{i=0}^{k-1} CA^i B u_{k-i-1} + Du_k.$$

So y is indeed the output for input u. By the closed graph theorem the map that sends an input to the corresponding output is in $\mathcal{L}(l^2(\mathbb{Z}^+, \mathscr{U}), l^2(\mathbb{Z}^+, \mathscr{Y}))$. This map obviously commutes with right-translations: if y is the output for the input u then $[0; y]$ is the output for the input $[0; u]$. Since the Z-transform is an isometric isomorphism between $l^2(\mathbb{Z}^+, \mathscr{H})$ and $H^2(\mathbb{D}, \mathscr{H})$ the map that sends $\hat{u}$ to $\hat{y}$ is bounded from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$. The shift-invariance in time-domain translates to commutation with multiplication by z in the frequency domain. Hence $\hat{u} \mapsto \hat{y}$ is a bounded linear map from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$ that commutes with multiplication by z. By Lemma A.4 it is given by multiplication by an $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function. This function coincides with the input-output function restricted to the unit disc. □

Corollary 3.23. *The dual system of an input-output stable system is input-output stable.*

Proof. This follows from Proposition 3.22 since $\mathsf{D} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ if and only if $\mathsf{D}^\dagger \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$. □

Proposition 3.24. *The Hankel map of an input-output stable discrete-time system has a unique extension to an element of $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), l^2(\mathbb{Z}^+, \mathscr{Y}))$.*

Proof. By Proposition 3.22 the transfer function of the system is an element of $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. It follows from Definition A.24 that it has a bounded Hankel operator. By Lemma A.26 the Hankel operator and the Hankel map are similar with as similarity operator the Z-transform. It follows that the Hankel map extends to a bounded operator from $l^2(\mathbb{Z}^-, \mathscr{U})$ to $l^2(\mathbb{Z}^+, \mathscr{Y})$ as desired. □

Example 3.25. The backward shift realization and the restricted backward shift realization of a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function from Remark 2.13 are input stable. From Lemma 2.4 we obtain $\mathcal{H}^*\mathcal{H} = \mathcal{B}^*\mathcal{C}^*\mathcal{C}\mathcal{B} = \mathcal{B}^* L_C \mathcal{B}$. From Example 3.9 we obtain $L_C = I$. From Proposition 3.24 we obtain that the Hankel map is bounded. It follows that $\mathcal{B}$ is bounded. Hence the system is input stable by Proposition 3.4.

The following result gives necessary and sufficient conditions for exponential stability.

Proposition 3.26. *The following are equivalent.*

1. *The discrete-time system is exponentially stable.*
2. *There exists a nonnegative self-adjoint operator $L \in \mathcal{L}(\mathscr{X})$ such that $A^*LA - L + I = 0$.*
3. *The spectral radius of the state operator is strictly smaller than one.*
4. *There exist $M \geq 0$ and $r \in [0,1)$ such that for all sequences x with $[0; x; y] \in \mathbb{B}$ we have $\|x_n\| \leq Mr^n \|x_0\|$ for all $n \geq 0$.*
5. *We have $r_{\mathsf{A}} \geq 1$ and the restriction of the state function to the open unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$.*

Proof. We will show that exponential stability implies 2 implies 3 implies 4 implies 5 implies exponential stability.

(i) exponential stability implies 2: this follows from Proposition 3.2 with $C = I$.

(ii) 2 implies 3. We will show that it follows from the Lyapunov equation that the approximate eigenvalues of A must lie in the open unit disc. Since the boundary of the spectrum of an operator consists of approximate eigenvalues (see Taylor and Lay [91, Theorem V.4.1 page 282]), this shows that the spectrum of A is contained in the open unit disc which is equivalent with the spectral radius being strictly smaller than one. Suppose λ is an approximate eigenvalue and x_n is a sequence of approximate eigenvectors; that is, $\|x_n\| = 1$ and $\|(\lambda I - A)x_n\| \to 0$. Using the Lyapunov equation we obtain

$$(\lambda I - A)^* L (\lambda I - A) - \lambda(\lambda I - A)^* L - \bar{\lambda} L (\lambda I - A) = (1 - |\lambda|^2)L - I.$$

By applying this to x_n and taking the inner product with x_n we obtain $(1-|\lambda|^2)\langle Lx_n, x_n\rangle \to 1$. Since L is nonnegative this implies that $1-|\lambda|^2 > 0$.

(ii) 3 implies 4. From the Gelfand formula $r(A) = \lim_{n\to\infty} \sqrt[n]{\|A^n\|}$ it follows that for $r := (1 + r(A))/2 \in (0,1)$ there exists a $N \in \mathbb{Z}^+$ such that for all $n \geq N$ we have $\|A^n\| \leq r^n$. Define $\tilde{M} := \max_{i=0,\ldots,N-1} \|A^i\|/r^i$ and $M = \max\{\tilde{M}, 1\}$. Then $\|A^n\| \leq Mr^n$ for all $n \geq 1$. Since $x_n = A^n x_0$ the assertion follows.

(iii) 4 implies 5. From the given inequality we conclude that Z-transform of x is holomorphic on the open disc with radius $1/r$. In particular, it follows that for all $x_0 \in \mathscr{X}$ the function $\mathsf{A}(\cdot)x_0$ is holomorphic in a neighborhood of the unit disc. It follows that for all $x_0 \in \mathscr{X}$ the function $z \mapsto \|\mathsf{A}(z)x_0\|^2$ is continuous on the closed unit disc. Since the closed unit disc is compact, this function is bounded. We conclude that for all $x_0 \in \mathscr{X}$ the function $\mathsf{A}(\cdot)x_0$ restricts to a function in $H^\infty(\mathbb{D}, \mathscr{X})$. It follows that $r_\mathsf{A} \geq 1$ and A restricted to the unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$.

(iv) 5 implies exponential stability. Let x be the state corresponding to initial state x_0 and zero input. Since the state function restricts to a function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$ we have that the Z-transform of the state, $\hat{x}(z) = \mathsf{A}(z)x_0$, restricted to the unit disc is in $H^\infty(\mathbb{D}, \mathscr{X})$. Since $H^\infty(\mathbb{D}, \mathscr{X})$ is contained in $H^2(\mathbb{D}, \mathscr{X})$ and the Z-transform is isometric from $l^2(\mathbb{Z}^+, \mathscr{X})$ onto $H^2(\mathbb{D}, \mathscr{X})$ we obtain that the state is in $l^2(\mathbb{Z}^+, \mathscr{X})$ and so the system is exponentially stable. □

Corollary 3.27. *The dual system of an exponentially stable system is exponentially stable.*

Proof. This follows from Proposition 3.26 since the spectral radius of an operator and its dual are equal. □

After having established equivalent conditions for the types of stability we have introduced, we are now ready to study their relationships to each other.

The following proposition shows that exponential stability implies all the other types of stability.

Proposition 3.28. *If a discrete-time system is exponentially stable, then it is strongly stable, output stable, input stable and input-output stable.*

Proof. (i) Exponential stability implies strong stability: any square summable sequence tend to zero.

(ii) Exponential stability implies output stability: since the input is assumed to be zero we have $y_n = Cx_n$ and so $\|y_n\| \leq \|C\| \, \|x_n\|$. Since x is square summable it follows that y is.

(iii) Exponential stability implies input stability: by Corollary 3.27 the dual system is exponentially stable so it follows by (ii) that the dual system is output stable, which shows that the original system is input stable.

(iv) Exponential stability implies input-output stability: by Proposition 3.26 part 5 we have $r_\mathsf{A} \geq 1$ which by Remark 2.5 implies $r_\mathsf{D} \geq 1$. From the same proposition we obtain that A restricted to the open unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$, using Remark 2.5 again we obtain that the restriction of D to the unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Proposition 3.22 now shows that the system is input-output stable. □

Remark 3.29. It follows similarly as in part (iv) of the proof of Proposition 3.28 that B and C restrict to functions in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{X}))$ and $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}, \mathscr{Y}))$, respectively, when the system is exponentially stable.

As the following example shows Proposition 3.28 is the only possible positive result on the connection between the different stability concepts.

Example 3.30.

1. An example of a system that is strongly stable, input stable, output stable and input-output stable, but not exponentially stable. Take $\mathscr{X} = l^2(\mathbb{Z}^+, \mathbb{C})$, $\mathscr{U} = \mathbb{C}$, $\mathscr{Y} = \mathbb{C}$, $B = 0$, $C = 0$, $D = 0$. It trivially follows that this system for any A is input stable, output stable and input-output stable. Define A as follows: $(Ax)_n := x_{n+1}$. Then A is strongly stable: we have

$$\|A^n x\|^2 = \sum_{k=0}^{\infty} \|(A^n x)_k\|^2 = \sum_{k=0}^{\infty} \|x_{k+n}\|^2 = \sum_{i=n}^{\infty} \|x_i\|^2,$$

and since $x \in l^2(\mathbb{Z}^+, \mathbb{C})$ this expression tends to zero as $n \to \infty$. Let $\{e_n\}$ be the standard basis of $l^2(\mathbb{Z}^+, \mathbb{C})$. The system is not exponentially stable: take e_0 as initial state, then the state at time n equals e_n. Since the state at any time instance has norm one it is not square summable over time.

2. An example of a system that is input stable, output stable and input-output stable, but not strongly stable. Take $\mathscr{X} = \mathscr{U} = \mathscr{Y} = \mathbb{C}$, $A = 1$, $B = 0$, $C = 0$, $D = 0$. It trivially follows that this system is input stable, output stable and input-output stable. Since $A^n x = x$ for all $n \in \mathbb{Z}^+$ the system is not strongly stable.

3. An example of a system that is strongly stable, input stable and input-output stable, but not output stable. Take $\mathscr{X} = l^2(\mathbb{Z}^+, \mathbb{C})$, $\mathscr{U} = \mathbb{C}$, $\mathscr{Y} = l^2(\mathbb{Z}^+, \mathbb{C})$, $B = 0$, $D = 0$. Define A as follows: $(Ax)_n := x_{n+1}$. Then, as in part 1, A is strongly stable. Since $B = 0$ the system is obviously input and input-output stable for any choice of C. Choose $C = I$. Then the state and the output coincide and it follows that the system is output stable if and only if it is exponentially stable. We saw in part 1 that A is not exponentially stable. It follows that the system is not output stable.

4. An example of a system that is strongly stable, output stable and input-output stable, but not input stable. The dual system of the system from part 3 provides such an example.

5. An example of a system that is strongly stable, input stable, output stable, but not input-output stable. The function $\mathsf{G} : \mathbb{D} \to \mathbb{C}$ defined by

$$\mathsf{G}(z) := \sum_{n=1}^{\infty} \frac{1}{n} z^n$$

is in $H^2(\mathbb{D}, \mathbb{C})$ since $\sum_{n=1}^{\infty} 1/n^2 < \infty$, but is not in $H^\infty(\mathbb{D}, \mathbb{C})$ since for $z \to 1$ we have that $|\mathsf{G}(z)|$ becomes arbitrarily large since $\sum_{n=1}^{\infty} 1/n$ diverges. It follows using Proposition 3.22 that any system with G as transfer function is not input-output stable. Define $\mathscr{X} := l^2(\mathbb{Z}^+, \mathbb{C})$, the operator $A \in \mathcal{L}(\mathscr{X})$ by $(Ax)_n = x_{n+1}$ and the operator $C \in \mathcal{L}(\mathscr{X}, \mathbb{C})$ by $Cx = x_0$. Define $B \in \mathcal{L}(\mathbb{C}, \mathscr{X})$ by $(Bu)_n = u/(n+1)$. This operator is bounded since

$$\|Bu\|_{\mathscr{X}}^2 = \left\| \frac{u}{n+1} \right\|_{l^2(\mathbb{Z}^+, \mathbb{C})}^2 = |u|^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

It is easily seen that $CA^nB = 1/(n+1)$ for all $n \in \mathbb{Z}^+$, from which it follows that G is the transfer function of the discrete-time system Σ with system operator $[A, B; C, 0]$. As in part 1 A is strongly stable. It is easily seen that the identity is a solution of the observation Lyapunov equation, from which it follows using Proposition 3.2 that Σ is output stable. It is easily computed that the output map of the dual system has, with respect to the standard basis of $l^2(\mathbb{Z}^+, \mathbb{C})$, the following matrix represention.

$$\begin{bmatrix} 1 & 1/2 & 1/3 & \dots \\ 1/2 & 1/3 & \dots & \dots \\ 1/3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

This matrix is called the infinite Hilbert matrix and is known to define a bounded operator on $l^2(\mathbb{Z}^+, \mathbb{C})$ with norm π (see Peller [75, page 6]). It follows that Σ is input stable.

Notes

Exponential stability and input-output stability have been the main stability concepts in systems and control theory in the last decades. Proposition 3.26 can be considered as the discrete-time version of a now classical continuous-time result of Datko [20]. Connections between Lyapunov equations and strong stability were investigated by Przyłuski [77]. Proposition 3.22 is classical, see for example Weiss [96] for more information on the continuous-time version.

Chapter 4

Stabilizability

Stabilizability is an important concept in systems theory. In this chapter we consider several forms of stabilizability.

Definition 4.1. Let S be the system operator of a discrete-time system and $[F, G] \in \mathcal{L}(\mathscr{X} \times \mathscr{U}, \mathscr{U})$. Then $[F, G]$ is called an **admissible feedback pair** if $I - G$ is boundedly invertible. The corresponding **closed-loop system** is the discrete-time system with system operator

$$S_{[F,G]} := \left[\begin{array}{c|c} A + B(I-G)^{-1}F & B(I-G)^{-1} \\ \hline (I-G)^{-1}F & (I-G)^{-1} \\ C + D(I-G)^{-1}F & D(I-G)^{-1} \end{array}\right]. \tag{4.1}$$

Remark 4.2. Definition 4.1 is motivated by the following. We first add the equations $v_n = Fx_n + Gu_n$ to the equations $x_{n+1} = Ax_n + Bu_n$, $y_n = Cx_n + Du_n$ that describe the system. We then choose the input u to be $u_n = v_n + r_n$, i.e. we feed the additional output v back. We consider r as the input and $[u; y]$ as the output of a new system. This new system is described by the system operator given in Definition 4.1.

Remark 4.3. Let $[F, G]$ be an admissible feedback pair. Then $[(I - G)^{-1}F, 0]$ is also an admissible feedback pair and the state operator and output operator of the respective closed-loop systems are equal. This explains why it is often assumed in the literature that the second component of an admissible feedback pair equals zero. It turns out that for making the connection with continuous-time systems using the Cayley transform it is however useful to work with general admissible feedback pairs.

Definition 4.4. Let S be the system operator of a discrete-time system and let $[L; K] \in \mathcal{L}(\mathscr{Y}, \mathscr{X} \times \mathscr{Y})$. Then $[L; K]$ is called an **admissible injection pair** if $I - K$ is boundedly invertible. The corresponding **closed-loop**

system is the discrete-time system with system operator

$$S^{[L;K]} := \left[\begin{array}{c|cc} A + L(I-K)^{-1}C & L(I-K)^{-1} & B + L(I-K)^{-1}D \\ \hline (I-K)^{-1}C & (I-K)^{-1} & (I-K)^{-1}D \end{array}\right]. \quad (4.2)$$

The following result shows that the notions of admissible feedback pair and admissible injection pair are dual.

Lemma 4.5. *Let S be the system operator of a discrete-time system. The closed-loop system of S with the admissible feedback pair $[F, G]$ is the dual of the closed-loop system of S^* with the admissible injection pair $[L; K] := [F, G]^*$.*

Proof. This is immediate. □

Definition 4.6. A discrete-time system is called

- **exponentially stabilizable** if there exists an admissible feedback pair such that the closed-loop system is exponentially stable.
- **exponentially detectable** if there exists an admissible injection pair such that the closed-loop system is exponentially stable.
- **output stabilizable** if there exists an admissible feedback pair such that the closed-loop system is output stable.
- **input stabilizable** if there exists an admissible injection pair such that the closed-loop system is input stable.

Proposition 4.7. *A discrete-time system is exponentially stabilizable if and only if its dual system is exponentially detectable. It is output stabilizable if and only if its dual system is input stabilizable.*

Proof. This follows using Lemma 4.5. □

Proposition 4.8. *If a discrete-time system is exponentially stabilizable, then it is output stabilizable. If a discrete-time system is exponentially detectable, then it is input stabilizable.*

Proof. If the system is exponentially stabilizable, then there exists an admissible feedback pair such that the closed-loop system is exponentially stable. By Proposition 3.28 this closed-loop system is also output stable. The second statement follows by duality. □

Remark 4.9. Note that an exponentially stable system is exponentially stabilizable (take F and G equal to zero) and exponentially detectable (take K and L equal to zero). An output stable system is output stabilizable (take F and G equal to zero) and an input stable system is input stabilizable (take K and L equal to zero).

Proposition 4.10. *Let S be the system operator of a discrete-time system and $[F, G]$ an admissible feedback pair. Define $\mathfrak{F}(z) = F(I - zA)^{-1}$ and $\mathfrak{G}(z) = G + Fz(I - zA)^{-1}B$, then the generalized resolvents of the closed-loop system are*

$$\left[\begin{array}{c|c} \mathfrak{A}^{\mathrm{cl}} & \mathfrak{B}^{\mathrm{cl}} \\ \hline \mathfrak{C}^{\mathrm{cl}} & \mathfrak{D}^{\mathrm{cl}} \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{A} + \mathfrak{B}(I - \mathfrak{G})^{-1}\mathfrak{F} & \mathfrak{B}(I - \mathfrak{G})^{-1} \\ \hline (I - \mathfrak{G})^{-1}\mathfrak{F} & (I - \mathfrak{G})^{-1} \\ \mathfrak{C} + \mathfrak{D}(I - \mathfrak{G})^{-1}\mathfrak{F} & \mathfrak{D}(I - \mathfrak{G})^{-1} \end{array}\right]. \tag{4.3}$$

Proof. This is easily computed. $\square$

The following concerns a relationship between stabilizability and stability.

Proposition 4.11. *If a discrete-time system is input-output stable and*

- *output stabilizable, then it is output stable.*
- *input stabilizable, then it is input stable.*
- *exponentially stabilizable and detectable, then it is exponentially stable.*

Proof. It follows from (4.3) that $\mathfrak{C} = \mathfrak{C}_2^{\mathrm{cl}} - \mathfrak{D}\mathfrak{C}_1^{\mathrm{cl}}$. It follows that $\mathsf{C} = \mathsf{C}_2^{\mathrm{cl}} - \mathsf{D}\mathsf{C}_1^{\mathrm{cl}}$ in a neighbourhood of zero. Since the closed-loop system is output stable we have that for every $x \in \mathscr{X}$ the function $\mathsf{C}^{\mathrm{cl}}x$ restricts to a function in $H^2(\mathbb{D}; \mathscr{U} \times \mathscr{Y})$. By input-output stability D restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. It follows that for every $x \in \mathscr{X}$ the function $\mathsf{C}x$ restricts to a function in $H^2(\mathbb{D}; \mathscr{Y})$. Hence the system is output stable.

We note that in the case that the system is exponentially stabilizable we have that C^{cl} restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}, \mathscr{U} \times \mathscr{Y}))$ by Remark 3.29. The argumention above then leads to the stronger conclusion $\mathsf{C} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}, \mathscr{Y}))$.

The second statement follows by duality.

From (4.3) we obtain that $\mathfrak{A} = \mathfrak{A}^{\mathrm{cl}} - \mathfrak{B}\mathfrak{C}_1^{\mathrm{cl}}$. It follows that $\mathsf{A} = \mathsf{A}^{\mathrm{cl}} - \mathsf{B}\mathsf{C}_1^{\mathrm{cl}}$ in a neighbourhood of zero. By exponential stabilizability A^{cl} restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}))$. That $\mathsf{C}_1^{\mathrm{cl}}$ restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}, \mathscr{U}))$ we already concluded in the second paragraph of this proof. Applying the second paragraph of this proof to the dual system, since the system is exponentially detectable this is justified, we see that B restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{X}))$. It follows that A restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}))$, which shows that the system is exponentially stable. $\square$

Proposition 4.12. *Let Σ be a discrete-time system, $[F, G]$ an admissible feedback pair and $\Sigma_{[F,G]}$ the corresponding closed-loop system. If Σ is exponentially stabilizable, then $\Sigma_{[F,G]}$ is. If Σ is exponentially detectable, then $\Sigma_{[F,G]}$ is.*

Proof. Since Σ is exponentially stabilizable, there exists an admissible feedback pair $[\underline{F}, \underline{G}]$ such that $A + B(I - \underline{G})^{-1}\underline{F}$ is exponentially stable. Define the admissible feedback pair $[\tilde{G}, \tilde{F}] := [0, (I - G)(I - \underline{G})^{-1}\underline{F} - F]$. It is easily seen that the state operator of the closed-loop system of $\Sigma_{[F,G]}$ with this admissible feedback pair equals $A + B(I - \underline{G})^{-1}\underline{F}$. It follows that $\Sigma_{[F,G]}$ is exponentially stabilizable.

Since Σ is exponentially stabilizable, there exists an admissible injection pair $[\underline{L}; \underline{K}]$ such that $A + \underline{L}(I - \underline{K})^{-1}C$ is exponentially stable. Define the admissible injection pair $[\tilde{L}; \tilde{K}]$ by $\tilde{L} := [-(B + \underline{L}(I - \underline{K})^{-1}D), \underline{L}(I - \underline{K})^{-1}]$ and $\tilde{K} := 0$. It is easily computed that the state operator of the closed-loop system of $\Sigma_{[F,G]}$ with this admissible injection pair equals$A + \underline{L}(I - \underline{K})^{-1}C$. It follows that $\Sigma_{[F,G]}$ is exponentially detectable. □

Corollary 4.13. *Let Σ be a discrete-time system, $[F, G]$ an admissible feedback pair and $\Sigma_{[F,G]}$ the corresponding closed-loop system. If Σ is exponentially stabilizable and detectable and $\Sigma_{[F,G]}$ is input-output stable, then $\Sigma_{[F,G]}$ is exponentially stable.*

Proof. It follows from Proposition 4.12 that $\Sigma_{[F,G]}$ is exponentially stabilizable and detectable. Proposition 4.11 now shows that $\Sigma_{[F,G]}$ is exponentially stable. □

Proposition 4.14. *Let $\check{\Sigma}$ be a discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Assume that $\check{D}_1$ is boundedly invertible. Define the system Σ as in Proposition 2.23. Then $[\check{C}_1; I - \check{D}_1]$ is an admissible feedback pair for Σ and the corresponding closed-loop system equals $\check{\Sigma}$.*

Proof. This is an easy computation. □

Corollary 4.15. *Use the notation and assumptions of Proposition 4.14. If $\check{\Sigma}$ is exponentially stable, then Σ is exponentially stabilizable. If $\check{\Sigma}$ is output stable, then Σ is output stabilizable.*

Proof. This follows immediately. □

Notes

The concept of stabilizability is classical. The notion of admissible feedback pair as given here is due to Staffans [89]; previously G was always taken equal to zero.

Chapter 5

Energy preserving systems

In this chapter we consider energy preserving systems. We will not go very deeply into the theory. Only those results that will be used in later chapters are discussed. For more on energy preserving systems we refer to Staffans [89, Chapter 11] and the references therein. We start with the definition of an energy preserving system.

Definition 5.1. A discrete-time system is called **energy preserving** if there exists an $L = L^* \in \mathcal{L}(\mathscr{X})$ such that for any trajectory $[u; x; y] \in \mathbb{B}$ and any $n \in \mathbb{Z}^+$ we have

$$\langle Lx_n, x_n\rangle + \sum_{k=0}^{n-1} \|y_k\|^2 = \langle Lx_0, x_0\rangle + \sum_{k=0}^{n-1} \|u_k\|^2. \tag{5.1}$$

The operator L is called the **storage operator**.

The idea of the definition is that the norms in the input and output spaces represent energy so that $\sum_{k=0}^{n-1} \|u_k\|^2$ is the amount of energy supplied to the system up to time n and $\sum_{k=0}^{n-1} \|y_k\|^2$ is the amount of energy extracted from the system up to time n; the quadratic form $\langle Lx, x\rangle$ is supposed to represent the energy stored in the system if it is in the state x; (5.1) then says that the amount of energy supplied to the system up to time n plus the energy stored inside the system initially is equal to the amount of energy extracted from the system up to time n plus the energy stored inside the system at time n. This physical interpretation will not be important in the sequel. Note that we allowed the storage operator L to be indefinite which means that the energy stored in the system can be negative.

Our first result concerns stability.

Proposition 5.2. *An energy preserving discrete-time system with nonnegative storage operator is both output stable and input-output stable.*

Proof. Let $x_0 \in \mathscr{X}$ and $u \in l^2(\mathbb{Z}^+, \mathscr{U})$. Adding the finite nonnegative quantity $\sum_{k=n}^{\infty} \|u_k\|^2$ to the right-hand side of (5.1) we obtain for all $n \in \mathbb{Z}^+$

$$\langle Lx_n, x_n \rangle + \sum_{k=0}^{n-1} \|y_k\|^2 \leq \langle Lx_0, x_0 \rangle + \|u\|^2.$$

Since L is nonnegative we obtain from this the inequality

$$\sum_{k=0}^{n-1} \|y_k\|^2 \leq \langle Lx_0, x_0 \rangle + \|u\|^2.$$

This inequality shows that for any initial condition and zero input the output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$, i.e the system is output stable; it also shows that for initial condition zero and input in $l^2(\mathbb{Z}^+, \mathscr{U})$ the output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$, i.e. the system is input-output stable. □

The next result gives algebraic conditions for a system to be energy preserving.

Proposition 5.3. *A discrete-time system is energy preserving if and only if there exists a $L = L^* \in \mathcal{L}(\mathscr{X})$ such that*

$$A^*LA - L + C^*C = 0, \quad B^*LB + D^*D = I, \quad B^*LA + D^*C = 0. \quad (5.2)$$

This L is then a storage operator.

Proof. We first note that a system is energy preserving with storage operator L if and only if for all $w \in \mathscr{X}$ and $v \in \mathscr{U}$ we have

$$\langle L(Aw + Bv), Aw + Bv \rangle + \langle Cw + Dv, Cw + Dv \rangle = \langle Lw, w \rangle + \langle v, v \rangle. \quad (5.3)$$

Indeed, this equation is (5.1) for $n = 1$ with $x_0 = w$ and $u_0 = v$ and so it is clearly implied by (5.1). It follows using induction that (5.1) is implied by (5.3). Equation (5.3) can be written in the following form

$$\left\langle \begin{bmatrix} A^*LA - L + C^*C & A^*LB + C^*D \\ B^*LA + D^*C & B^*LB + D^*D - I \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle = 0,$$

which is equivalent to (5.2). □

Note that we already met the first equation of (5.2) in Chapter 3 and called it the observation Lyapunov equation (Definition 3.6).

Proposition 5.4. *Consider a discrete-time system for which the observability gramian is a storage operator. Let* $[u;x;y] \in \mathbb{B}$ *with* $x_0 = 0$ *and* $u \in l^2(\mathbb{Z}^+, \mathscr{U})$*. Then* $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$.

Proof. First assume that $u : \mathbb{Z}^+ \to \mathscr{U}$ is finitely nonzero. We have

$$\langle L_C x_n, x_n\rangle + \sum_{k=0}^{n-1} \|y_k\|^2 = \sum_{k=0}^{n-1} \|u_k\|^2.$$

By Proposition 3.10 we have $\langle L_C x_n, x_n\rangle \to 0$ as $n \to \infty$. So we obtain $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$ in case u is finitely nonzero. Using that the finitely nonzero sequences are dense in $l^2(\mathbb{Z}^+, \mathscr{U})$ and that the system is input-output stable by Proposition 5.2 we obtain the general case. □

Lemma 5.5. *Consider an output stable discrete-time system with the property that if* $[u;x;y] \in \mathbb{B}$ *with* $x_0 = 0$ *and* $u \in l^2(\mathbb{Z}^+, \mathscr{U})$*, then* $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$*. Then* $B^*L_C B + D^*D = I$.

Proof. Define the sequence u by $u_0 = v$ and $u_i = 0$ if $i > 0$. Let y denote the corresponding output for initial condition zero. Then, since $(y_n)_{n\geq 1}$ is the output for initial condition Bv and zero input and $y_0 = Dv$, we have (here $\mathcal{C}$ is the output map of the system)

$$\|y\|^2_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|\mathcal{C}Bv\|^2 + \|Dv\|^2 = \langle B^*L_C Bv, v\rangle + \langle D^*Dv, v\rangle.$$

Since $\|u\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})} = \|v\|^2$ we obtain the desired equality. □

Lemma 5.6. *Consider an approximately controllable output stable discrete-time system with the property that if* $[u;x;y] \in \mathbb{B}$ *with* $x_0 = 0$ *and* $u \in l^2(\mathbb{Z}^+, \mathscr{U})$*, then* $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$*. Then* $B^*L_C A + D^*C = 0$.

Proof. We first note that in a real Hilbert space we have

$$\langle h_1, h_2\rangle = \left(\|h_1 + h_2\|^2 - \|h_1\|^2 - \|h_2\|^2\right)/2,$$

and that in a complex Hilbert space a similar equation expressing the inner product in terms of the norm exists. This shows that from $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$ it follows that $\langle y^1, y^2\rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \langle u^1, u^2\rangle_{l^2(\mathbb{Z}^+,\mathscr{U})}$, where y^i is the output for input u^i and initial condition zero. By shift-invariance we obtain

$$\langle \mathcal{D}u^1, \mathcal{D}u^2\rangle_{l^2(\mathbb{Z},\mathscr{Y})} = \langle u^1, u^2\rangle_{l^2(\mathbb{Z},\mathscr{U})} \tag{5.4}$$

when both u^1 and u^2 have support bounded to the left. Here $\mathcal{D}$ is the input-output map of the system.

Let $u^1 : \mathbb{Z} \to \mathscr{U}$ have support bounded to the left and be zero on $\mathbb{Z}^+$. Let $v \in \mathscr{U}$ and define $u^2 : \mathbb{Z} \to \mathscr{U}$ by $u^2_0 = v$, $u^2_i = 0$ for $i \neq 0$. Since obviously $\langle u^1, u^2 \rangle_{l^2(\mathbb{Z},\mathscr{U})} = 0$ we obtain using (5.4) that $\langle \mathcal{D}u^1, \mathcal{D}u^2 \rangle_{l^2(\mathbb{Z},\mathscr{Y})} = 0$. Since u^2 equals zero on $\mathbb{Z}^-$ we have that $\mathcal{D}u^2$ equals zero on $\mathbb{Z}^-$ and using this we see that

$$\langle \mathcal{D}u^1, \mathcal{D}u^2 \rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = 0. \tag{5.5}$$

Since u^1 is zero on $\mathbb{Z}^+$, we have that $\mathcal{D}u^1$ restricted to $\mathbb{Z}^+$ equals $\mathcal{H}u^1$, where $\mathcal{H}$ is the Hankel map of the system. It follows that

$$\langle \mathcal{H}u^1, \mathcal{D}u^2 \rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = 0. \tag{5.6}$$

Define $w := \mathcal{B}u^1$, where $\mathcal{B}$ is the input map of the system. Then we see that $\mathcal{H}u^1 = \mathcal{C}\mathcal{B}u^1$, where $\mathcal{C}$ is the output map of the system, using Lemma 2.4. Since u^2_n equals zero for $n \geq 1$ we have $(\mathcal{D}u^2)_n = (\mathcal{C}Bv)_n$ for $n \geq 1$. Separating the first term in (5.6) we obtain

$$\langle Cw, Dv \rangle_{\mathscr{Y}} + \langle \mathcal{C}Aw, \mathcal{C}Bv \rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = 0.$$

Since this holds for all $v \in \mathscr{U}$ this implies $(B^* L_C A + D^* C)w = 0$. Since by approximate controllability $\mathcal{B}$ has dense range we obtain $B^* L_C A + D^* C = 0$ on a dense set. Hence $B^* L_C A + D^* C = 0$ by continuity. □

Combining the last two lemmas we obtain the following result.

Proposition 5.7. *An approximately controllable output stable discrete-time system with the property that if $[u; x; y] \in \mathbb{B}$ with $x_0 = 0$ and $u \in l^2(\mathbb{Z}^+, \mathscr{U})$, then $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$ is energy preserving with the observability gramian as storage operator.*

Proof. This follows immediately from Lemmas 5.5 and 5.6 combined with the algebraic conditions for energy preservation from Proposition 5.3. □

Proposition 5.8. *A discrete-time system that is energy preserving and input stable is input-output stable.*

Proof. Let $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and denote the output for this input and initial condition zero by y. Since the system is input stable the input map is bounded from $l^2(\mathbb{Z}^-, \mathscr{U})$ to $\mathscr{X}$ and we have

$$\|x_n\| \leq \|\mathcal{B}\| \; \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}.$$

Using that the system is energy preserving we obtain

$$\sum_{k=0}^{n-1} \|y_k\|^2 = \sum_{k=0}^{n-1} \|u_k\|^2 - \langle Lx_n, x_n \rangle \leq (1 + \|L\| \; \|\mathcal{B}\|^2) \; \|u\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})},$$

and so $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. □

Notes

The concept of energy-preserving, or more generally passive or dissipative, systems is well-established within systems and control theory. We refer to Staffans [89, Chapter 11] for more information on energy-preserving infinite-dimensional systems. Propositions 5.2 and 5.3 are rather obvious and well-known. We took most of the other results in this chapter from Curtain and Opmeer [16] and Opmeer and Curtain [71], but we make no priority claim, these results may have appeared elsewhere earlier.

Chapter 6

The linear quadratic optimal control problem

In this chapter we consider the best-studied problem in systems theory, the **linear quadratic optimal control problem**. This problem is also known as the linear quadratic regulator or **LQR-problem**. We first review several well-known results. Many of these are available in the literature only under stronger assumptions then the ones we impose. Towards the end of this chapter some completely new results are presented that are of crucial importance in later chapters.

Problem 6.1. For a given discrete-time system, consider the cost function

$$J(x_0, u) = \|u\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})} + \|y\|^2_{l^2(\mathbb{Z}^+,\mathscr{Y})}, \tag{6.1}$$

where y is the output for initial state x_0 and input u. The goal is the minimize this cost function over all inputs.

An obvious condition on the underlying system is that, for each initial state, there should exist an input that makes the cost finite. We formalize this in the following definition.

Definition 6.2. A discrete-time system satisfies the **finite cost condition** if the following holds. For each initial state $x_0 \in \mathscr{X}$ there exists an input $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ such that the corresponding output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$.

The principal ingredient in the solution of the LQR-problem is the following well-known result, which is often referred to as the **orthogonal projection lemma**.

Proposition 6.3. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{K}$ a nonempty closed subspace of $\mathscr{H}$. Define, for $h_0 \in \mathscr{H}$, the affine set*

$$\mathscr{K}(h_0) := \{h \in \mathscr{H} : h = h_0 + k \text{ for some } k \in \mathscr{K}\}.$$

Then there exists a unique $h_{\min} \in \mathscr{K}(h_0)$ *such that*

$$\|h_{\min}\| = \min_{h \in \mathscr{K}(h_0)} \|h\|.$$

$h_{\min}$ *is characterized by the fact that it is the unique fixed point in* $\mathscr{K}(h_0)$ *of the orthogonal projection onto* $\mathscr{K}^{\perp}$.

Proof. See for example Kreyszig [47, Section 3.3]. □

We will first analyze a certain set associated with the system. For a discrete-time system consider the set of **stable input-output pairs**

$$\mathscr{V}(x_0) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} l^2(\mathbb{Z}^+, \mathscr{U}) \\ l^2(\mathbb{Z}^+, \mathscr{Y}) \end{bmatrix} : y \text{ satisfies } (2.1) \right\}. \tag{6.2}$$

Note that $\mathscr{V}(x_0)$ is nonempty for every $x_0 \in \mathscr{X}$ if and only if the finite cost condition is satisfied. $\mathscr{V}(x_0)$ will play the role of $\mathscr{K}(h_0)$ in the orthogonal projection lemma.

Lemma 6.4. $\mathscr{V}(0)$ *is a closed linear subspace of* $l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$.

Proof. If $[u; y] \in \mathscr{V}(0)$, then

$$y_n = \sum_{k=0}^{n-1} CA^k B u_{n-k-1} + D u_n. \tag{6.3}$$

From this it is easily seen that $\mathscr{V}(0)$ is a linear space. We now prove that $\mathscr{V}(0)$ is closed. Let $[u^m; y^m] \in \mathscr{V}(0)$ and assume that there exist $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$ such that $u^m \to u$ in $l^2(\mathbb{Z}^+, \mathscr{U})$ and $y^m \to y$ in $l^2(\mathbb{Z}^+, \mathscr{Y})$. Then $u_n^m \to u_n$ in $\mathscr{U}$, from which we obtain

$$y_n^m = \sum_{k=0}^{n-1} CA^k B u_{n-k-1}^m + D u_n^m \to \sum_{k=0}^{n-1} CA^k B u_{n-k-1} + D u_n,$$

since we also have $y_n^m \to y_n$ in $\mathscr{Y}$ we obtain that y is the output corresponding to u. This shows that $\mathscr{V}(0)$ is closed. □

The next result establishes existence and uniqueness of the minimizing input.

Proposition 6.5. *If the finite cost condition is satisfied, then, for every* $x_0 \in \mathscr{X}$, *there exists a unique element in* $\mathscr{V}(x_0)$ *with minimal norm. This element is characterized by the fact that it is the unique fixed point in* $\mathscr{V}(x_0)$ *of the orthogonal projection onto* $\mathscr{V}(0)^{\perp}$.

Proof. We apply Proposition 6.3 with $\mathscr{H} = l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$ and $\mathscr{K} = \mathscr{V}(0)$.

Note that if $(u^1, y^1), (u^2, y^2) \in \mathscr{V}(x_0)$, then $(u^1 - u^2, y^1 - y^2) \in \mathscr{V}(0)$. So $\mathscr{V}(x_0)$ is a translation of the closed subspace $\mathscr{V}(0)$ just like $\mathscr{K}(h_0)$ is a translation of the closed set $\mathscr{K}$. $\mathscr{V}(0)$ is nonempty since it contains zero. That $\mathscr{V}(0)$ is a closed convex subset follows from Lemma 6.4. The above shows that all the conditions of Proposition 6.3 are fulfilled. This proposition now gives the desired result. □

Definition 6.6. Define for a system that satisfies the finite cost condition the operator

$$\mathcal{I}^+ : \mathscr{X} \to l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}), \qquad \mathcal{I}^+ w := \begin{bmatrix} u_w^{\min} \\ y_w^{\min} \end{bmatrix},$$

that assigns to $w \in \mathscr{X}$ the element of $\mathscr{V}(w)$ with minimal norm. This operator is called the **minimizing operator** of the system.

Proposition 6.7. *The minimizing operator is linear.*

Proof. Let $w_1, w_2 \in \mathscr{X}$. We shall prove that $\mathcal{I}^+(w_1 + w_2) = \mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$. Since the system is linear, we have that the output for initial state $w_1 + w_2$ and input $u_{w_1}^{\min} + u_{w_2}^{\min}$ is $y_{w_1}^{\min} + y_{w_2}^{\min}$. Hence $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2 \in \mathscr{V}(w_1 + w_2)$. Let P be the orthogonal projection onto $\mathscr{V}(0)^\perp$. Since $\mathcal{I}^+ w_1$ and $\mathcal{I}^+ w_2$ are both fixed points of P, it follows that $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$ is. So $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$ is a fixed point of P in $\mathscr{V}(w_1 + w_2)$. Since by Proposition 6.5 the element of $\mathscr{V}(\cdot)$ with minimal norm is the unique fixed point of P in this set, it follows that $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$ is the element of minimal norm in $\mathscr{V}(w_1 + w_2)$. Hence $\mathcal{I}^+(w_1 + w_2) = \mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$. □

Proposition 6.8. *The minimizing operator is bounded.*

Proof. We show that the minimizing operator is closed. It then follows from the closed graph theorem that it is bounded. Let $w^k \in \mathscr{X} \to w^\infty$ in $\mathscr{X}$, $\mathcal{I}^+ w^k = [u_{w^k}^{\min}; y_{w^k}^{\min}] \to [u^\infty; y^\infty]$ in $l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$. We need to show that $[u_{w^\infty}^{\min}; y_{w^\infty}^{\min}] = [u^\infty; y^\infty]$.

The output y for initial condition w and input u is given by

$$y_n = CA^n w + \sum_{i=0}^{n-1} CA^i B u_{n-1-i} + D u_n.$$

Applying this with $w = w^k$ and $u = u_{w^k}^{\min}$ we obtain

$$\left(y_{w^k}^{\min}\right)_n = CA^n w^k + \sum_{i=0}^{n-1} CA^i B \left(u_{w^k}^{\min}\right)_{n-1-i} + D \left(u_{w^k}^{\min}\right)_n .$$

Taking the limit for $k \to \infty$ we obtain

$$y_n^\infty = CA^n w^\infty + \sum_{i=0}^{n-1} CA^i B u_{n-1-i}^\infty + D u_n^\infty.$$

This shows that the output for initial state w^∞ and input u^∞ is y^∞. This shows that $[u^\infty; y^\infty] \in \mathscr{V}(w)$. We show that $[u_{w^\infty}^{\min}; y_{w^\infty}^{\min}] = [u^\infty; y^\infty]$ by proving the latter is a fixed point of the projection onto $\mathscr{V}(0)^\perp$. Since $[u_{w^k}^{\min}; y_{w^k}^{\min}]$ is the element with minimal norm in $\mathscr{V}(w^k)$, we have

$$P_{\mathscr{V}(0)^\perp} \begin{bmatrix} u_{w^k}^{\min} \\ y_{w^k}^{\min} \end{bmatrix} = \begin{bmatrix} u_{w^k}^{\min} \\ y_{w^k}^{\min} \end{bmatrix}.$$

Letting $k \to \infty$ we obtain

$$P_{\mathscr{V}(0)^\perp} \begin{bmatrix} u^\infty \\ y^\infty \end{bmatrix} = \begin{bmatrix} u^\infty \\ y^\infty \end{bmatrix}.$$

So $[u^\infty; y^\infty]$ is indeed a fixed point of the projection onto $\mathscr{V}(0)^\perp$. Since $[u^\infty; y^\infty] \in \mathscr{V}(w)$, and by the uniqueness of the fixed point, we have $[u_{w^\infty}^{\min}; y_{w^\infty}^{\min}] = [u^\infty; y^\infty]$. $\square$

Definition 6.9. Define the following sesquilinear form for a system that satisfies the finite cost condition

$$q^{\min} : \mathscr{X} \times \mathscr{X} \to \mathbb{C}, \quad q^{\min}(w_1, w_2) = \langle \mathcal{I}^+ w_1, \mathcal{I}^+ w_2 \rangle.$$

This sesquilinear form is called the **optimal cost sesquilinear form** of the system.

Note that $q^{\min}(w, w)$ is the optimal cost for the initial condition w. We remind the reader that a sesquilinear form (linear in the first variable and anti-linear in the second) f is called hermitian if $f(x, y) = \overline{f(y, x)}$ for all x and y, nonnegative if $f(x, x) \geq 0$ for all x and positive if $f(x, x) > 0$ for all nonzero f.

Proposition 6.10. *The optimal cost sesquilinear form is continuous, hermitian and nonnegative.*

Proof. The optimal cost sesquilinear form is continuous, since the minimizing operator is continuous by Proposition 6.8. That it is hermitian, nonnegative follows immediately from the definition. $\square$

Definition 6.11. For a system that satisfies the finite cost condition define the bounded self-adjoint nonnegative linear operator $Q^{\min} \in \mathcal{L}(\mathscr{X})$ by

$$q^{\min}(w_1, w_2) = \langle Q^{\min} w_1, w_2 \rangle.$$

This operator is called the **optimal cost operator** of the system.

Proposition 6.12. *Assume that Σ satisfies the finite cost condition. Σ is approximately observable if and only if the optimal cost sesquilinear form is positive (or equivalently, the optimal cost operator is positive).*

Proof. Assume that the optimal cost sesquilinear form is not positive. Then there exists a nonzero $w \in \mathscr{X}$ with zero optimal cost. It follows that the output for initial state w and zero input is zero. This contradicts approximate observability.

Assume that Σ is not approximately observable. Then there exists a nonzero $w \in \mathscr{X}$ such that with w as initial state and zero input the output is zero. It follows that the optimal cost with w as initial state is zero. Hence the optimal cost sesquilinear form is not positive. □

Definition 6.13. For a system that satisfies the finite cost condition define the operator

$$F^{\min} : \mathscr{X} \to \mathscr{U}, \quad F^{\min} w = \left(u_w^{\min}\right)_0.$$

This operator is called the **optimal cost feedback operator**.

Proposition 6.14. *The optimal cost feedback operator is linear and bounded.*

Proof. Denote by P the projection from $l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$ onto the $\mathscr{U}$-component of the zero-th coordinate. Then $F^{\min} = P\mathcal{I}^+$. Since both P and $\mathcal{I}^+$ are linear and bounded it follows that $F^{\min}$ is. □

Proposition 6.15. *For every initial state $x_0 \in \mathscr{X}$ and input $u : \mathbb{Z}^+ \to \mathscr{U}$ we have*

$$q^{\min}(x_0) \leq \|u_0\|^2 + \|y_0\|^2 + q^{\min}(x_1), \tag{6.4}$$

where y is the output and x the state. Equality holds if and only if $u_0 = (u_{x_0}^{\min})_0$.

Proof. For notational simplicity we denote $u_{x_0}^{\min}$ by $u^{\min}$ in this proof. The input $[u_0, u_1^{\min}, u_2^{\min}, \ldots]$ is denoted by v. If (6.4) would not hold, then the input v would have a strictly lower cost than $u^{\min}$, which is impossible by definition of $u^{\min}$. So (6.4) must hold. If we have an equality in (6.4), then $u^{\min}$ and v give rise to the same cost. By the uniqueness of the optimal input we have $u^{\min} = v$ and so $u_0 = (u_{x_0}^{\min})_0$. □

Proposition 6.16. *For every initial state $x_0 \in \mathscr{X}$ and input $u : \mathbb{Z}^+ \to \mathscr{U}$ we have*

$$
\begin{aligned}
& q^{\min}(x_0) \le \langle Cx_0, Cx_0\rangle + q^{\min}(Ax_0) \\
& -\langle S^{-1}(B^*Q^{\min}A + D^*C)x_0, (B^*Q^{\min}A + D^*C)x_0\rangle \qquad (6.5)\\
& +\langle u_0 + S^{-1}(B^*Q^{\min}A + D^*C)x_0, S\left(u_0 + S^{-1}(B^*Q^{\min}A + D^*C)x_0\right)\rangle,
\end{aligned}
$$

where $S := I + D^*D + B^*Q^{\min}B$. Equality holds if and only if $u_0 = u^{\min}_{x_0}$.

Proof. Some elementary algebraic manipulations show that the right-hand side of (6.5) is identical to the right-hand side of (6.4). The statement then follows from Proposition 6.15. □

Proposition 6.17. *Given $x_0 \in \mathscr{X}$ we have equality in* (6.5) *if and only if $u_0 = \left(I + D^*D + B^*Q^{\min}B\right)^{-1}(B^*Q^{\min}A + D^*C)x_0$.*

Proof. According to Proposition 6.16, for every $u_0 \in \mathscr{U}$ the inequality (6.5) holds and for exactly one we have equality. It follows that this u_0 is the one that minimizes

$$
\begin{aligned}
&\langle Cx_0, Cx_0\rangle + q^{\min}(Ax_0) - \langle S^{-1}(B^*Q^{\min}A + D^*C)x_0, (B^*Q^{\min}A + D^*C)x_0\rangle \\
&\quad +\langle u_0 + S^{-1}(B^*Q^{\min}A + D^*C)x_0, S\left(u_0 + S^{-1}(B^*Q^{\min}A + D^*C)x_0\right)\rangle.
\end{aligned}
$$

The first three terms do not depend on u_0. It follows that equality holds only for that u_0 that minimizes

$$
\langle u_0 + S^{-1}(B^*Q^{\min}A + D^*C)x_0, S\left(u_0 + S^{-1}(B^*Q^{\min}A + D^*C)x_0\right)\rangle.
$$

This function is nonnegative since S is nonnegative. It is zero if and only if $u_0 = -\left(I + D^*D + B^*Q^{\min}B\right)^{-1}(B^*Q^{\min}A + D^*C)x_0$. It follows that with this u_0 and only with this u_0, we have equality in (6.5). □

Proposition 6.18. *The optimal cost feedback operator can be written in terms of the optimal cost operator as follows:*

$$
F^{\min} = -\left(I + D^*D + B^*Q^{\min}B\right)^{-1}(B^*Q^{\min}A + D^*C).
$$

Proof. This follows from Proposition 6.16 which says that we have equality in (6.5) if and only if $u_0 = u^{\min}_{x_0}$ and Proposition 6.17 which says that we have equality in (6.5) if and only if $u_0 = -\left(I + D^*D + B^*Q^{\min}B\right)^{-1}(B^*Q^{\min}A + D^*C)x_0$. □

Proposition 6.19. *The optimal cost operator satisfies*

$$\begin{aligned}&A^*Q^{\min}A - Q^{\min} + C^*C\\&-(A^*Q^{\min}B + D^*C)(I + D^*D + B^*Q^{\min}B)^{-1}(AQ^{\min}B^* + DC^*) = 0.\end{aligned}$$

Proof. This follows from substituting u_0 from Proposition 6.17 into (6.5). □

Definition 6.20. The equation

$$\begin{aligned}&A^*QA - Q + C^*C\\&-(C^*D + A^*QB)(I + D^*D + B^*QB)^{-1}(D^*C + B^*QA) = 0.\end{aligned} \tag{6.6}$$

is called the **control algebraic Riccati equation**. We consider only bounded self-adjoint nonnegative solutions of this equation. With a nonnegative self-adjoint solution $Q \in \mathcal{L}(\mathscr{X})$ we associate the following operators:

$$S := I + D^*D + B^*QB, \quad F := -S^{-1}(D^*C + B^*QA). \tag{6.7}$$

For a bounded nonnegative seld-adjoint solution Q of the control algebraic Riccati equation and S as above, define the sesquilinear forms q and s by $q(x_1, x_2) := \langle Qx_1, x_2\rangle_{\mathscr{X}}$ and $s(u_1, u_2) := \langle Su_1, u_2\rangle_{\mathscr{U}}$, respectively. The triple (q, s, F) is called a **control Riccati triple**.

The next two propositions give alternative characterizations of control Riccati triples.

Proposition 6.21. *The triple (q, s, F) is a control Riccati triple if and only if*

- *$q : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ is a bounded nonnegative hermitian sesquilinear form.*
- *$s : \mathscr{U} \times \mathscr{U} \to \mathbb{C}$ is a bounded nonnegative hermitian sesquilinear form.*
- *$F : \mathscr{X} \to \mathscr{U}$ is a bounded linear operator.*
- *For all $w \in \mathscr{X}$, $u \in \mathscr{U}$ we have*

$$\begin{aligned} q(Aw) + \|Cw\|^2_{\mathscr{Y}} &= q(w) + s(Fw),\\ s(u) &= \|u\|^2_{\mathscr{U}} + \|Du\|^2_{\mathscr{Y}} + q(Bu),\\ -s(Fw, u) &= \langle Cw, Du\rangle_{\mathscr{Y}} + q(Aw, Bu).\end{aligned} \tag{6.8}$$

Proof. The second equation of (6.8) is easily seen to be equivalent to the definition of S in (6.7). The third equation of (6.8) is then seen to be equivalent to the definition of F in (6.7). Finally it follows that the first equation of (6.8) is equivalent to Q satisfying the control algebraic Riccati equation. □

Proposition 6.22. *The triple (q, s, F) is a control Riccati triple if and only if*

- $q : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ *is a bounded nonnegative hermitian sesquilinear form.*
- $s : \mathscr{U} \times \mathscr{U} \to \mathbb{C}$ *is a bounded nonnegative hermitian sesquilinear form.*
- $F : \mathscr{X} \to \mathscr{U}$ *is a bounded linear operator.*
- *For all $w \in \mathscr{X}$, $u \in \mathscr{U}$ we have*

$$q(Aw + Bu) + \|Cw + Du\|^2_{\mathscr{Y}} + \|u\|^2_{\mathscr{U}} = q(w) + s(Fw - u). \quad (6.9)$$

Proof. Writing out (6.9) shows that it is equivalent to

$$\begin{aligned} q(Aw) + q(Bu) + q(Aw, Bu) + q(Bu, Aw) + \|Cw\|^2 + \|Du\|^2 \\ + \langle Cw, Du\rangle + \langle Du, Cw\rangle + \|u\|^2 \\ = q(w) + s(Fw) + s(u) - s(Fw, u) - s(u, Fw). \end{aligned} \quad (6.10)$$

Using equations (6.8) we see that this holds. The first equation of (6.8) is (6.9) with $u = 0$, the second with $w = 0$. Using these first two equations we obtain that (6.10) reads

$$-[s(Fw, u) + s(u, Fw)] = q(Aw, Bu) + q(Bu, Aw) + \langle Cw, Du\rangle + \langle Du, Cw\rangle,$$

which is equivalent to

$$-\mathrm{Re}(s(Fw, u)) = \mathrm{Re}(\langle Cw, Du\rangle + q(Aw, Bu)).$$

Applying the above with iw instead of w gives equality of the imaginary parts of the third equation of (6.8). □

From Proposition 6.22 we obtain the following by induction.

Proposition 6.23. *If (q, s, F) is a control Riccati triple for the system Σ and $[u; x; y] \in \mathbb{B}$, then*

$$q(x_n) + \sum_{k=0}^{n-1} \|u_k\|^2 + \|y_k\|^2 = q(x_0) + \sum_{k=0}^{n-1} s(Fx_k - u_k).$$

Proof. This follows from (6.9) using induction. □

Proposition 6.24. *Let (q, s, F) be a control Riccati triple. Then for the input defined by $u_n := Fx_n$ we have*

$$J(x_0, u) \le q(x_0),$$

where J is the cost function (6.1).

Proof. Proposition 6.23 with $u_k = Fx_k$ gives

$$q(x_n) + \sum_{k=0}^{n-1} \|u_k\|^2 + \|y_k\|^2 = q(x_0).$$

Since $q \geq 0$ we obtain from this

$$\sum_{i=0}^{n-1} \|u_k\|^2 + \|y_k\|^2 \leq q(x_0).$$

Letting $n \to \infty$ gives the desired result. □

Proposition 6.25. *If a discrete-time system has a bounded nonnegative self-adjoint solution to its control algebraic Riccati equation, then the discrete-time system satisfies the finite cost condition.*

Proof. Proposition 6.24 shows that, for given $x_0 \in \mathscr{X}$, the input defined by $u_n := Fx_n$ gives rise to a finite cost. □

Proposition 6.26. *Assume that the discrete-time system Σ satisfies the finite cost condition. Let (q, s, F) be a control Riccati triple of Σ. Then $q^{\min} \leq q$, where $q^{\min}$ is the optimal cost sesquilinear form of Σ.*

Proof. This follows from Proposition 6.24 since

$$q^{\min}(x_0) \leq J(x_0, u) \leq q(x_0),$$

where u is the input defined in Proposition 6.24. □

Corollary 6.27. *The optimal cost operator is the smallest bounded nonnegative self-adjoint solution of the control algebraic Riccati equation.*

Proof. This is a reformulation of Proposition 6.26. □

Proposition 6.28. *Let Σ satisfy the finite cost condition and let $[u; x; y] \in \mathbb{B}$ with $[u; y] \in \mathscr{V}(x_0)$. Then*

$$\lim_{n\to\infty} q^{\min}(x_n) = 0.$$

Proof. Since $q^{\min}(x_n)$ is the optimal cost when starting from state x_n we have

$$q^{\min}(x_n) \leq J(x_n, [u_n, u_{n+1}, \ldots]) = \sum_{k=n}^{\infty} \|u_k\|^2 + \|y_k\|^2.$$

The right hand side converges to zero since $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. It follows that the left-hand side converges to zero as desired. □

Combining Propositions 6.23 and 6.28 we obtain the following.

Proposition 6.29. *Let Σ satisfy the finite cost condition and let $[u; x; y] \in \mathbb{B}$ with $[u; y] \in \mathscr{V}(x_0)$. Then*

$$\sum_{k=0}^{\infty} \|u_k\|^2 + \|y_k\|^2 = q^{\min}(x_0) + \sum_{k=0}^{\infty} s^{\min}(F^{\min} x_k - u_k).$$

Proof. This follows by letting $n \to \infty$ in Proposition 6.23 and using Proposition 6.28. □

Proposition 6.30. *Let Σ satisfy the finite cost condition and let $[u; x; y] \in \mathbb{B}$ with $[u; y] \in \mathscr{V}(x_0)$. Then $(F^{\min} x_k)_{k \geq 0}$ is in $l^2(\mathbb{Z}^+, \mathscr{U})$.*

Proof. It follows from Proposition 6.29 that

$$\sum_{k=0}^{\infty} s^{\min}(F^{\min} x_k - u_k) < \infty.$$

We also have

$$\sum_{k=0}^{\infty} \|F^{\min} x_k - u_k\|^2 \leq \sum_{k=0}^{\infty} \|(S^{\min})^{-1}\| \; s^{\min}(F^{\min} x_k - u_k)$$

and so $(F^{\min} x_k - u_k) \in l^2(\mathbb{Z}^+, \mathscr{U})$. Since $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ we have $(F^{\min} x_k) \in l^2(\mathbb{Z}^+, \mathscr{U})$. □

The following proposition gives another alternative characterization of control Riccati triples.

Proposition 6.31. *The equation* (6.9) *is equivalent to the following triple of equations.*

$$\begin{aligned} q(w) &= q((A + BF)w) + \|(C + DF)w\|^2_{\mathscr{Y}} + \|Fw\|^2_{\mathscr{U}}, \\ \|u\|^2_{\mathscr{U}} &= \|S^{-1/2}u\|^2_{\mathscr{U}} + \|DS^{-1/2}u\|^2_{\mathscr{Y}} + q(BS^{-1/2}u), \\ 0 &= \langle (C + DF)w, DS^{-1/2}u\rangle_{\mathscr{Y}} \\ &\quad + \langle Fw, S^{-1/2}u\rangle_{\mathscr{U}} + q((A + BF)w, BS^{-1/2}u). \end{aligned} \tag{6.11}$$

Proof. The second equation of (6.11) is easily seen to be equivalent to the formula for S in (6.7). The third equation of (6.11) is then seen to be equivalent to the formula for F in (6.7). Using this the first equation is seen to be equivalent to the control algebraic Riccati equation. □

To investigate the connection between the control algebraic Riccati equation and output stabilizability we introduce the following concept.

Definition 6.32. The **Riccati closed-loop system** associated with a control Riccati triple (q, s, F) is defined through its system operator

$$\left[\begin{array}{c|c} A+BF & BS^{-1/2} \\ \hline F & S^{-1/2} \\ C+DF & DS^{-1/2} \end{array}\right]. \tag{6.12}$$

In the case that $(q, s, F) = (q^{\min}, s^{\min}, F^{\min})$ the Riccati closed-loop system is called the **optimal closed loop system**.

Proposition 6.33. *Let (q, s, F) be a control Riccati triple. Then $[S^{1/2}F, I - S^{1/2}]$ is an admissible feedback pair and the corresponding closed-loop system is the Riccati closed-loop system.*

Proof. This is elementary. □

Proposition 6.34. *Let (q, s, F) be a control Riccati triple for the system Σ. Then the Riccati closed-loop system is energy preserving with storage operator Q. Hence the Riccati closed-loop system is output stable and input-output stable and Σ is output stabilizable.*

Proof. The necessary and sufficient conditions for energy preservation from Proposition 5.3 applied to the Riccati closed-loop system are exactly the equations (6.11). It follows from Proposition 5.2 that the Riccati closed-loop system is output stable and input-output stable. Since the Riccati closed-loop system is obtained from Σ by an admissible feedback pair, it follows that Σ is output stabilizable. □

In the case of the optimal closed-loop system we can say a bit more.

Proposition 6.35. *The observability gramian of the optimal closed loop system is $Q^{\min}$.*

Proof. Let $\mathcal{C}^{\min}$ be the output map of the optimal closed-loop system. We have

$$\langle L_C x_0, x_0 \rangle = \|\mathcal{C}^{\min} x_0\|^2_{l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})} = \|u^{\min}\|^2_{l^2(\mathbb{Z}^+, \mathscr{U})} + \|y^{\min}\|^2_{l^2(\mathbb{Z}^+, \mathscr{Y})} = q^{\min}(x_0).$$

It follows that $L_C = Q^{\min}$. □

Proposition 6.36. *The following are equivalent statements about a discrete-time system Σ.*

1. *Σ satisfies the finite cost condition.*
2. *Σ is output stabilizable.*
3. *The control algebraic Riccati equation of Σ has a bounded nonnegative self-adjoint solution.*

Proof. (1) implies (3) follows from Proposition 6.19 which shows that the optimal cost operator is a solution of the control algebraic Riccati equation. (3) implies (2) is contained in Proposition 6.34. (2) implies (1) follows by choosing $u_n := (I - G)^{-1} F x_n$ where $[F, G]$ is the output stabilizing admissible feedback pair. Since the feedback pair is output stabilizing, it follows that, for each $x_0 \in \mathscr{X}$, $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and the corresponding output $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. Hence for each $x_0 \in \mathscr{X}$ the set of stable input-output pairs $\mathscr{V}(x_0)$ is nonempty. □

Definition 6.37. The triple (p, r, L) is called a **filter Riccati triple** of Σ if it is a control Riccati triple for the dual system of Σ.

All the results obtained for control Riccati triples have obvious counterparts for filter Riccati triples. In particular, the existence of a filter Riccati triple is equivalent to the following **filter algebraic Riccati equation** having a nonnegative self-adjoint solution $P \in \mathcal{L}(\mathscr{X})$

$$\begin{aligned} &APA^* - P + BB^* \\ &-(APC^* + BD^*)(I + DD^* + CPC^*)^{-1}(CPA^* + DB^*) = 0. \end{aligned} \tag{6.13}$$

In the proofs of the next few results (Proposition 6.38 up to Proposition 6.46) we need some algebraic calculations involving the control algebraic Riccati equation and the filter algebraic Riccati equation that can be found in Appendix B.

Proposition 6.38. *Let Σ be an input and output stabilizable discrete-time system. Assume there exists a control Riccati triple (q, s, F) such that the main operator of the corresponding Riccati closed-loop system is strongly stable. Then (q, s, F) is the unique control Riccati triple of Σ.*

Proof. For the proof we need the following algebraic relations, which are proven in Appendix B (Lemmas B.4 and B.5). Lemma B.4 gives the following relation between the main operator A_Q of the Riccati closed-loop system corresponding to an arbitrary control Riccati triple (q, s, F) and $A_P := A - (BD^* + APC^*)(I + DD^* + CPC^*)^{-1}C$, where P is a bounded nonnegative self-adjoint solution of the filter algebraic Riccati equation:

$$(I + PQ)A_Q = A_P(I + PQ). \tag{6.14}$$

The following algebraic relation is also proven in Appendix B (Lemma B.5). If Q_1 and Q_2 are bounded nonnegative self-adjoint solutions of the control algebraic Riccati equation and A_{Q_1} and A_{Q_2} denote the main operators of the corresponding Riccati closed-loop systems, then

$$Q_1 - Q_2 = A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}. \tag{6.15}$$

By induction it follows that for all $n \in \mathbb{Z}^+$ we have

$$Q_1 - Q_2 = A_{Q_2}^{*n}(Q_1 - Q_2)A_{Q_1}^n. \tag{6.16}$$

Using these facts we now prove the proposition. Since Σ is input stabilizable, there exists a bounded nonnegative self-adjoint solution P of the filter algebraic Riccati equation. Since A_Q is assumed to be strongly stable and (6.14) shows that A_P is similar to A_Q, we have that A_P is also strongly stable. Now let $\tilde{Q}$ be an arbitrary bounded nonnegative self-adjoint solution of the control algebraic Riccati equation. According to (6.14), $A_{\tilde{Q}}$ is similar to the strongly stable operator A_P and hence is strongly stable. Since $A_{\tilde{Q}}$ is strongly stable there exists for every $x \in \mathscr{X}$ a real number c_x such that for every $n \in \mathbb{Z}^+$ we have $\|A_{\tilde{Q}}^n x\| \leq c_x$. By the uniform boundedness theorem this implies that there exists a real number c such that for every $n \in \mathbb{Z}^+$ we have $\|A_{\tilde{Q}}^n\| \leq c$. Using (6.16) with $Q_1 = Q$ and $Q_2 = \tilde{Q}$ we have for all $x \in \mathscr{X}$ and $n \in \mathbb{Z}^+$

$$\|(Q-\tilde{Q})x\| = \|A_{\tilde{Q}}^{*n}(Q-\tilde{Q})A_Q^n x\| \leq \|A_{\tilde{Q}}^{*n}\| \, \|Q-\tilde{Q}\| \, \|A_Q^n x\| \leq c \, \|Q-\tilde{Q}\| \, \|A_Q^n x\|.$$

Since A_Q is strongly stable, the right-hand side converges to zero as $n \to \infty$. This implies that the left-hand side is zero and so $\tilde{Q} = Q$. □

Proposition 6.39. *Let Σ be a discrete-time system. Assume that its control algebraic Riccati equation has a bounded nonnegative self-adjoint solution Q and that its filter algebraic Riccati equation has a bounded nonnegative self-adjoint solution P. Then the control Lyapunov equation of the Riccati closed-loop system corresponding to Q has a solution $L_b := (I + PQ)^{-1}P = P^{1/2}(I + P^{1/2}QP^{1/2})^{-1}P^{1/2} \geq 0$.*

Proof. This is proven in Appendix B on page 180. □

Corollary 6.40. *Let Σ be an input and output stabilizable discrete-time system. Then the Riccati closed-loop system associated with any solution of the control algebraic Riccati equation is input, output and input-output stable.*

Proof. That the Riccati closed-loop system is input and input-output stable follows from Proposition 6.34. From Proposition 6.39 we obtain that the control Lyapunov equation of the Riccati closed-loop system has a bounded nonnegative self-adjoint solution. It follows from Corollary 3.4 that the Riccati closed-loop system is input stable. □

Corollary 6.41. *Let Σ be an input and output stabilizable discrete-time system. Then the Hankel map of the Riccati closed-loop system associated with any solution of the control algebraic Riccati equation has norm strictly smaller than one.*

Proof. Using Propositions 6.34 and 6.39 we obtain solutions $L_c = Q$ and $L_b := (I + PQ)^{-1}P$ of the Lyapunov equations of the Riccati closed-loop system. So we have $L_c L_b = PQ(I + PQ)^{-1}$. We prove that the spectral radius of $L_c L_b$ is strictly smaller than one. Lemmas 3.18 and 3.19 then give the result. We have $r(L_c L_b) = r(Q^{1/2}PQ^{1/2}(I + Q^{1/2}PQ^{1/2})^{-1})$. It is easily seen that for any nonnegative self-adjoint operator T we have $T(I+T)^{-1} < I$. Denote $T := Q^{1/2}PQ^{1/2}$. Then from the above we obtain $r(L_c L_b) < 1$. □

The following lemma on square roots of operators is needed in the proof of Proposition 6.43.

Lemma 6.42. *Let $P, Q \in \mathcal{L}(\mathscr{H})$ be nonnegative self-adjoint. Define $L := (I + PQ)^{-1}P$. Then, for all $h \in \mathscr{H}$,*

$$\|L^{1/2}h\| \leq \|P^{1/2}h\| + \frac{2}{\pi}\ (2 + \|L\|)\ \|Q\|\ \|Ph\|.$$

Proof. According to Kato [42, Lemma V.3.43 page 284] we have the following representation for the square root of a bounded nonnegative self-adjoint operator T:

$$T^{1/2}h = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(\lambda I + T)^{-1}Th\, d\lambda$$

and we have the following resolvent estimate [42, equation (V.3.38) page 279]

$$\|(\lambda I + T)^{-1}\| \leq \frac{1}{\lambda}$$

for $\lambda > 0$. Applying this with L and P we obtain

$$L^{1/2}h - P^{1/2}h = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\left[(\lambda I + L)^{-1}L - (\lambda I + P)^{-1}P\right] h\, d\lambda$$

and some rewriting of the integrand shows that this equals

$$\frac{1}{\pi}\int_0^\infty \lambda^{1/2}(\lambda I + L)^{-1}LQ(\lambda I + P)^{-1}Ph\, d\lambda.$$

Using the above resolvent estimate we obtain

$$\|\lambda^{1/2}(\lambda I + L)^{-1}LQ(\lambda I + P)^{-1}Ph\| \leq \lambda^{-3/2}\ \|L\|\ \|Q\|\ \|Ph\|$$

and so

$$\| \int_1^\infty \lambda^{1/2} (\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph \, d\lambda \| \leq 2\|L\| \; \|Q\| \; \|Ph\|.$$

Since $(\lambda I + L)^{-1} L = I - \lambda(\lambda I + L)^{-1}$ we obtain from the above resolvent estimate $\|(\lambda I + L)^{-1} L\| \leq 2$ and so

$$\|\lambda^{1/2} (\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph\| \leq 2\lambda^{-1/2} \; \|Q\| \; \|Ph\|,$$

which gives

$$\| \int_0^1 \lambda^{1/2} (\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph \, d\lambda \| \leq 4 \; \|Q\| \; \|Ph\|.$$

Combining the above two estimates we obtain

$$\| \int_0^\infty \lambda^{1/2} (\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph \, d\lambda \| \leq 2 \; (2 + \|L\|) \; \|Q\| \; \|Ph\|$$

and so

$$\|L^{1/2} h - P^{1/2} h\| \leq \frac{2}{\pi} \; (2 + \|L\|) \; \|Q\| \; \|Ph\|,$$

which gives

$$\|L^{1/2} h\| \leq \|P^{1/2} h\| + \frac{2}{\pi} \; (2 + \|L\|) \; \|Q\| \; \|Ph\|,$$

as desired. □

Proposition 6.43. *Let Σ be an input and output stabilizable discrete-time system. Let Q be a solution of the control algebraic Riccati equation of Σ and denote the optimal cost operator of the dual system of Σ by $P^{\min}$. Then the controllability gramian of the Riccati closed-loop system associated with Q is $L_B = (I + P^{\min} Q)^{-1} P^{\min}$.*

Proof. Proposition 6.35 applied to the dual system Σ_{dual} of Σ shows that $P^{\min}$ is the observability gramian of the optimal closed-loop system of Σ_{dual}. It follows from Lemma 3.7 that for all $h \in \mathscr{X}$ we have $(P^{\min})^{1/2} A_d^n h \to 0$ as $n \to \infty$, where A_d is the main operator of the optimal closed-loop system of Σ_{dual}. The operator A_d is given explicitely by

$$A_d = A^* - C^* (I + DD^* + C^* P^{\min} C)^{-1} (DB^* + CP^{\min} A^*).$$

Lemma B.4 gives

$$(I + P^{\min}Q)A_Q = A_{P^{\min}}(I + P^{\min}Q), \tag{6.17}$$

where $A_{P^{\min}} = A_d^*$ and A_Q is the main operator of the Riccati closed-loop system of Σ associated with Q. Since A_d is the adjoint of $A_{P^{\min}}$ it follows that $(P^{\min})^{1/2}A^{*n}_{P^{\min}}h \to 0$. Using (6.17) we obtain that

$$(P^{\min})^{1/2}(I + QP^{\min})^{-1}A_Q^{*n}(I + QP^{\min})h \to 0.$$

Using that $(I + (P^{\min})^{1/2}Q(P^{\min})^{1/2})^{-1}(P^{\min})^{1/2} = (P^{\min})^{1/2}(I + QP^{\min})^{-1}$ it follows that $(P^{\min})^{1/2}A_Q^{*n}w \to 0$ for all $w \in \mathscr{X}$.

Define $L := (I + P^{\min}Q)^{-1}P^{\min}$. It follows from Proposition 6.39 that L is a solution of the control Lyapunov equation of the Riccati closed-loop system. Lemma 6.42 gives

$$\|L^{1/2}h\| \le \|(P^{\min})^{1/2}h\| + \frac{2}{\pi}\,(2 + \|L\|)\,\|Q\|\,\|P^{\min}h\|.$$

With $h = A_Q^{*n}w$ we obtain from this that $L^{1/2}A_Q^{*n}w \to 0$ for all $w \in \mathscr{X}$. By Lemma 3.13 we obtain that L is the controllability gramian. □

Proposition 6.44. *Let Σ be an input and output stabilizable discrete-time system. Let $Q^{\min}$ be the optimal cost operator of Σ and denote the optimal cost operator of the dual system of Σ by $P^{\min}$. Denote the Hankel map of the optimal closed-loop system by $\mathcal{H}$. Then $\|\mathcal{H}\|^2 = r((I + P^{\min}Q^{\min})^{-1}P^{\min}Q^{\min})$.*

Proof. By Lemma 3.18 we have $\|\mathcal{H}\|^2 = r(L_BL_C)$, where L_B is the controllability gramian and L_C the observability gramian of the optimal closed-loop system. Proposition 6.35 shows that $L_B = Q^{\min}$ and Proposition 6.43 shows that $L_C = (I + P^{\min}Q^{\min})^{-1}P^{\min}$. The desired result follows. □

Proposition 6.45. *Let $\check{\Sigma}$ be an energy preserving discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Assume that $\check{D}_1$ has a bounded inverse and that the storage operator L is nonnegative self-adjoint. Define the system Σ as in Proposition 2.23. Then L is a solution of the control algebraic Riccati equation of Σ.*

Proof. Define $Q := L$, $S := \check{D}_1^{-*}\check{D}_1^{-1}$, $F := \check{C}_1$. One easily checks the equations (6.11) using the equations from Proposition 5.3 applied to $\check{\Sigma}$. □

Proposition 6.46. *Let $\check{\Sigma}$ be an energy preserving discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Assume that $\check{D}_1$ has a bounded inverse and that the storage operator L_c is nonnegative self-adjoint. Further* *assume that $\check{\Sigma}$ is input stable. Define the system Σ as in Proposition 2.23. Let L_b be a solution of the control Lyapunov equation of $\check{\Sigma}$ and assume that $1 \notin \sigma(L_b L_c)$. Then $P := (I - L_b L_c)^{-1} L_b$ is a solution of the filter algebraic Riccati equation of Σ.*

Proof. This is proven in Appendix B on page 183. □

In the following four Propositions 6.47-6.50, we compare the closed-loop systems associated with different control Riccati triples. These propositions are used in the chapter on coprime factorization (Chapter 7) to show that all Riccati closed-loop systems provide a strongly right-coprime factorization using the the optimal closed-loop system does.

The first of these propositions shows the relation between the transfer functions.

Proposition 6.47. *Let Σ be an output stabilizable discrete-time system. Let Σ_i $(i = 1, 2)$ be the Riccati closed-loop system associated with the control Riccati triple (q_i, s_i, F_i). Let S_i and Q_i be the operators corresponding to the sesquilinear forms s_i and q_i, respectively. Let Σ_s be the discrete-time system with system operator*

$$\left[\begin{array}{c|c} A + BF_1 & BS_1^{-1/2} \\ \hline S_2^{1/2}(F_1 - F_2) & S_2^{1/2} S_1^{-1/2} \end{array}\right].$$

Then $\mathsf{D}_1 = \mathsf{D}_2 \mathsf{D}_s$ in a neighbourhood of zero, where D_i is the transfer function of Σ_i and D_s is the transfer function of Σ_s.

Proof. Using Proposition 2.20 we see that once we prove that the transfer function of the series interconnection of Σ_s and Σ_2 equals the transfer function of Σ_1, then we are done.

We write down a realization of the transfer function of the series interconnection of Σ_s and Σ_2 using Lemma 2.21:

$$\left[\begin{array}{cc|c} A + BF_1 & 0 & BS_1^{-1/2} \\ 0 & A + BF_2 & 0 \\ \hline F_1 & F_2 & S_1^{-1/2} \\ C + DF_1 & C + DF_2 & DS_1^{-1/2} \end{array}\right].$$

Since the state operator is diagonal and the input operator has zero as its second component, the transfer function is equal to the transfer function of

$$\left[\begin{array}{c|c} A + BF_1 & BS_1^{-1/2} \\ \hline F_1 & S_1^{-1/2} \\ C + DF_1 & DS_1^{-1/2} \end{array}\right],$$

which is Σ_1. $\square$

Proposition 6.48. *Let Σ be an output stabilizable discrete-time system. The system Σ_s from Proposition 6.47 is energy preserving with storage operator $\Delta := Q_1 - Q_2$.*

Proof. It is straightforward to check the necessary and sufficient conditions (5.2) using that Q_1 and Q_2 satisfy the control algebraic Riccati equation. $\square$

Proposition 6.49. *Let Σ be an input and output stabilizable discrete-time system. Then the system Σ_s from Proposition 6.47 is input stable and input-output stable.*

Proof. It follows from Proposition 6.39 that any Riccati closed-loop system of Σ is input stable. Since the state operator and input operator of Σ_s are equal to those of the Riccati closed-loop system associated with the control Riccati triple (q_1, s_1, F_1) it follows that Σ_s is input stable. Propositions 5.8 and 6.48 now show that Σ_s is input-output stable. $\square$

Proposition 6.50. *Let Σ be an input and output stabilizable discrete-time system. Then the transfer function of the system Σ_s from Proposition 6.47 has an inverse in $H^\infty(\mathbb{D}, \mathscr{U})$.*

Proof. Using Proposition 2.22 it is easily seen that a realization of the inverse of the transfer function of Σ_s is a system of the same form as Σ_s, but with the indices 1 and 2 interchanged. It follows from Proposition 6.49 that this realization is input-output stable. Hence D_s^{-1} is in $H^\infty(\mathbb{D}, \mathscr{U})$. $\square$

Notes

The LQR problem for discrete-time systems was studied by Lee, Chow and Barr [51] and Zabczyk [100], [101], [102]. Our approach to this problem, based on the set of stable input-output pairs, follows Curtain and Zwart [18]. The properties of the Riccati closed-loop system given in this chapter are mainly taken from Opmeer and Curtain [71]. Proposition 6.43 is well-known in the case of exponentially stabilizable and detectable systems, see Curtain and Zwart [18, Lemma 9.4.10]. It was first proven in the generality considered here in Curtain and Opmeer [16]. Propositions 6.47-6.50 were also first proven in Curtain and Opmeer [16].

Chapter 7

Coprime factorization

In this chapter and the following two chapters we consider the following set of holomorphic functions.

Definition 7.1. Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces. The set $H_0(\mathscr{U}, \mathscr{Y})$ consists of functions $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ that are holomorphic with $0 \in D(\mathsf{G})$.

Remark 7.2. Note the transfer function of a discrete-time system is always in our set of holomorphic functions. Moreover, it follows from Proposition 2.12 that any function in this set is the transfer function of some discrete-time system.

In this chapter we study coprime factorization over H^∞. We study both a strong and a weak form of coprimeness. Since we are dealing with operator-valued functions, we have to distinguish between right coprimeness and left coprimeness.

Definition 7.3. Let $\mathsf{M} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ and $\mathsf{N} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_3))$.

The functions M and N are called **weakly right-coprime** if for every Z-transformable sequence $h : \mathbb{Z}^+ \to \mathscr{H}_1$ with $[\mathsf{M}\hat{h}; \mathsf{N}\hat{h}] \in H^2(\mathbb{D}, \mathscr{H}_2 \times \mathscr{H}_3)$ we have $h \in l^2(\mathbb{Z}^+, \mathscr{H}_1)$.

The functions M and N are called **strongly right-coprime** if $[\mathsf{M}; \mathsf{N}]$ has a left-inverse in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_2 \times \mathscr{H}_3, \mathscr{H}_1))$, meaning if there exist $\tilde{\mathsf{X}} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ and $\tilde{\mathsf{Y}} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_3, \mathscr{H}_1))$ such that

$$\tilde{\mathsf{X}}(z)\mathsf{M}(z) - \tilde{\mathsf{Y}}(z)\mathsf{N}(z) = I_{\mathscr{H}_1} \quad \forall z \in \mathbb{D}. \tag{7.1}$$

The functions $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ are called **right Bezout factors** for the pair (M, N).

Let $\tilde{\mathsf{M}} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ and $\tilde{\mathsf{N}} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_3, \mathscr{H}_2))$.

The functions $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{N}}$ are called **strongly left-coprime** if $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ has a right-inverse in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1 \times \mathscr{H}_3))$, that is to say, if there exist $\mathsf{X} \in$

$H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2,\mathscr{H}_1))$ and $\mathsf{Y}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2,\mathscr{H}_3))$ such that

$$\tilde{\mathsf{M}}(z)\mathsf{X}(z)-\tilde{\mathsf{N}}(z)\mathsf{Y}(z)=I_{\mathscr{H}_2}\quad\forall z\in\mathbb{D}. \tag{7.2}$$

The functions X and Y are called **left Bezout factors** for the pair $(\tilde{\mathsf{M}},\tilde{\mathsf{N}})$.

Definition 7.4. Let $\mathsf{G}:D(\mathsf{G})\subset\mathbb{C}\to\mathcal{L}(\mathscr{U},\mathscr{Y})$ be holomorphic with $0\in D(\mathsf{G})$.

G has a **right factorization** if there exist $\mathsf{M}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U}))$ and $\mathsf{N}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ such that $\mathsf{M}(z)$ is invertible for z in a neighbourhood of zero and $\mathsf{G}(z)=\mathsf{N}(z)\mathsf{M}(z)^{-1}$ for z in a neighbourhood of zero. The factor $[\mathsf{M};\mathsf{N}]$ provides a **weakly right-coprime factorization** if M and N are weakly right-coprime and a **strongly right-coprime factorization** if M and N are strongly right-coprime. The right factor $[\mathsf{M};\mathsf{N}]$ is called **normalized** when multiplication with $[\mathsf{M};\mathsf{N}]$ is an isometry from $H^2(\mathbb{D},\mathscr{U})$ into $H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$.

G has a **left factorization** if there exist $\tilde{\mathsf{M}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y}))$ and $\tilde{\mathsf{N}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ such that $\tilde{\mathsf{M}}(z)$ is invertible for z in a neighbourhood of zero and $\mathsf{G}(z)=\tilde{\mathsf{M}}(z)^{-1}\tilde{\mathsf{N}}(z)$ for z in a neighbourhood of zero. $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ is a **strongly left-coprime factor** if $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{N}}$ are strongly left-coprime. The left factor $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ is called **normalized** when multiplication with $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ is a co-isometry from $H^2(\mathbb{D},\mathscr{Y}\times\mathscr{U})$ into $H^2(\mathbb{D},\mathscr{Y})$.

G has a **doubly coprime factorization** if it has a left factorization and a right factorization and there exist $\tilde{\mathsf{X}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U}))$, $\tilde{\mathsf{Y}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y},\mathscr{U}))$, $\mathsf{X}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y}))$ and $\mathsf{Y}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y},\mathscr{U}))$ such that

$$\begin{bmatrix}\tilde{\mathsf{X}} & -\tilde{\mathsf{Y}}\\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}}\end{bmatrix}\begin{bmatrix}\mathsf{M} & \mathsf{Y}\\ \mathsf{N} & \mathsf{X}\end{bmatrix}=I=\begin{bmatrix}\mathsf{M} & \mathsf{Y}\\ \mathsf{N} & \mathsf{X}\end{bmatrix}\begin{bmatrix}\tilde{\mathsf{X}} & -\tilde{\mathsf{Y}}\\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}}\end{bmatrix}. \tag{7.3}$$

The doubly coprime factorization is called **normalized** when both the right factor $[\mathsf{M};\mathsf{N}]$ and the left factor $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ are normalized.

Remark 7.5. In this chapter we will prove results for right factorizations. However, all results translate to left factorizations by considering $\mathsf{G}^\dagger$.

The next proposition is a first step towards relating state space closed-loop systems (see Definition 4.1) and factorizations.

Proposition 7.6. *Let* $\mathsf{G}:D(\mathsf{G})\subset\mathbb{C}\to\mathcal{L}(\mathscr{U},\mathscr{Y})$ *be holomorphic with* $0\in D(\mathsf{G})$*. Let* Σ *be a realization of* G *and let* $[F,G]$ *be an admissible feedback pair for* Σ*. Denote the transfer function of the closed-loop system by* $[\mathsf{M};\mathsf{N}]$*. Then* $\mathsf{M}(z)$ *is invertible for* z *in a neighbourhood of zero and* $\mathsf{G}(z)=\mathsf{N}(z)\mathsf{M}(z)^{-1}$ *in a neighbourhood of zero.*

Proof. This follows from Proposition 2.23. □

The following proposition provides a fundamental property of weakly right-coprime functions.

Proposition 7.7. *Assume that the functions* $\mathsf{M} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ *and* $\mathsf{N} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_3))$ *are weakly right-coprime. If for a holomorphic* $\mathsf{R} : D(\mathsf{R}) \to \mathcal{L}(\mathscr{H}_4, \mathscr{H}_1)$ *with* $0 \in D(\mathsf{R})$ *we have* $[\mathsf{MR}; \mathsf{NR}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_2 \times \mathscr{H}_3))$*, then* $\mathsf{R} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_1))$.

Proof. Let $h \in H^2(\mathbb{D}, \mathscr{H}_4)$. Then since $[\mathsf{MR}; \mathsf{NR}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_2 \times \mathscr{H}_3))$ we have $[\mathsf{MR}h; \mathsf{NR}h] \in H^2(\mathbb{D}, \mathscr{H}_2 \times \mathscr{H}_3)$. Since $[\mathsf{M}; \mathsf{N}]$ is weakly right-coprime it follows that $\mathsf{R}h \in H^2(\mathbb{D}, \mathscr{H}_1)$. So multiplication by R maps $H^2(\mathbb{D}, \mathscr{H}_4)$ into $H^2(\mathbb{D}, \mathscr{H}_1)$. It is easily shown that a multiplication operator is closed from H^2 to H^2. By the closed graph theorem it follows that multiplication with R is a continuous operator from $H^2(\mathbb{D}, \mathscr{H}_4)$ to $H^2(\mathbb{D}, \mathscr{H}_1)$. By Lemma A.4 it follows that $\mathsf{R} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_1))$. $\square$

The following lemma gives additional conditions under which a weakly right-coprime factor is strongly right coprime (see Corollary 7.9). This will be useful in Chapter 8.

Lemma 7.8. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. If* $[\mathsf{M}; \mathsf{N}]$ *is a weakly right-coprime factor of* G*,* $I - \mathsf{G}(0)$ *has a bounded inverse and* $(I - \mathsf{G})^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$*, then* $\mathsf{M}(0) - \mathsf{N}(0)$ *has a bounded inverse and* $(\mathsf{M} - \mathsf{N})^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$.

Proof. We have $\mathsf{M} - \mathsf{N} = (I - \mathsf{G})\mathsf{M}$, which shows that $\mathsf{M}(0) - \mathsf{N}(0)$ has a bounded inverse. We have $\mathsf{M}(\mathsf{M} - \mathsf{N})^{-1} = (I - \mathsf{G})^{-1}$ and $\mathsf{N}(\mathsf{M} - \mathsf{N})^{-1} = \mathsf{G}(I - \mathsf{G})^{-1} = (I - \mathsf{G})^{-1} - I$. Proposition 7.7 now shows that $(\mathsf{M} - \mathsf{N})^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$. $\square$

Corollary 7.9. *Under the assumptions of Lemma 7.8 we have that* $[\mathsf{M}; \mathsf{N}]$ *is strongly right-coprime.*

Proof. We can choose the Bezout factors $\tilde{\mathsf{X}} = \tilde{\mathsf{Y}} = (\mathsf{M} - \mathsf{N})^{-1}$. $\square$

Weak right-coprimeness is connected to the linear quadratic optimal control problem as the following proposition shows. The set $\hat{\mathscr{V}}(x_0)$ is defined as the set of Z-transforms of sequences in $\mathscr{V}(x_0)$, which was defined in (6.2).

Proposition 7.10. *Let* D *be the transfer function of the discrete-time system* Σ *and let* $[\mathsf{M}; \mathsf{N}]$ *be a right factor. Then multiplication by* $[\mathsf{M}; \mathsf{N}]$ *is an injection from* $H^2(\mathbb{D}, \mathscr{U})$ *into* $\hat{\mathscr{V}}(0)$*. The factorization is weakly right-coprime if and only if multiplication by* $[\mathsf{M}; \mathsf{N}]$ *is a bijection from* $H^2(\mathbb{D}, \mathscr{U})$ *onto* $\hat{\mathscr{V}}(0)$.

Proof. That multiplication with $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ into $H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$ follows from the fact that $[\mathsf{M};\mathsf{N}]\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U},\mathscr{U}\times\mathscr{Y}))$. Let $r\in l^2(\mathbb{Z}^+,\mathscr{U})$, we show that $[\mathsf{M};\mathsf{N}]\hat{r}\in\hat{\mathscr{V}}(0)$. Define $\hat{u}:=\mathsf{M}\hat{r}$. We have to show that $\mathsf{N}\hat{r}=\mathsf{D}\mathsf{M}\hat{r}$. This follows since $\mathsf{D}\mathsf{M}\hat{r}=\mathsf{N}\mathsf{M}^{-1}\mathsf{M}\hat{r}=\mathsf{N}\hat{r}$. It follows that multiplication by $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ into $\hat{\mathscr{V}}(0)$.

We show that multiplication with $[\mathsf{M};\mathsf{N}]$ is injective. Suppose that there are two Z-transformable sequences $r_i:\mathbb{Z}^+\to\mathscr{U}$ $(i=1,2)$ with $[\mathsf{M};\mathsf{N}]\hat{r}_1=[\mathsf{M};\mathsf{N}]\hat{r}_2$. Then $\mathsf{M}(z)\hat{r}_1(z)=\mathsf{M}(z)\hat{r}_2(z)$ in a neighbourhood of zero and since $\mathsf{M}(z)$ is invertible for z in a neighbourhood of zero we have $\hat{r}_1(z)=\hat{r}_2(z)$ in a neighbourhood of zero. This shows that $r_1=r_2$. Hence multiplication with $[\mathsf{M};\mathsf{N}]$ is injective.

Multiplication with $[\mathsf{M};\mathsf{N}]$ is onto if and only if for every $[u;y]\in\mathscr{V}(0)$ there exists an $r\in l^2(\mathbb{Z}^+,\mathscr{U})$ such that $[\hat{u};\hat{y}]=[\mathsf{M};\mathsf{N}]\hat{r}$. Suppose that multiplication with $[\mathsf{M};\mathsf{N}]$ is onto, and let $h:\mathbb{Z}^+\to\mathscr{U}$ be a Z-transformable sequence with $[\mathsf{M};\mathsf{N}]\hat{h}\in H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$. Then $[\mathsf{M};\mathsf{N}]\hat{h}\in\hat{\mathscr{V}}(0)$ and since multiplication by $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ onto $\hat{\mathscr{V}}(0)$ there exists an $r\in l^2(\mathbb{Z}^+,\mathscr{U})$ such that $[\mathsf{M};\mathsf{N}]\hat{h}=[\mathsf{M};\mathsf{N}]\hat{r}$. Since multiplication by $[\mathsf{M};\mathsf{N}]$ is injective as proven above it follows that $h=r$. Hence $h\in l^2(\mathbb{Z}^+,\mathscr{U})$ and so $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime. Suppose that $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime. Let $[u;y]\in\mathscr{V}(0)$. Define $r:\mathbb{Z}^+\to\mathscr{U}$ through its Z-transform: $\hat{r}(z):=\mathsf{M}(z)^{-1}\hat{u}(z)$ for z in a neighbourhood of zero. We then have $[\mathsf{M};\mathsf{N}]\hat{r}=[\hat{u};\hat{y}]$. So $[\mathsf{M};\mathsf{N}]\hat{r}\in H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$. By weak right-coprimeness we have $r\in l^2(\mathbb{Z}^+,\mathscr{U})$. This shows that muliplication by $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ onto $\hat{\mathscr{V}}(0)$. □

The following result connects the existence of normalized weakly right-coprime factorizations to the linear quadratic optimal control problem.

Proposition 7.11. *Let Σ be an output stabilizable discrete-time system. Then the transfer function of its optimal closed-loop system provides a normalized weakly right-coprime factorization of the transfer function of Σ.*

Proof. That the transfer function $[\mathsf{M};\mathsf{N}]$ of the optimal closed-loop system satisfies $\mathsf{D}(z)=\mathsf{N}(z)\mathsf{M}(z)^{-1}$ in a neighbourhood of zero follows from Proposition 7.6. Combining Propositions 6.34 and 6.35 we see that optimal closed-loop system is energy-preserving with as storage operator the observability gramian. By Proposition 5.2 the optimal closed-loop system is input-output stable, so $[\mathsf{M};\mathsf{N}]\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U},\mathscr{U}\times\mathscr{Y}))$. Proposition 5.4, and the fact that $l^2(\mathbb{Z}^+,\mathscr{U})$ and $H^2(\mathbb{D},\mathscr{U})$ are isometrically isomorphic under the Z-transform, shows that the factorization is normalized. We show that it is weakly right-coprime. Let $[u;y]\in\mathscr{V}(0)$. Let x be the corresponding state for initial state zero and define $r_k:=-(S^{\min})^{1/2}F^{\min}x_k+(S^{\min})^{1/2}u_k$. Then by Proposition

6.30 we have $r \in l^2(\mathbb{Z}^+, \mathscr{U})$. The sequence r is the output for input u of the system Σ_- defined by its system operator $[A, B; -(S^{\min})^{1/2} F^{\min}, (S^{\min})^{1/2}]$. Using Proposition 2.22 we see that the inverse of the transfer function of Σ_- has a realization $[A + BF^{\min}, B(S^{\min})^{-1/2}; F, (S^{\min})^{-1/2}]$. But this is a realization of M and so we conclude that Σ_- has M^{-1} as its transfer function. So $\hat{r}(z) = \mathsf{M}(z)^{-1}\hat{u}(z)$. It follows that $[\hat{u}(z); \hat{y}(z)] = [\mathsf{M}(z); \mathsf{N}(z)]\hat{r}(z)$. Hence each element of $\mathscr{V}(0)$ is in the range of the operator of multiplication by $[\mathsf{M}; \mathsf{N}]$. By Proposition 7.10 the pair (M, N) is weakly right-coprime. □

The existence of a right factorization and of an output stabilizable realization are equivalent as the following proposition shows.

Proposition 7.12. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Then the following are equivalent:*

1. *G has a right factorization.*
2. *G has an output stabilizable realization.*

Proof. If G has an output stabilizable realization, then by Proposition 7.11 it has a right factorization. Assume that G has a right factor $[\mathsf{M}; \mathsf{N}]$. This right factor has a realization $\check{\Sigma}$ that is output stable (for example the backward shift realization from Remark 2.13 which is output stable by Example 3.3). Since $\mathsf{M}(0)$ has a bounded inverse, we can use Proposition 2.23 to obtain a realization Σ of G. It follows from Corollary 4.15 that Σ is output stabilizable. □

The following proposition shows that existence of a right factorization implies the existence of a normalized weakly right-coprime factorization.

Proposition 7.13. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. If G has a right factorization, then it has a normalized weakly right-coprime factorization.*

Proof. From Proposition 7.12 we see that G has an output stabilizable realization Σ. Proposition 7.10 shows that the optimal closed-loop system of Σ provides a normalized weakly right-coprime factorization of G. □

The following proposition gives a parametrization of all right factorizations.

Proposition 7.14. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$ and assume that G has a right factorization. Let $[\mathsf{M}_0, \mathsf{N}_0]$ be a weakly right-coprime factor. Then all right factors are parametrized as follows:*

$$\mathsf{M} = \mathsf{M}_0\mathsf{V}, \qquad \mathsf{N} = \mathsf{N}_0\mathsf{V},$$

where V *runs through the set of* $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$ *functions that have a bounded inverse in zero. The weakly right-coprime factors are exactly those for which* V^{-1} *is in* $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$ *as well.*

Proof. The above M and N obviously provide a factorization. Assume that $[\mathsf{M}_1;\mathsf{N}_1]$ is a right factor. Define $\mathsf{V} := \mathsf{M}_0^{-1}\mathsf{M}_1$. Then $\mathsf{M}_1 = \mathsf{M}_0\mathsf{V}$ and $\mathsf{N}_1 = \mathsf{G}\mathsf{M}_1 = \mathsf{G}\mathsf{M}_0\mathsf{V} = \mathsf{N}_0\mathsf{V}$. By Proposition 7.7 we have that $\mathsf{V}\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$.

If V has an inverse in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$, then from $[\mathsf{M};\mathsf{N}]h = [\mathsf{M}_0;\mathsf{N}_0]\mathsf{V}h \in H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$ we obtain $h = \mathsf{V}^{-1}\mathsf{V}h \in H^2(\mathbb{D},\mathscr{U})$. This shows that in this case $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime. If $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime, then it follows from symmetry considerations that V must have an inverse in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$. $\square$

Proposition 7.15. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ *be holomorphic with* $0\in D(\mathsf{G})$ *and assume that* G *has a right factorization. Let* $[\mathsf{M}_0,\mathsf{N}_0]$ *be a normalized weakly right-coprime factor. Then all normalized weakly right-coprime factors are parametrized as follows:*

$$\mathsf{M} = \mathsf{M}_0\mathsf{V}, \qquad \mathsf{N} = \mathsf{N}_0\mathsf{V},$$

where $\mathsf{V}\in\mathcal{L}(\mathscr{U})$ *is unitary.*

Proof. That the above M and N provide a normalized weakly right-coprime factorization is obvious. Assume that the pair $[\mathsf{M};\mathsf{N}]$ is a normalized weakly right-coprime factor. From Proposition 7.14 we obtain that a normalized weakly right-coprime factor must be of the indicated form, but we only know that V and its inverse are in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$. So we still need to show that this function is constant and that this constant is a unitary operator. Since the factorizations are normalized we have $\mathsf{M}^*\mathsf{M}+\mathsf{N}^*\mathsf{N} = I$ and $\mathsf{M}_0^*\mathsf{M}_0+\mathsf{N}_0^*\mathsf{N}_0 = I$ almost everywhere on the unit circle by Lemmas A.18 and A.20. Since $\mathsf{M} = \mathsf{M}_0\mathsf{V}$ (on the open unit disc, but this extends to almost everywhere on the unit circle) it follows that $\mathsf{V}^*\mathsf{V} = I$ almost everywhere on the unit circle. Since V has an inverse in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$ its boundary function has an inverse in $L^\infty(\mathbb{T},\mathcal{L}(\mathscr{U}))$ and since $\mathsf{V}^*\mathsf{V} = I$, this inverse must equal V^*. Hence V^* is the boundary function of a function in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$, namely of V^{-1}. Define $\mathsf{V}_- : \mathbb{D}^+ \to \mathcal{L}(\mathscr{U})$ by $\mathsf{V}_-(z) = \mathsf{V}(1/\bar{z})^*$. Then $\mathsf{V}^- \in H^\infty(\mathbb{D}^+,\mathcal{L}(\mathscr{U}))$ since it is obviously holomorphic and

$$\sup_{z\in\mathbb{D}^+}\|\mathsf{V}_-(z)\| = \sup_{z\in\mathbb{D}^+}\|\mathsf{V}(1/\bar{z})^*\| = \sup_{z\in\mathbb{D}^+}\|\mathsf{V}(1/\bar{z})\| = \sup_{s\in\mathbb{D}}\|\mathsf{V}(s)\| = \|\mathsf{V}\|_\infty.$$

The boundary function of V_- equals V^*. Hence V^* is the boundary function of a function in $H^\infty(\mathbb{D}^+,\mathcal{L}(\mathscr{U}))$, namely of V_-. So V^* is the boundary function of both a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ function and a $H^\infty(\mathbb{D}^+, \mathcal{L}(\mathscr{U}))$ function. It follows from Corollary A.14 that V^* is constant. Hence V is constant. It follows from the earlier established $\mathsf{V}^*\mathsf{V} = I$ almost everywhere on the unit circle and the fact that V has an inverse that V is unitary. □

Proposition 7.16. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. If G has a strongly right-coprime factorization, then all weakly right-coprime factorizations are strongly right-coprime.*

Proof. Assume that $[\mathsf{M}_0; \mathsf{N}_0]$ is a strongly right-coprime factor. Let $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ be right Bezout factors. According to Proposition 7.14 all weakly right-coprime factors are of the form $[\mathsf{M}_0; \mathsf{N}_0]\mathsf{V}$ with both V and its inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$. It is easily seen that $\mathsf{V}^{-1}\tilde{\mathsf{X}}$ and $\mathsf{V}^{-1}\tilde{\mathsf{Y}}$ are right Bezout factors for $[\mathsf{M}_0; \mathsf{N}_0]\mathsf{V}$. It follows that $\mathsf{M}_0\mathsf{V}$ and $\mathsf{N}_0\mathsf{V}$ are strongly right-coprime. □

In the following proposition we need the **Hankel operator** which is defined in Definition A.24. We further note that a H^∞ function is called inner if the corresponding multiplication operator is an isometry (Definition A.19, see also Lemma A.20).

Proposition 7.17. *Let $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. Assume that G is inner and that it has a left inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$. Then the norm of the associated Hankel operator is strictly less than one.*

Proof. We have that the Hankel operator has norm less than or equal to one, since it is the composition of an isometric operator with two projections, each of which have norm smaller than or equal to one.

We show that the norm of the Hankel operator cannot be one. Suppose it is. Then there exists a sequence $h^n \in L^2(\mathbb{T}, \mathscr{H}_1)$ with norm one such that $\|P_+ L_\mathsf{G} P_- h_n\| \to 1$. Here P_- is the projection from $L^2(\mathbb{T}, \mathscr{H}_1)$ onto the subspace of functions whose nonnegative Fourier coefficients are zero, P_+ is the projection from $L^2(\mathbb{T}, \mathscr{H}_2)$ onto the subspace of functions whose negative Fourier coefficients are zero and L_G is the operator multiplication with G (see Definition A.15). We can assume without loss of generality that the h^n have zero nonnegative Fourier coefficients. Define $f^n := L_\mathsf{G} P_- h_n$, $f^n_+ := P_+ f^n$, $f^n_- := P_- f^n$. Then since G is inner we have $\|f^n\| = 1$ and we have $\|f^n\|^2 = \|f^n_+\|^2 + \|f^n_-\|^2$. Since by assumption $\|f^n_+\| \to 1$ it follows that $\|f^n_-\| \to 0$. By assumption there exists a $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ such that $\mathsf{HG} = I$. We then have $h^n = L_\mathsf{H} L_\mathsf{G} h^n = L_\mathsf{H} f^n = L_\mathsf{H} f^n_+ + L_\mathsf{H} f^n_-$. Since $L_\mathsf{H} f^n_+$ has zero negative Fourier coefficients and h^n has zero nonnegative Fourier coefficients we have $\langle h^n, L_\mathsf{H} f^n_+ \rangle = 0$. So

$$0 = \langle h^n, L_\mathsf{H} f^n_+ \rangle = \langle h^n, h^n \rangle - \langle h^n, L_\mathsf{H} f^n_- \rangle \to 1.$$

This contradiction shows that the Hankel operator must have norm strictly smaller than one. □

The following proposition complements the previous one.

Proposition 7.18. *Let $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. Assume that G is inner and that the norm of the associated Hankel operator is strictly less than one. Then G has a left inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$.*

Proof. We apply Proposition A.27 (the Nehari theorem) to G^*. Since $\|H_{\mathsf{G}}\| < 1$, this gives the existence of a $\mathsf{K} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ such that

$$\|\mathsf{G}^* + \mathsf{K}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} < 1.$$

Since G is inner we have $\mathsf{G}^*\mathsf{G} = I$ almost everywhere on the unit circle from Proposition A.18. From this we obtain $I + \mathsf{KG} = \mathsf{G}^*\mathsf{G} + \mathsf{KG} = (\mathsf{G}^* + \mathsf{K})\mathsf{G}$ almost everywhere on the unit circle, which gives

$$\|I + \mathsf{KG}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1))} \leq \|\mathsf{G}^* + \mathsf{K}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \, \|\mathsf{G}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))} < 1.$$

Since $\mathsf{KG} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1))$, which is a Banach algebra, we obtain that KG has an inverse R in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1))$ from the geometric series theorem. In particular $\mathsf{RKG} = I$, which implies that RK is a left inverse of G. □

Combining Propositions 7.17 and 7.18 we obtain the following.

Corollary 7.19. *Assume $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ is inner. Then G has a left inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ if and only if the norm of the Hankel operator of G is strictly less than one.*

The following result connects the existence of normalized strongly right-coprime factorizations to the linear quadratic optimal control problem.

Proposition 7.20. *Let Σ be an input and output stabilizable discrete-time system. Then the transfer function of the optimal closed-loop system of Σ is a normalized strongly right-coprime factor of the transfer function of Σ.*

Proof. From Proposition 7.11 we obtain that the transfer function of the optimal closed-loop system of Σ is a normalized right factor. Corollary 6.41 shows that the Hankel map of this system has norm strictly smaller than one. Since the Hankel map and the Hankel operator have the same norm by Lemma A.26, Proposition 7.18 then gives the result. □

The following proposition shows that not only the optimal closed-loop system provides a strongly right-coprime factorization, but that every Riccati closed-loop system does. Note that we may not obtain a normalized factorization in this case.

Proposition 7.21. *Let Σ be an input and output stabilizable discrete-time system. Then the transfer function of any Riccati closed-loop system of Σ is a strongly right-coprime factor of the transfer function of Σ.*

Proof. That we obtain a factorization follows from Propositions 6.34 and 7.6. Application of Propositions 6.47, 6.49 and 6.50 shows that the transfer function of an arbitrary Riccati closed-loop system of Σ can be obtained by multiplying the transfer function of the optimal closed-loop system from the right with a function that is in H^∞ and whose inverse is H^∞. Using that by Proposition 7.20 the transfer function of the optimal closed-loop system is strongly right-coprime it then easily follows that the transfer function of any Riccati closed-loop system of Σ is a strongly right-coprime. □

The following proposition shows that existence of a strongly right-coprime factorization and the existence of an input and output stabilizable realization are equivalent.

Proposition 7.22. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Then the following are equivalent:*

1. *G has an input and output stabilizable realization.*
2. *G has a normalized strongly right-coprime factorization.*
3. *G has a strongly right-coprime factorization.*

Proof. If G has an input and output stabilizable realization, then by Proposition 7.20 it has a normalized strongly right-coprime factorization. Assume that G has a strongly right-coprime factorization. It follows from Propositions 7.13 and 7.16 that G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. By Proposition 7.17 the norm of the Hankel operator associated to $[\mathsf{M}; \mathsf{N}]$ is strictly smaller than one. The function $[\mathsf{M}; \mathsf{N}]$ has an approximately controllable input and output stable realization $\check{\Sigma}$ (for example the restricted backward shift realization from Remark 2.13 which is output stable by Example 3.3 and input stable by Example 3.25). From Proposition 5.7 we obtain that $\check{\Sigma}$ is energy preserving with the observability gramian as storage operator (note that the condition on the equality of the norm of the input and output in Proposition 5.7 is satisfied since the factorization is normalized). We use Proposition 2.23 to obtain the corresponding realization Σ of G. It follows from Corollary 4.15 that Σ is output stabilizable. It follows from Lemma 3.18 combined with Lemma A.26 that the spectral radius of $L_B L_C$, the product of the controllability and the observability gramian of $\check{\Sigma}$, is strictly smaller than one. This implies that the operator $I - L_B L_C$ has a bounded inverse. Proposition 6.46 now shows that $P := (I - L_B L_C)^{-1} L_B$ provides a solution of the filter algebraic Riccati equation of Σ. The dual version of Proposition 6.36 now shows that Σ is input stabilizable. □

The following lemma shows that we can always pick right Bezout factors with a nice property.

Lemma 7.23. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$ and assume that G has a strongly right-coprime factorization. For every strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ there exists a pair of right Bezout factors with $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$.*

Proof. Let $[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]$ be an arbitrary pair of Bezout factors. Define $\tilde{\mathsf{Y}}(z) := (I - \mathsf{M}(0)^{-1}\mathsf{M}(z))\tilde{\mathsf{Y}}_1(z)$ and $\tilde{\mathsf{X}}(z) = \mathsf{M}(0)^{-1} + (I - \mathsf{M}(0)^{-1}\mathsf{M}(z))\tilde{\mathsf{X}}_1(z)$. Then obviously $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$ and it is not hard to see that $\tilde{\mathsf{X}}$, $\tilde{\mathsf{Y}}$ is a right Bezout pair. □

The following proposition shows that the existence of a doubly coprime factorization follows from the existence of a strongly right-coprime factorization.

Proposition 7.24. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. The following are equivalent:*

1. *G has a normalized strongly right-coprime factorization.*
2. *G has a normalized strongly left-coprime factorization.*
3. *G has a normalized doubly coprime factorization.*

Moreover, any given normalized strongly right-coprime factorization and normalized strongly left-coprime factorization can be embedded in a normalized doubly coprime factorization.

Proof. That (1) and (2) are equivalent follows from Proposition 7.22 noting that the second condition in that proposition holds for G if and only if it holds for $\mathsf{G}^\dagger$. It is clear that (3) implies (1) and (2). We show that (1) implies (3).

Now assume that G has the normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ with corresponding right Bezout factors $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$ and the normalized strongly left-coprime factor $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ with the corresponding left Bezout factors $[\mathsf{X}_1; \mathsf{Y}_1]$. By Propostition 7.23 we can assume that $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$. Define $\Delta := \tilde{\mathsf{X}}\mathsf{Y}_1 - \tilde{\mathsf{Y}}\mathsf{X}_1$ and $\mathsf{Y} := -\mathsf{M}\Delta + \mathsf{Y}_1$, $\mathsf{X} := -\mathsf{N}\Delta + \mathsf{X}_1$. It is now easily verified that with this X and Y we have

$$\begin{bmatrix} \tilde{\mathsf{X}} & -\tilde{\mathsf{Y}} \\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}} \end{bmatrix} \begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} = I. \tag{7.4}$$

Next we show that

$$\begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{X}} & -\tilde{\mathsf{Y}} \\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}} \end{bmatrix} = I.$$

Since $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$ we have that

$$\begin{bmatrix} \tilde{\mathsf{X}}(0) & -\tilde{\mathsf{Y}}(0) \\ -\tilde{\mathsf{N}}(0) & \tilde{\mathsf{M}}(0) \end{bmatrix}$$

has the bounded inverse

$$\begin{bmatrix} \tilde{\mathsf{M}}(0) & 0 \\ \tilde{\mathsf{M}}(0)^{-1}\tilde{\mathsf{N}}(0)\tilde{\mathsf{M}}(0) & \tilde{\mathsf{M}}(0)^{-1} \end{bmatrix}.$$

The function $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}; -\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$ is holomorphic at zero which implies that it has a realization Σ. Since the function value at zero has a bounded inverse operator, it follows from Proposition 2.22 that $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}; -\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$ is invertible in a neighbourhood of zero. It follows from (7.4) that $[\mathsf{M}, \mathsf{Y}; \mathsf{N}, \mathsf{X}]$ equals this inverse. By the identity theorem for holomorphic functions we have that (7.3) holds on $\mathbb{D}$. Hence G has a doubly coprime factorization. This doubly coprime factorization is obviously normalized. By construction both the given normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ and the given normalized strongly left-coprime factor $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ are embedded in the doubly coprime factor. □

The following result gives a parametrization of all right Bezout factors.

Proposition 7.25. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$. *Then it has a strongly left-coprime factor* $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$. *Let* $\tilde{\mathsf{X}}_0$, $\tilde{\mathsf{Y}}_0$ *be right Bezout factors for* $[\mathsf{M}; \mathsf{N}]$ *and let* $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$. *Then* $\tilde{\mathsf{X}} := \tilde{\mathsf{X}}_0 + \mathsf{V}\tilde{\mathsf{N}}$, $\tilde{\mathsf{Y}} := \tilde{\mathsf{Y}}_0 + \mathsf{V}\tilde{\mathsf{M}}$ *are right Bezout factors for* $[\mathsf{M}; \mathsf{N}]$. *Moreover, all right Bezout factors for* $[\mathsf{M}; \mathsf{N}]$ *are of this form.*

Proof. That G has a strongly left-coprime factorization follows from (the proof of) Proposition 7.24. That the indicated functions are right Bezout factors is easily checked. We show that all right Bezout factors are of this form. Let $\tilde{\mathsf{X}}_0$, $\tilde{\mathsf{Y}}_0$ be arbitrarty right Bezout factors for $[\mathsf{M}; \mathsf{N}]$. Define V in a neighbourhood of zero by $\mathsf{V} = (\tilde{\mathsf{Y}} - \tilde{\mathsf{Y}}_0)\tilde{\mathsf{M}}^{-1}$. It follows that $\tilde{\mathsf{Y}} = \tilde{\mathsf{Y}}_0 + \mathsf{V}\tilde{\mathsf{M}}$ in a neighbourhood of zero. Using the Bezout equation (7.1) we have $(\tilde{\mathsf{X}} - \tilde{\mathsf{X}}_0)\mathsf{M} = (\tilde{\mathsf{Y}} - \tilde{\mathsf{Y}}_0)\mathsf{N}$. Using the above equation for $\tilde{\mathsf{Y}}$ we see that this equals $\mathsf{V}\tilde{\mathsf{M}}\mathsf{N}$ in a neighbourhood of zero. Since $\tilde{\mathsf{M}}\mathsf{N} = \tilde{\mathsf{N}}\mathsf{M}$ we obtain $(\tilde{\mathsf{X}} - \tilde{\mathsf{X}}_0)\mathsf{M} = \mathsf{V}\tilde{\mathsf{N}}\mathsf{M}$ in a neighbourhood of zero. It follows that $\tilde{\mathsf{X}} = \tilde{\mathsf{X}}_0 + \mathsf{V}\tilde{\mathsf{N}}$ in a neighbourhood of zero. The only thing left to show is that $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$. This follows since

$$\mathsf{V} = \mathsf{V}(\tilde{\mathsf{M}}\mathsf{X} - \tilde{\mathsf{N}}\mathsf{Y}) = (\tilde{\mathsf{Y}} - \tilde{\mathsf{Y}}_0)\mathsf{X} - (\tilde{\mathsf{X}}_0 - \tilde{\mathsf{X}})\mathsf{Y},$$

where X, Y are left Bezout factors for $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$. □

The set of all strongly right-coprime pairs is open as the following proposition shows.

Proposition 7.26. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Then there exists a $\varepsilon > 0$ such that for all $\Delta = [\Delta_\mathsf{M}; \Delta_\mathsf{N}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ with $\|\Delta\|_\infty < \varepsilon$ the functions $\mathsf{M} + \Delta_\mathsf{M}$ and $\mathsf{N} + \Delta_\mathsf{N}$ are strongly right-coprime.*

Proof. From Proposition 7.24 we obtain the existence of $\mathsf{X} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}))$ and $\mathsf{Y} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ such that $[\mathsf{M}, \mathsf{Y}; \mathsf{N}, \mathsf{X}]$ is invertible in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))$. The result follows using that the invertible elements in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))$ form an open set. □

In Proposition 7.32 we give an explicit ε under which the result of Proposition 7.26 holds under the assumption that $\mathscr{U}$ is finite-dimensional. The following results (Lemma 7.27 up to Proposition 7.31) are used in the proof of Proposition 7.32.

Lemma 7.27. *Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces, $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ and $S \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$. Assume that $TS = I_{\mathscr{H}_2}$ and $ST = I_{\mathscr{H}_1}$. Then*

$$\inf_{h \in \mathscr{H}_2 : \|h\| = 1} \|Sh\| = \frac{1}{\|T\|}.$$

Proof. We have for each $h \in \mathscr{H}_2$ that $\|h\| = \|TSh\| \leq \|T\| \, \|Sh\|$. This implies

$$\inf_{h \in \mathscr{H}_2 : \|h\| = 1} \|Sh\| \geq \frac{1}{\|T\|}.$$

There exist $f_n \in \mathscr{H}_1$ with norm one such that $\|Tf_n\| \to \|T\|$. Define $h_n := Tf_n / \|Tf_n\|$. Then $\|h_n\| = 1$ and $Sh_n = f_n / \|Tf_n\|$. So $\|Sh_n\| = 1/\|Tf_n\|$. For $n \to \infty$ we have $\|Sh_n\| \to 1/\|T\|$. This implies that $1/\|T\|$ is not only a lower bound, but the largest lower bound, i.e. it is the desired infimum. □

Lemma 7.28. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$. Then*

$$\inf_{\|x\|=1} \|Tx\| = \inf_{\|y\|=1} \|T^*y\|,$$

provided that both are positive.

Proof. We have

$$\inf_{\|x\|=1} \|Tx\|^2 = \inf_{\|x\|=1} \langle T^*Tx, x\rangle.$$

It is well-known (see for example Kreyzsig [47, p467]) that the number on the right-hand side is the smallest spectral value of T^*T. Similarly, $\inf_{\|y\|=1} \|T^*y\|^2$ is the smallest spectral value of TT^*. It follows from Lemma 3.16 that the spectra of T^*T and TT^* are equal, with the possible exception of zero. The result follows. □

Lemma 7.29. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G that has a normalized doubly coprime factorization. Denote the normalized strongly left-coprime factor by $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$, the normalized strongly right-coprime factor by $[\mathsf{M}; \mathsf{N}]$ and the left Bezout factor by $[\mathsf{X}; \mathsf{Y}]$. Denote the Hankel operator of $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ by $H_{[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]}$. Then*

$$\inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))} \left\| \begin{bmatrix} \mathsf{Y} \\ \mathsf{X} \end{bmatrix} - \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} \mathsf{V} \right\| = \frac{1}{\sqrt{1 - \|H_{[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]}\|^2}}. \tag{7.5}$$

Proof. Let $T_{[\mathsf{M};\mathsf{N}]} : H^2(\mathbb{D}, \mathscr{U}) \to H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y})$ be the operator of multiplication by $[\mathsf{M}; \mathsf{N}]$. Since $T_{[\mathsf{M};\mathsf{N}]}$ is an isometry its range is closed and we have the orthogonal decomposition

$$H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y}) = \mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]}) \oplus \mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp. \tag{7.6}$$

Denote by $P_{\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp}$ the orthogonal projection onto the second component in this decomposition. Define $T_{[\mathsf{Y};\mathsf{X}]}$ similarly to $T_{[\mathsf{M};\mathsf{N}]}$. Define $T : H^2(\mathbb{D}, \mathscr{Y}) \to H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y})$ by

$$T := P_{\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp} T_{[\mathsf{Y};\mathsf{X}]}. \tag{7.7}$$

We obtain from Corollary A.23 that the infimum on the left-hand side of (7.5) equals $\|T\|$. Define $S : \mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp \subset H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y}) \to H^2(\mathbb{D}, \mathscr{Y})$ as the restriction to $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$ of multiplication by $[-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$, i.e.

$$S = T_{[-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]}|_{\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp}.$$

We show that S is the inverse of T. First note that for any $y \in H^2(\mathscr{Y})$ there exists a $u \in H^2(\mathscr{U})$ such that

$$Ty = \begin{bmatrix} \mathsf{Y} \\ \mathsf{X} \end{bmatrix} y + \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} u.$$

It follows using (7.3) that $STy = y$ for all $y \in H^2(\mathscr{Y})$. From (7.3) we also obtain

$$\begin{bmatrix} \mathsf{Y} \\ \mathsf{X} \end{bmatrix} [-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}] + \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} [\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}] = I.$$

Restricting to $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$ and projecting onto $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$ shows that TS equals the identity operator on $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$.

Using Lemma 7.27 we obtain

$$\inf_{w \in \mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp : \|w\|=1} \|Sw\|_{H^2(\mathbb{D},\mathscr{Y})} = \frac{1}{\|T\|}.$$

Let $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} : H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y}) \to H^2(\mathbb{D}, \mathscr{Y})$ be the Toeplitz operator of $[-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$. From (7.3) we obtain that $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} T_{[\mathsf{M};\mathsf{N}]} = 0$. So $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}$ is zero on $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})$. It follows that $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}$ splits with respect to the decomposition (7.6) as

$$T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} = [0, S].$$

Since $T^*_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} = T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]}$ we have, with respect to the decomposition (7.6),

$$T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]} = \begin{bmatrix} 0 \\ S^* \end{bmatrix}.$$

It follows that

$$\inf_{y \in H^2(\mathbb{D},\mathscr{Y}) : \|y\|=1} \|T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]} y\|_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} = \inf_{y \in H^2(\mathbb{D},\mathscr{Y}) : \|y\|=1} \|S^* y\|_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})}. \tag{7.8}$$

Let $y \in H^2(\mathscr{Y})$. Since $[-\tilde{\mathsf{N}}^*; \tilde{\mathsf{M}}^*]$ is inner we have

$$\begin{aligned} \|y\|^2_{H^2(\mathbb{D},\mathscr{U})} &= \left\| \begin{bmatrix} -\tilde{\mathsf{N}}^* \\ \tilde{\mathsf{M}}^* \end{bmatrix} y \right\|^2_{L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y})} \\ &= \left\| P_{H^2(\mathscr{U}\times\mathscr{Y})} \begin{bmatrix} -\tilde{\mathsf{N}}^* \\ \tilde{\mathsf{M}}^* \end{bmatrix} y \right\|^2_{L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y})} \\ &\quad + \left\| P_{H^2(\mathscr{U}\times\mathscr{Y})^\perp} \begin{bmatrix} -\tilde{\mathsf{N}}^* \\ \tilde{\mathsf{M}}^* \end{bmatrix} y \right\|^2_{L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y})} \\ &= \|T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]} y\|^2 + \|H^*_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} y\|^2, \end{aligned}$$

where

$$H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} := P_{H^2(\mathbb{D},\mathscr{Y})} L_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} P_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})^\perp} : L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y}) \to L^2(\mathbb{T},\mathscr{Y})$$

is the Hankel operator of $[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]$. It follows that

$$\inf_{y\in H^2(\mathbb{D},\mathscr{Y}):\|y\|=1} \|T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]}y\|^2_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} = 1 - \|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\|^2. \tag{7.9}$$

Combining (7.8) and (7.9) we obtain

$$\inf_{y\in H^2(\mathbb{D},\mathscr{Y}):\|y\|=1} \|S^*y\|^2_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} = 1 - \|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\|^2. \tag{7.10}$$

Using the dual version of Proposition 7.17 we conclude from the fact that $\tilde{\mathsf{N}}$ and $\tilde{\mathsf{M}}$ are strongly left-coprime that $\|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\| < 1$, so that the number in (7.10) is positive. We use Lemma 7.28 to conclude that

$$\inf_{w\in\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp:\|w\|=1} \|Sw\|_{H^2(\mathbb{D},\mathscr{Y})} = \inf_{y\in H^2(\mathbb{D},\mathscr{Y}):\|y\|=1} \|S^*y\|_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} \tag{7.11}$$

Note that we have already established that both sides of (7.11) are positive so that Lemma 7.28 is indeed applicable. We earlier established that the left-hand side of (7.11) equals $1/\|T\|$ and that this equals one over the infimum in the statement of the lemma. The right-hand side of (7.11) we have shown to be equal to $\sqrt{1-\|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\|^2}$. Noting that $\|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\| = \|H_{[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]}\|$ gives the desired result. □

Applying Lemma 7.29 to $\mathsf{G}^\dagger$ we obtain the following.

Corollary 7.30. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized doubly coprime factorization. Denote the normalized strongly left-coprime factor by $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$, the normalized strongly right-coprime factor by $[\mathsf{M};\mathsf{N}]$ and the right Bezout factor by $[\tilde{\mathsf{X}};\tilde{\mathsf{Y}}]$. Denote the Hankel operator of $[\mathsf{M},\mathsf{N}]$ by $H_{[\mathsf{M},\mathsf{N}]}$. Then*

$$\inf_{\mathsf{V}\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{Y},\mathscr{U}))} \left\| \begin{bmatrix} \tilde{\mathsf{Y}} \\ \tilde{\mathsf{X}} \end{bmatrix} - \mathsf{V} \begin{bmatrix} \tilde{\mathsf{M}} \\ \tilde{\mathsf{N}} \end{bmatrix} \right\|_{H^\infty(\mathbb{D},\mathcal{L}(\mathscr{Y}\times\mathscr{U},\mathscr{U}))} = \frac{1}{\sqrt{1-\|H_{[\mathsf{M},\mathsf{N}]}\|^2}}.$$

Proof. This follows from applying Lemma 7.29 to $\mathsf{G}^\dagger$. □

Proposition 7.31. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized doubly coprime factorization. Denote the Hankel operator of a normalized strongly right-coprime factor $[\mathsf{M};\mathsf{N}]$ by $H_{[\mathsf{M};\mathsf{N}]}$. Then for all $z \in \mathbb{D}$ and $u \in \mathscr{U}$*

$$\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\|^2 \geq (1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2)\ \|u\|^2.$$

Proof. Denote $\eta := \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$. We have $\eta \in (0, 1]$ by Proposition 7.17. Let $\delta \in (0, \eta^2)$. Define

$$\varepsilon := \frac{1}{\sqrt{\eta^2 - \delta}} - \frac{1}{\eta}.$$

It easily follows that $\varepsilon > 0$.

Denote a Bezout factor of $[\mathsf{M}; \mathsf{N}]$ by $[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]$. We have for $z \in \mathbb{D}$ and $u \in \mathscr{U}$

$$\|u\| = \|[\tilde{\mathsf{X}}_1(z), \tilde{\mathsf{Y}}_1(z)][\mathsf{M}(z); \mathsf{N}(z)]u\| \leq \|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty \; \|[\mathsf{M}(z); \mathsf{N}(z)]u\|.$$

From this we obtain

$$\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\|^2 \geq \frac{1}{\|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty^2} \|u\|^2. \tag{7.12}$$

It is easily computed that if $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$ is a Bezout factor, then so is $[\tilde{\mathsf{X}} - \mathsf{V}\tilde{\mathsf{N}}, \tilde{\mathsf{Y}} - \mathsf{V}\tilde{\mathsf{M}}]$ for any $\mathsf{V} \in H^\infty$. Using this we obtain from Corollary 7.30 that for each $\tilde{\varepsilon} > 0$ there exists a right Bezout factor $[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]$ with

$$\|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty \leq \frac{1}{\eta} + \tilde{\varepsilon}. \tag{7.13}$$

In particular we can choose $\tilde{\varepsilon} = \varepsilon$, where ε is as above. With that choice the right-hand side of (7.13) equals $1/\sqrt{\eta^2 - \delta}$. It follows that

$$\frac{1}{\|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty^2} \geq \eta^2 - \delta. \tag{7.14}$$

Combining (7.12) and (7.14) we obtain

$$\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\|^2 \geq \left(1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2 - \delta\right) \; \|u\|^2.$$

Since this holds for every $\delta \in (0, \eta^2)$ we obtain the desired result. $\square$

The following proposition provides an explicit ball around a strongly right-coprime factor that only contains strongly right-coprime pairs.

Proposition 7.32. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$ *and that* $\mathscr{U}$ *is finite-dimensional. Denote the Hankel operator of* $[\mathsf{M}; \mathsf{N}]$ *by* $H_{[\mathsf{M};\mathsf{N}]}$. *If* $\Delta = [\Delta_\mathsf{M}; \Delta_\mathsf{N}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ *is such that* $\|\Delta\|_\infty < \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$, *then the functions* $\mathsf{M} + \Delta_\mathsf{M}$ *and* $\mathsf{N} + \Delta_\mathsf{N}$ *are strongly right-coprime.*

Proof. Define $\varepsilon := \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2} - \|\Delta\|_\infty > 0$. Using that

$$\begin{aligned}\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\| &\leq \left\| \begin{bmatrix} \mathsf{M}(z) + \Delta_{\mathsf{M}}(z) \\ \mathsf{N}(z) + \Delta_{\mathsf{N}}(z) \end{bmatrix} u \right\| + \|\Delta(z) u\| \\ &\leq \left\| \begin{bmatrix} \mathsf{M}(z) + \Delta_{\mathsf{M}}(z) \\ \mathsf{N}(z) + \Delta_{\mathsf{N}}(z) \end{bmatrix} u \right\| + \|\Delta\|_\infty \|u\|,\end{aligned}$$

we have

$$\left\| \begin{bmatrix} \mathsf{M}(z) + \Delta_{\mathsf{M}}(z) \\ \mathsf{N}(z) + \Delta_{\mathsf{N}}(z) \end{bmatrix} u \right\| \geq \left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\| - \|\Delta\|_\infty \, \|u\| \geq \varepsilon \, \|u\|,$$

where we have also used Proposition 7.31. The Corona Theorem (Proposition A.29) then shows that $\mathsf{M} + \Delta_{\mathsf{M}}$ and $\mathsf{N} + \Delta_{\mathsf{N}}$ are strongly right-coprime. □

The following proposition will be useful in the next two chapters.

Proposition 7.33. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a normalized doubly coprime factorization. Define* $\mathsf{W} : \mathbb{T} \to \mathcal{L}(\mathscr{U} \times \mathscr{Y})$ *(almost everywhere) by*

$$\mathsf{W}(z) = \begin{bmatrix} \mathsf{M}(z) & -\tilde{\mathsf{N}}(z)^* \\ \mathsf{N}(z) & \tilde{\mathsf{M}}(z)^* \end{bmatrix}.$$

Then $\mathsf{W}(z)$ *is unitary for almost all* $z \in \mathbb{T}$.

Proof. We first show that $\mathsf{W}(z)$ is an isometry, i.e. that $\mathsf{W}(z)^* \mathsf{W}(z) = I$ for almost all $z \in \mathbb{T}$. We have

$$\begin{aligned}\mathsf{W}(z)^* \mathsf{W}(z) &= \begin{bmatrix} \mathsf{M}(z)^* & \mathsf{N}(z)^* \\ -\tilde{\mathsf{N}}(z) & \tilde{\mathsf{M}}(z) \end{bmatrix} \begin{bmatrix} \mathsf{M}(z) & -\tilde{\mathsf{N}}(z)^* \\ \mathsf{N}(z) & \tilde{\mathsf{M}}(z)^* \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{M}(z)^* \mathsf{M}(z) + \mathsf{N}(z)^* \mathsf{N}(z) & \tilde{\mathsf{N}}(z)^* \tilde{\mathsf{M}}(z)^* - \mathsf{M}(z)^* \tilde{\mathsf{N}}(z)^* \\ \tilde{\mathsf{M}}(z) \mathsf{N}(z) - \tilde{\mathsf{N}}(z) \mathsf{M}(z) & \tilde{\mathsf{M}}(z) \tilde{\mathsf{M}}(z)^* + \tilde{\mathsf{N}}(z) \tilde{\mathsf{N}}(z)^* \end{bmatrix}.\end{aligned}$$

The diagonal entries equal the identity since both the right and the left factorization is normalized. The off-diagonal entries are zero by (7.3). We show that $\mathsf{W}(z)$ is surjective. Since a surjective isometry is unitary this proves the proposition. We use that $\mathsf{W}(z)$ is surjective if and only if its range is closed and $\mathsf{W}(z)^*$ is injective. We first show that the range of any isometry T is closed. Let $y_n \in \operatorname{Im}(T)$ and assume that y_n converges to y. Let x_n be such that $y_n = T x_n$ and define $x = T^* y$. Then

$$y \leftarrow T x_n = T T^* T x_n \to T T^* y = T x.$$

So $y \in \mathrm{Im}(T)$ from which it follows that the range of T is closed. We now show that $\mathsf{W}(z)^*$ is injective. We use (7.3) and the normalization property to obtain

$$\begin{aligned}
[\mathsf{M}^*, \mathsf{N}^*] &= [\mathsf{M}^*, \mathsf{N}^*] \begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{X}} & -\tilde{\mathsf{Y}} \\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}} \end{bmatrix} \\
&= [\tilde{\mathsf{X}} - \mathsf{M}^*\mathsf{Y}\tilde{\mathsf{N}} - \mathsf{N}^*\mathsf{X}\tilde{\mathsf{N}},\ -\tilde{\mathsf{Y}} + \mathsf{M}^*\mathsf{Y}\tilde{\mathsf{M}} + \mathsf{N}^*\mathsf{X}\tilde{\mathsf{M}}],
\end{aligned}$$

on the unit circle. Assume $[u; y] \in \ker \mathsf{W}(z)^*$. Then $\mathsf{M}^*u + \mathsf{N}^*y = 0$ and $-\tilde{\mathsf{N}}u + \tilde{\mathsf{M}}y = 0$. We obtain from the above $0 = \tilde{\mathsf{X}}u - \tilde{\mathsf{Y}}y$. Using (7.3) we obtain from $-\tilde{\mathsf{N}}u + \tilde{\mathsf{M}}y = 0$ and $\tilde{\mathsf{X}}u - \tilde{\mathsf{Y}}y = 0$ that $[u; y] = 0$. It follows that $\mathsf{W}(z)^*$ is injective. This completes the proof. □

Notes

An excellent account of the use of coprime factorizations in systems and control theory is Vidyasagar [94]. The relation with state space systems was made by Khargonekar and Sontag [43] and Nett, Jacobson and Balas [58] in the case of rational functions. The relation between state space systems and normalized coprime factorizations of rational functions was established in Meyer and Franklin [55].

The concept of weak coprimeness as used here is due to Mikkola [56]. The results presented here on weakly coprime factorizations are also due to Mikkola [56]. Our proofs differ only slightly from his. Proposition 7.17 is due to Glover and McFarlane [36] in the rational case. Earlier generalizations to the general, not necessarily rational, case can be found in Curtain and Zwart [18, Lemma 9.4.7] and Oostveen [64, Lemma 7.2.4].

Propositions 7.18 to 7.22 were first given by Curtain and Opmeer [16] for continuous-time systems. This sequence of propositions constitutes our main original contribution on coprime factorizations. The sequence of propositions establishes a long sought after necessary and sufficient state space condition for existence of strongly coprime factorizations over H^∞. Partial result in this direction were obtained in, among others, Curtain and Zwart [19], Curtain, Weiss and Weiss [10], Curtain and Oostveen [12] and Staffans [90]. We note that in [16] also state space formulas for the Bezout factors are given for the continuous-time case. These are based on state space formulas for the continuous-time suboptimal Nehari problem obtained in Curtain and Opmeer [15]. Similar state space formulas can be obtained in discrete-time using the same approach.

Lemma 7.29 is due to Glover and McFarlane [36] for rational functions. The nonrational case was proven by Georgiou and Smith [34] for $\mathscr{U}$ and $\mathscr{Y}$

finite-dimensional. Our proof, also valid for $\mathscr{U}$ and $\mathscr{Y}$ infinite-dimensional, does not significantly differ from the one given by Georgiou and Smith.

Proposition 7.33 is due to Glover and McFarlane [36] for the rational case and to Curtain [11] for the general case considered here.

For a different viewpoint on coprime factorizations for not necessarily rational functions we refer to Quadrat [78], [79], [80], [81], [82], [83].

Chapter 8

Robust stabilization

In this chapter we consider so-called frequency domain feedback controller design. We are mainly interested in feedback controllers that provide a certain type of robustness. The following definition is basic for this chapter.

Definition 8.1. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. We say that K is an **admissible feedback function** for G if $\mathsf{K} : D(\mathsf{K}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{Y}, \mathscr{U})$ is holomorphic with $0 \in D(\mathsf{K})$ and $I - \mathsf{KG}$ has a bounded inverse in zero.

Lemma 8.2. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$ and let K be an admissible feedback function for G. Then $I - \mathsf{GK}$ has a bounded inverse in zero.*

Proof. This follows from Lemma 3.16 with $Z = \mathsf{G}(0)$, $T = \mathsf{K}(0)$ and $\lambda = 1$. □

Remark 8.3. Note that it follows from Lemma 8.2 that K is an admissible feedback function for G if and only if G is an admissible feedback function for K.

Definition 8.4. An admissible feedback function K for G is called **stabilizing** if

$$\begin{bmatrix} (I - \mathsf{KG})^{-1} & \mathsf{K}(I - \mathsf{GK})^{-1} \\ \mathsf{G}(I - \mathsf{KG})^{-1} & (I - \mathsf{GK})^{-1} \end{bmatrix} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U} \times \mathscr{Y})). \quad (8.1)$$

Remark 8.5. Note that the function in (8.1) is the inverse of

$$\begin{bmatrix} I & -\mathsf{K} \\ -\mathsf{G} & I \end{bmatrix}.$$

The intuition behind the above definitions is that G is the transfer function of the plant and that K is the transfer function of the controller. The interconnection shown in figure 8.1 of the two systems is well-defined when K is an admissible feedback function for G. The condition (8.1) is equivalent to the transfer function from $[u_1; u_2]$ to $[e_1; e_2]$ being in H^∞.

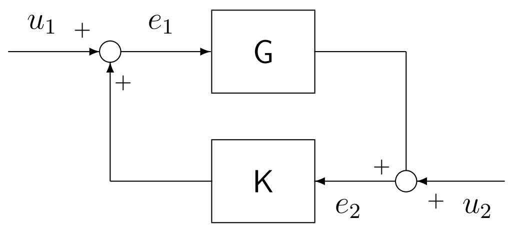

Figure 8.1: Feedback interconnection of G and K.

The following proposition, and most of the other propositions in this chapter, are formulated for right factorizations. By applying them to $\mathsf{G}^\dagger$ one obtains the obvious left versions of the results.

Proposition 8.6. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Let $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$ be a right Bezout pair for this factorization and assume that $\tilde{\mathsf{X}}$ has a bounded inverse in zero. Define $\mathsf{K} := \tilde{\mathsf{X}}^{-1}\tilde{\mathsf{Y}}$. Then K is a stabilizing admissible feedback function for G.*

Proof. We have

$$I - \mathsf{K}\mathsf{G} = I - \tilde{\mathsf{X}}^{-1}\tilde{\mathsf{Y}}\mathsf{N}\mathsf{M}^{-1} = \tilde{\mathsf{X}}^{-1}\left(\tilde{\mathsf{X}}\mathsf{M} - \tilde{\mathsf{Y}}\mathsf{N}\right)\mathsf{M}^{-1} = \tilde{\mathsf{X}}^{-1}\mathsf{M}^{-1},$$

which shows that $I - \mathsf{K}\mathsf{G}$ has a bounded inverse in zero. Furthermore, we have

$$\begin{bmatrix} (I-\mathsf{K}\mathsf{G})^{-1} & \mathsf{K}(I-\mathsf{G}\mathsf{K})^{-1} \\ \mathsf{G}(I-\mathsf{K}\mathsf{G})^{-1} & (I-\mathsf{G}\mathsf{K})^{-1} \end{bmatrix} = \begin{bmatrix} \mathsf{M}\tilde{\mathsf{X}} & \mathsf{M}\tilde{\mathsf{Y}} \\ \mathsf{N}\tilde{\mathsf{X}} & I + \mathsf{N}\tilde{\mathsf{Y}} \end{bmatrix},$$

which is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$. To obtain the formula for $(I - \mathsf{G}\mathsf{K})^{-1}$ we have used that

$$(I - \mathsf{G}\mathsf{K})^{-1} = I + \mathsf{G}(I - \mathsf{K}\mathsf{G})^{-1}\mathsf{K},$$

which is easily checked. □

Proposition 8.7. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that there exists a stabilizing admissible feedback function* K *for* G. *Then* G *has a strongly right-coprime factorization.*

Proof. Define $\mathsf{M}_1 := (I - \mathsf{KG})^{-1}$ and $\mathsf{N}_1 := \mathsf{G}(I - \mathsf{KG})^{-1}$. Since the feedback function is stabilizing we have $\mathsf{M}_1, \mathsf{N}_1 \in H^\infty$ and obviously $\mathsf{G} = \mathsf{N}_1\mathsf{M}_1^{-1}$. So G has a right factorization. By Proposition 7.13 G has a weakly right-coprime factorization $[\mathsf{M}; \mathsf{N}]$. Similarly K has a weakly right-coprime factorization: $\mathsf{K} = \mathsf{YX}^{-1}$. Clearly,

$$\mathsf{N}_2 := \begin{bmatrix} 0 & \mathsf{Y} \\ \mathsf{N} & 0 \end{bmatrix}, \quad \mathsf{M}_2 := \begin{bmatrix} \mathsf{M} & 0 \\ 0 & \mathsf{X} \end{bmatrix}$$

provides a weakly right-coprime factorization of $\mathsf{G}_2 := [0, \mathsf{K}; \mathsf{G}, 0]$. Since the feedback function is stabilizing we have $(I - \mathsf{G}_2)^{-1} \in H^\infty$. It follows from Lemma 7.8 that $(\mathsf{M}_2 - \mathsf{N}_2)^{-1} \in H^\infty$. Denote the upper row of $(\mathsf{M}_2 - \mathsf{N}_2)^{-1}$ by $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$, then $\tilde{\mathsf{X}}\mathsf{M} - \tilde{\mathsf{Y}}\mathsf{N} = I$. Hence $[\mathsf{M}; \mathsf{N}]$ is a strongly right-coprime factor. $\square$

Corollary 8.8. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Any stabilizing feedback function for* G *has a strongly right-coprime factorization.*

Proof. The symmetry mentioned in remark 8.3 and Proposition 8.7 give the result. $\square$

Proposition 8.9. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Then the following are equivalent:*

1. G *has a strongly right-coprime factorization.*

2. *There exists a stabilizing admissible feedback function for* G.

Proof. (2) implies (1) is Proposition 8.7. (1) implies (2) follows from Proposition 8.6 using Lemma 7.23. $\square$

Proposition 8.10. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a doubly coprime factorization. Then all stabilizing admissible feedback functions are given by*

$$\mathsf{K} = (\mathsf{Y} + \mathsf{MV})(\mathsf{X} + \mathsf{NV})^{-1}, \tag{8.2}$$

where $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ *is such that* $\mathsf{X} + \mathsf{NV}$ *has a bounded inverse in zero, but is otherwise arbitrary.*

Proof. We first show that the function defined by (8.2) is a stabilizing admissible feedback function. Define $\underline{\mathsf{Y}} := \mathsf{Y} + \mathsf{MV}$, $\underline{\mathsf{X}} := \mathsf{X} + \mathsf{NV}$. Then

$$\begin{bmatrix} \mathsf{M} & \underline{\mathsf{Y}} \\ \mathsf{N} & \underline{\mathsf{X}} \end{bmatrix} = \begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} \begin{bmatrix} I & \mathsf{V} \\ 0 & I \end{bmatrix},$$

from which it easily follows that with $\underline{\tilde{\mathsf{X}}} := \tilde{\mathsf{X}} + \mathsf{V}\tilde{\mathsf{N}}$ and $\underline{\tilde{\mathsf{Y}}} := \tilde{\mathsf{Y}} + \mathsf{V}\tilde{\mathsf{M}}$ we obtain (7.3) with the Bezout factors replaced by the underlined versions. It follows from the left version of Proposition 8.6 that if $\underline{\mathsf{X}}$ is invertible at zero, then $\underline{\mathsf{Y}}\underline{\mathsf{X}}^{-1}$ is a stabilizing admissible feedback function.

Assume that K is a stabilizing admissible feedback function. We will show that it is of the form (8.2). Since there exists a stabilizing admissible feedback function, G has a doubly coprime factorization by Propositions 7.24 and 8.7. From Proposition 7.24 and Corollary 8.8 we obtain that K has a strongly left-coprime factorization: $\mathsf{K} = \tilde{\mathsf{W}}^{-1}\tilde{\mathsf{Z}}$ and $\tilde{\mathsf{W}}\mathsf{S} - \tilde{\mathsf{Z}}\mathsf{R} = I$. Define $\Delta := \tilde{\mathsf{Z}}\mathsf{N} - \tilde{\mathsf{W}}\mathsf{M}$. Since $\Delta = \tilde{\mathsf{W}}(\mathsf{KG} - I)\mathsf{M}$ we have that Δ is invertible in zero. It is easily calculated that the matrix in (8.1) can be written as

$$\begin{bmatrix} (I - \mathsf{KG})^{-1} & \mathsf{K}(I - \mathsf{GK})^{-1} \\ \mathsf{G}(I - \mathsf{KG})^{-1} & (I - \mathsf{GK})^{-1} \end{bmatrix} = \begin{bmatrix} -\mathsf{M}\Delta^{-1}\tilde{\mathsf{W}} & -\mathsf{M}\Delta^{-1}\tilde{\mathsf{Z}} \\ -\mathsf{N}\Delta^{-1}\tilde{\mathsf{W}} & I - \mathsf{N}\Delta^{-1}\tilde{\mathsf{Z}} \end{bmatrix}. \quad (8.3)$$

Using the above Bezout equation for the strongly left-coprime factorization of K we obtain $\mathsf{M}\Delta^{-1} = \mathsf{M}\Delta^{-1}\tilde{\mathsf{W}}\mathsf{S} - \mathsf{M}\Delta^{-1}\tilde{\mathsf{Z}}\mathsf{R}$, which is in H^∞. Similarly we obtain $\mathsf{N}\Delta^{-1} \in H^\infty$. Using Proposition 7.7 we see that $\Delta^{-1} \in H^\infty$. Define $\mathsf{V} := \Delta^{-1}(\tilde{\mathsf{W}}\mathsf{Y} - \tilde{\mathsf{Z}}\mathsf{X}) \in H^\infty$. We show that $\mathsf{K}(\mathsf{X} + \mathsf{NV}) = \mathsf{Y} + \mathsf{MV}$. Using (8.3) we obtain

$$\begin{aligned} \mathsf{Y} + \mathsf{MV} &= \mathsf{Y} + \mathsf{M}\Delta^{-1}(\tilde{\mathsf{W}}\mathsf{Y} - \tilde{\mathsf{Z}}\mathsf{X}) = (I + \mathsf{M}\Delta^{-1}\tilde{\mathsf{W}})\mathsf{Y} - \mathsf{M}\Delta^{-1}\tilde{\mathsf{Z}}\mathsf{X} \\ &= (I - (I - \mathsf{KG})^{-1}))\mathsf{Y} + \mathsf{K}(I - \mathsf{GK})^{-1}\mathsf{X}, \end{aligned}$$

and

$$\begin{aligned} \mathsf{X} + \mathsf{NV} &= \mathsf{X} + \mathsf{N}\Delta^{-1}(\tilde{\mathsf{W}}\mathsf{Y} - \tilde{\mathsf{Z}}\mathsf{X}) = \mathsf{N}\Delta^{-1}\tilde{\mathsf{W}}\mathsf{Y} + (I - \mathsf{N}\Delta^{-1}\tilde{\mathsf{Z}})\mathsf{X} \\ &= -\mathsf{G}(I - \mathsf{KG})^{-1}\mathsf{Y} + (I - \mathsf{GK})^{-1}\mathsf{X}. \end{aligned} \quad (8.4)$$

From this it easily follows that $\mathsf{K}(\mathsf{X} + \mathsf{NV}) = \mathsf{Y} + \mathsf{MV}$. So the only thing left to show is that $\mathsf{X} + \mathsf{NV}$ has a bounded inverse in zero. Using (8.4) we obtain

$$\tilde{\mathsf{M}}(I - \mathsf{GK})(\mathsf{X} + \mathsf{NV}) = -\tilde{\mathsf{N}}\mathsf{Y} + \tilde{\mathsf{M}}\mathsf{X} = I,$$

where the last identity follows from (7.3). Since $\tilde{\mathsf{M}}(I - \mathsf{GK})$ has a bounded inverse in zero it follows that $\mathsf{X} + \mathsf{NV}$ does and that this inverse equals $\tilde{\mathsf{M}}(I - \mathsf{GK})$. □

Lemma 8.11. *If in a Banach algebra we have* $\|x\|^2 + \|y\|^2 \le \alpha^2 < 1$*, then* $I - y$ *is invertible and* $\|(I-y)^{-1}x\|^2 \le \alpha^2/(1-\alpha^2)$.

Proof. That $I - y$ has a bounded inverse follows from the Neumann series theorem. From this theorem we also obtain $\|(I-y)^{-1}\| \le 1/(1-\|y\|)$. It follows that $\|(I-y)^{-1}x\|^2 \le \|x\|^2/(1-\|y\|)^2$. Denote $x_1 := \|x\|$ and $y_1 := \|y\|$. Using elementary vector calculus one sees that the function $x_1^2/(1-y_1)^2$ under the constraint $x_1^2 + y_1^2 \le \alpha^2 < 1$ has the maximum $\alpha^2/(1-\alpha^2)$. The desired result follows. $\square$

We now focus our attention on feedback functions that provide a certain robustness.

Definition 8.12. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Suppose that G has a normalized weakly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Let $\varepsilon > 0$. The function G_Δ is called a ε-**right factor admissible perturbation** of G if G_Δ has a right factor $[\mathsf{M}_\Delta; \mathsf{N}_\Delta]$ with $\|[\mathsf{M}_\Delta; \mathsf{N}_\Delta] - [\mathsf{M}; \mathsf{N}]\|_\infty < \varepsilon$.

Note that, due to Proposition 7.15, the set of ε-right factor admissible perturbations does not depend on the specific normalized weakly right-coprime factor chosen to define it.

Definition 8.13. Suppose that G has a normalized weakly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Let $\varepsilon > 0$. An admissible feedback function K for G is called ε-**robust right factor stabilizing** if it is a stabilizing admissible feedback function for all ε-right factor admissible perturbations of G.

Proposition 8.14. *Suppose that* G *has a normalized doubly coprime factorization. Let* $\varepsilon \in (0,1)$*. Suppose that* $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))$ *satisfies*

$$\left\| [\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}] \right\| \le \sqrt{1-\varepsilon^2},$$

and that $\tilde{\mathsf{V}}$ *has a bounded inverse in zero. Then* $\mathsf{K} := \tilde{\mathsf{V}}^{-1}\tilde{\mathsf{U}}$ *is a* ε*-robust right factor stabilizing admissible feedback function for* G.

Proof. Let $\mathsf{W} : \mathbb{T} \to \mathcal{L}(\mathscr{U} \times \mathscr{Y})$ be the function from Proposition 7.33, i.e.

$$\mathsf{W}(z) = \begin{bmatrix} \mathsf{M}(z) & -\tilde{\mathsf{N}}(z)^* \\ \mathsf{N}(z) & \tilde{\mathsf{M}}(z)^* \end{bmatrix}.$$

Define $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))$ by

$$\mathsf{F} := \Big([\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}] \Big) \mathsf{W} = [I - \tilde{\mathsf{V}}\mathsf{M} + \tilde{\mathsf{U}}\mathsf{N}, \tilde{\mathsf{V}}\tilde{\mathsf{N}}^* + \tilde{\mathsf{U}}\tilde{\mathsf{M}}^*], \tag{8.5}$$

where we have used (7.3). Since $\mathsf{W}(z)$ is unitary we have

$$\|\mathsf{F}\|_\infty \leq \sqrt{1-\varepsilon^2}.$$

It follows that $\|I - \tilde{\mathsf{V}}\mathsf{M} + \tilde{\mathsf{U}}\mathsf{N}\|_\infty < 1$. Since $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ is a Banach algebra it follows that $\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N}$ has an inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$. Denote an arbitrary ε-right factor admissible perturbation of G by G_Δ. Denote a right factor of G_Δ as in Definition 8.12 by $[\mathsf{M}_\Delta; \mathsf{N}_\Delta]$. Further denote $\Delta = [\Delta_\mathsf{M}; \Delta_\mathsf{N}] = [\mathsf{M}_\Delta; \mathsf{N}_\Delta] - [\mathsf{M}; \mathsf{N}]$. We have

$$\begin{aligned}
I - \mathsf{K}\mathsf{G}_\Delta &= I - \tilde{\mathsf{V}}^{-1}\tilde{\mathsf{U}}(\mathsf{N} + \Delta_\mathsf{N})(\mathsf{M} + \Delta_\mathsf{M})^{-1} \\
&= \tilde{\mathsf{V}}^{-1}\left(\tilde{\mathsf{V}}\mathsf{M} + \tilde{\mathsf{V}}\Delta_\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N} - \tilde{\mathsf{U}}\Delta_\mathsf{N}\right)(\mathsf{M} + \Delta_\mathsf{M})^{-1} \\
&= \tilde{\mathsf{V}}^{-1}\left(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N} + [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}]\begin{bmatrix} \Delta_\mathsf{M} \\ \Delta_\mathsf{N} \end{bmatrix}\right)(\mathsf{M} + \Delta_\mathsf{M})^{-1} \\
&= \tilde{\mathsf{V}}^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})\,(I + \mathsf{S}\Delta)\,(\mathsf{M} + \Delta_\mathsf{M})^{-1},
\end{aligned}$$

where

$$\mathsf{S} := (\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}[\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}].$$

It follows that $I - \mathsf{K}\mathsf{G}_\Delta$ has an inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ if and only if $I + \mathsf{S}\Delta$ does. The latter is true if $\|\mathsf{S}\|_\infty < 1/\varepsilon$. We have

$$\begin{aligned}
\|\mathsf{S}\|_\infty^2 &= \|\mathsf{S}\mathsf{W}\|_\infty^2 = \|[I, -(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}(\tilde{\mathsf{V}}\tilde{\mathsf{N}}^* + \tilde{\mathsf{U}}\tilde{\mathsf{M}}^*)]\|^2 \\
&= 1 + \|(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}(\tilde{\mathsf{V}}\tilde{\mathsf{N}}^* + \tilde{\mathsf{U}}\tilde{\mathsf{M}}^*)]\|^2 = 1 + \|(I - \mathsf{F}_1)^{-1}\mathsf{F}_2\|^2,
\end{aligned}$$

where $\mathsf{F} = [\mathsf{F}_1, \mathsf{F}_2]$ is the function from (8.5). From Lemma 8.11 we obtain

$$\|\mathsf{S}\|_\infty^2 \leq \frac{1}{\varepsilon^2},$$

as desired.

To check that K stabilizes G_Δ we have to show that

$$\begin{bmatrix} (I - \mathsf{K}\mathsf{G}_\Delta)^{-1} & \mathsf{K}(I - \mathsf{G}_\Delta\mathsf{K})^{-1} \\ \mathsf{G}_\Delta(I - \mathsf{K}\mathsf{G}_\Delta)^{-1} & (I - \mathsf{G}_\Delta\mathsf{K})^{-1} \end{bmatrix} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y})).$$

We already saw that $(I - \mathsf{K}\mathsf{G}_\Delta)^{-1} \in H^\infty$. We compute

$$\mathsf{G}_\Delta(I - \mathsf{K}\mathsf{G}_\Delta)^{-1} = (\mathsf{N} + \Delta_\mathsf{N})(I + \mathsf{S}\Delta)^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}\tilde{\mathsf{V}},$$

which is clearly in H^∞. Similarly we have

$$\mathsf{K}(I - \mathsf{G}_\Delta\mathsf{K})^{-1} = (I - \mathsf{K}\mathsf{G}_\Delta)^{-1}\mathsf{K} = (\mathsf{M} + \Delta_\mathsf{M})(I + \mathsf{S}\Delta)^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}\tilde{\mathsf{U}},$$

and

$$(I - \mathsf{G}_\Delta\mathsf{K})^{-1} = I + \mathsf{G}_\Delta(I - \mathsf{K}\mathsf{G}_\Delta)^{-1}\mathsf{K} = I + (\mathsf{N} + \Delta_\mathsf{N})(I + \mathsf{S}\Delta)^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}\tilde{\mathsf{U}},$$

which are both in H^∞. $\square$

Proposition 8.15. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Suppose that* G *has a normalized doubly coprime factorization. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of the normalized strongly right-coprime factor* $[\mathsf{M};\mathsf{N}]$. *Let* $\varepsilon < \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$. *Then there exists a* $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))$ *such that*

$$\left\| [\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}] \right\| \leq \sqrt{1 - \varepsilon^2}.$$

If the input space $\mathscr{U}$ *is finite-dimensional, then* $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}]$ *can be chosen such that* $\tilde{\mathsf{V}}$ *has a bounded inverse in zero.*

Proof. The existence of $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}]$ such that the desired inequality is satisfied follows easily from Nehari's theorem (Lemma A.27). We now show that if $\mathscr{U}$ is finite-dimensional we can choose $\tilde{\mathsf{V}}$ such that it has a bounded inverse in zero. First choose a $\tilde{\varepsilon} \in (\varepsilon, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$ and find a $[\tilde{\mathsf{V}}_1, \tilde{\mathsf{U}}_1]$ such that the desired inequality is satisfied with $\tilde{\varepsilon}$ instead of ε. Define $[\tilde{\mathsf{V}}_\delta, \tilde{\mathsf{U}}_\delta] := [\tilde{\mathsf{V}}_1, \tilde{\mathsf{U}}_1] - \delta[I, 0]$. Then this satisfies

$$\left\| [\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}_\delta, -\tilde{\mathsf{U}}_\delta] \right\| \leq \sqrt{1 - \tilde{\varepsilon}^2} + \delta.$$

It follows that for $\delta \in (0, \sqrt{1 - \varepsilon^2} - \sqrt{1 - \tilde{\varepsilon}^2})$ the desired inequality is satisfied. Since $\mathscr{U}$ is finite-dimensional, this interval must contain a point in the resolvent set of $\tilde{\mathsf{V}}_1(0)$. For such a δ we have that $\tilde{\mathsf{V}}_\delta$ has a bounded inverse in zero. $\square$

Remark 8.16. If $\mathsf{G}(0) = 0$, then the conclusion of Proposition 8.15 is also true without the assumption that the input space $\mathscr{U}$ is finite-dimensional. We indicate why this is true. From the proof of Proposition 8.14 we obtain that $\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N}$ has an inverse in H^∞. It follows that $(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})(0)$ has a bounded inverse. We have $\mathsf{N}(0) = \mathsf{G}(0)\mathsf{M}(0) = 0$ and so $\tilde{\mathsf{V}}(0)\mathsf{M}(0) = (\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})(0)$ has a bounded inverse. Since $\mathsf{M}(0)$ has a bounded inverse it follows that $\tilde{\mathsf{V}}$ has a bounded inverse in zero as desired.

Proposition 8.17. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Suppose that* G *has a normalized doubly coprime factorization and that* $\mathscr{U}$ *is finite-dimensional. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of the normalized strongly right-coprime factor* $[\mathsf{M};\mathsf{N}]$. *Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$. *Then there exists an* ε*-robust right factor stabilizing admissible feedback function for* G.

Proof. This follows from combining Propositions 8.14 and 8.15. $\square$

Remark 8.18. The assumption in Proposition 8.17 that $\mathscr{U}$ is finite-dimensional is made because that assumption had to be made in Proposition 8.15 to obtain invertiblity of $\tilde{\mathsf{V}}$ in zero. We can avoid making this assumption in Proposition 8.17 by considering controllers with internal loop as in Curtain, Weiss and Weiss [17].

Corollary 8.19. *Suppose that* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *with* $0 \in D(\mathsf{G})$ *is a matrix-valued rational function. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of a normalized strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$*. Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$*. Then there exists a rational* ε*-robust right factor stabilizing admissible feedback function for* G*.*

Proof. This follows from the proof of Proposition 8.17, using that in the proof of Proposition 8.15 we can choose $[\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}]$ rational by Lemma A.28. □

Notes

We refer to Vidyagagar [94], Zhou, Doyle and Glover [103], Doyle, Francis and Tannenbaum [22] and Francis [28] for general information on stabilizing feedback functions for rational functions.

For finite-dimensional input and output spaces Proposition 8.9 is originally due to Inouye [40]. An independent proof was given by Smith [88]. The general case, also valid for infinite-dimensional input and output spaces, was first proven by Mikkola [56], whose proof we followed. Proposition 8.10 is due to Youla, Jabr and Bongiorno [98] in the case of rational functions. Using controllers with internal loop this result in the general case is due to Curtain, Weiss and Weiss [17]. The results presented here on robust right factor stabilizing feedback functions are due to McFarlane and Glover [36], [54] in the rational case. Continuous-time versions of the general case are given in Curtain and Zwart [18, Section 9.4], Oostveen [64, Section 7.2] and Curtain [11]. All of these references closely follow the arguments in McFarlane and Glover as do we.

Chapter 9

The gap metric

In this chapter we provide an alternative view towards ε-admissible right factor perturbations. This is done in terms of the gap metric.

We first consider the gap metric as a metric on the set of closed subspaces of a given Hilbert space. The relevant definition is as follows.

Definition 9.1. Let $\mathscr{K}_i$ $(i = 1, 2)$ be closed subspaces of the Hilbert space $\mathscr{H}$. Denote by P_i the orthogonal projection onto $\mathscr{K}_i$. Define

$$\delta(\mathscr{K}_1, \mathscr{K}_2) = \|P_1 - P_2\|.$$

The function δ is called the **gap metric**.

Lemmas 9.2 through 9.7 give some basic properties of the gap metric.

Lemma 9.2. *The gap metric is a metric on the set of closed subspaces of a given Hilbert space.*

Proof. Symmetry is obvious. It is also obvious that $\delta(\mathscr{K}_1, \mathscr{K}_2) = 0$ implies $\mathscr{K}_1 = \mathscr{K}_2$. The triangle inequality follows from the triangle inequality in $\mathcal{L}(\mathscr{H})$. □

Definition 9.3. Let $\mathscr{K}_i$ $(i = 1, 2)$ be closed subspaces of the Hilbert space $\mathscr{H}$. Denote by P_i the orthogonal projection onto $\mathscr{K}_i$. Define

$$\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2) = \|(I - P_2)P_1\|.$$

The function $\vec{\delta}$ is called the **directed gap**.

Lemma 9.4. *We have*

$$\delta(\mathscr{K}_1, \mathscr{K}_2) = \max\{\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2), \vec{\delta}(\mathscr{K}_2, \mathscr{K}_1)\}.$$

Proof. We have

$$(P_1 - P_2)x = P_1(I - P_2)x - (I - P_1)P_2x = P_1(I - P_2)^2x - (I - P_1)P_2^2x.$$

Using that $P_1(I - P_2)^2x$ and $(I - P_1)P_2^2x$ are orthogonal we obtain from this that

$$\begin{aligned}
\|(P_1 - P_2)x\|^2 &= \|P_1(I - P_2)^2x - (I - P_1)P_2^2x\|^2 \\
&= \|P_1(I - P_2)^2x\|^2 + \|(I - P_1)P_2^2x\|^2 \\
&\leq \|P_1(I - P_2)\|^2\|(I - P_2)x\|^2 + \|(I - P_1)P_2\|^2 \ \|P_2x\|^2.
\end{aligned}$$

Since $\|(I - P_2)x\|^2 + \|P_2x\|^2 = \|x\|^2$ we obtain from this

$$\|(P_1 - P_2)x\|^2 \leq \max\{\|P_1(I - P_2)\|^2, \|(I - P_1)P_2\|^2\} \ \|x\|^2.$$

Since the adjoint of $(I - P_1)P_2$ equals $P_2(I - P_1)$ we have

$$\|(P_1 - P_2)x\|^2 \leq \max\{\|P_1(I - P_2)\|^2, \|P_2(I - P_1)\|^2\} \ \|x\|^2.$$

It follows that $\delta(\mathscr{K}_1, \mathscr{K}_2) \leq \max\{\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2), \vec{\delta}(\mathscr{K}_2, \mathscr{K}_1)\}$. The converse inequality follows from

$$\|(I - P_2)P_1\| = \|(P_1 - P_2)P_1\| \leq \|P_1 - P_2\|$$

and the similar inequality with the roles of P_1 and P_2 reversed. □

Lemma 9.5. *We have $\delta(\mathscr{K}_1, \mathscr{K}_2) < 1$ if and only if P_1 restricts to a bijection from $\mathscr{K}_2$ onto $\mathscr{K}_1$. In this case we have $\delta(\mathscr{K}_1, \mathscr{K}_2) = \vec{\delta}(\mathscr{K}_1, \mathscr{K}_2) = \vec{\delta}(\mathscr{K}_2, \mathscr{K}_1)$.*

Proof. Assume that $\delta(\mathscr{K}_1, \mathscr{K}_2) < 1$. This implies that $\|P_1 - P_2\| < 1$, from which it follows that $T := I - P_1 + P_2$ has a bounded inverse. We have $P_1T = P_1P_2$. Since T maps $\mathscr{H}$ onto $\mathscr{H}$ and P_1 maps $\mathscr{H}$ onto $\mathscr{K}_1$, it follows that P_1P_2 maps $\mathscr{H}$ onto $\mathscr{K}_1$. Obviously it then follows that P_1 maps $\mathscr{K}_2$ onto $\mathscr{K}_1$. If $h \in \mathscr{K}_2$ is such that $P_1h = 0$, then $\|h\| = \|(I - P_1)h\|$, since $[P_1; I - P_1]$ is an isometry. Since $h \in \mathscr{K}_2$ we have $P_2h = h$ and so we obtain $\|h\| = \|(I - P_1)P_2h\| \leq \|P_2 - P_1\| \ \|h\|$. Since by assumption $\|P_2 - P_1\| < 1$, this can only hold if $h = 0$. It follows that P_1 restricted to $\mathscr{K}_2$ is injective.

Now assume that P_1 restricts to a bijection from $\mathscr{K}_2$ onto $\mathscr{K}_1$. Denote this restriction by P_1^r. Define

$$\tau_1 := \inf_{h \in \mathscr{K}_2, \|h\|=1} \|P_1^r h\|.$$

Since P_1^r has a bounded inverse, we have $\tau_1 > 0$. Using that $\|(I - P_1)P_2h\|^2 = \|P_2h\|^2 - \|P_1P_2h\|^2$ (which follows from the Pythagorean Theorem), we have

$$\sup_{h \in \mathscr{K}_2, \|h\|=1} \|(I - P_1)P_2h\|^2 = 1 - \inf_{h \in \mathscr{K}_2, \|h\|=1} \|P_1h\|^2 = 1 - \tau_1^2.$$

From this we obtain $\|(I-P_1)P_2\|^2 = 1-\tau_1^2$. By interchanging the role of P_1 and P_2 we obtain $\|(I-P_2)P_1\|^2 = 1-\tau_2^2$, where $\tau_2 := \inf_{h\in\mathscr{K}_1, \|h\|=1} \|P_2^r h\|$ and P_2^r is the restriction of P_2 to $\mathscr{K}_1$. We now show that the adjoint of P_1^r equals P_2^r. Let $h_1 \in \mathscr{K}_1$ and $h_2 \in \mathscr{K}_2$, then

$$\langle P_1^r h_2, h_1\rangle_{\mathscr{K}_1} = \langle P_1 h_2, h_1\rangle_{\mathscr{H}} = \langle h_2, P_1 h_1\rangle_{\mathscr{H}} = \langle h_2, h_1\rangle_{\mathscr{H}}.$$

Similarly we obtain $\langle h_2, h_1\rangle_{\mathscr{H}} = \langle h_2, P_2^r h_1\rangle_{\mathscr{K}_2}$. We conclude that the adjoint of $P_1^r \in \mathcal{L}(\mathscr{K}_2, \mathscr{K}_1)$ is $P_2^r \in \mathcal{L}(\mathscr{K}_1, \mathscr{K}_2)$. From Lemma 7.28 we obtain $\tau_1 = \tau_2$. So we obtain $\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2) = \vec{\delta}(\mathscr{K}_2, \mathscr{K}_1) < 1$. □

Lemma 9.6. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$. Assume that T is an isometry. Then the orthogonal projection onto the range of T is given by $P := TT^*$.*

Proof. For this we have to show three things:

1. P is a projection, i.e. $P = P^2$.
2. The projection is orthogonal, i.e. $P = P^*$.
3. P maps onto the image of T.

$P^2 = TT^*TT^* = TT^* = P$, where we have used that $T^*T = I$ since T is an isometry. That P is self-adjoint is obvious, so the projection is orthogonal. We have

$$\mathrm{Im}(P) = \mathrm{Im}(TT^*) \subset \mathrm{Im}(T) = \mathrm{Im}(TT^*T) \subset \mathrm{Im}(TT^*) = \mathrm{Im}(P),$$

where we have again used that $T^*T = I$. So $\mathrm{Im}(P) = \mathrm{Im}(T)$. □

Lemma 9.7. *Let $T_1 \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ be an isometry and $\mathscr{K}_2$ a closed subspace of $\mathscr{H}_2$. Define $\mathscr{K}_1 := \mathrm{Im}(T_1)$. Then $\mathscr{K}_1$ is a closed subspace of $\mathscr{H}_2$ and*

$$\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2) = \|(I-P_2)T_1\|.$$

Proof. That the image of an isometry is closed is easily proven. Using Lemma 9.6 we obtain $P_1 = T_1 T_1^*$, so

$$\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2) = \|(I-P_2)T_1 T_1^*\|.$$

So we need to show that $\|(I-P_2)T_1T_1^*\| = \|(I-P_2)T_1\|$. We show that in general for $S \in \mathcal{L}(\mathscr{H}_2)$ we have $\|ST_1T_1^*\| = \|ST_1\|$. We have

$$\|ST_1T_1^*\| \le \|ST_1\|\,\|T_1\| = \|ST_1\| = \|ST_1T_1^*T_1\| \le \|ST_1T_1^*\|\,\|T_1\| = \|ST_1T_1^*\|.$$

□

We use the space $\mathscr{V}(0)$ of stable input-output pairs for initial condition zero (see (6.2)) to define a distance between discrete-time systems.

Definition 9.8. Let Σ_i ($i = 1, 2$) be discrete-time systems with the same input and output spaces. The **gap** $\delta(\Sigma_1, \Sigma_2)$ is defined as $\delta(\mathscr{V}_1(0), \mathscr{V}_2(0))$. The **directed gap** $\vec{\delta}(\Sigma_1, \Sigma_2)$ is defined as $\vec{\delta}(\mathscr{V}_1(0), \mathscr{V}_2(0))$.

Remark 9.9. Note that discrete-time systems with the same transfer function have gap zero. So the gap is not a metric on the set of discrete-time systems. Let $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ and $\mathsf{G}(z) = \sqrt{z - \alpha}$. For $\alpha \neq 0$ this function is holomorphic at zero (with an appropriate choice of the branch cut). This implies that G can be realized as the transfer function of a discrete-time system. If $\hat{u} \in H^2$, then $\hat{y} := \mathsf{G}\hat{u}$ can never be in H^2 unless $\hat{u} = 0$ since otherwise $\mathsf{G} = \hat{y}/\hat{u}$ would be meromorphic in $\mathbb{D}$, which it clearly is not if $|\alpha| < 1$. This implies that for all realizations of G we have $\mathscr{V}(0) = \{0\}$. Since this is true for all $\alpha \neq 0$ with $|\alpha| < 1$ it is not even true that the gap is a distance on equivalence classes of discrete-time systems with the same transfer function.

Definition 9.10. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a right factor $[\mathsf{M}; \mathsf{N}]$. Define the space $Z_{[\mathsf{M};\mathsf{N}]} \subset H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y})$ by $Z_{[\mathsf{M};\mathsf{N}]} = \{(\mathsf{M}_i v; \mathsf{N}_i v) : v \in H^2(\mathbb{D}, \mathscr{U})\}$.

Proposition 9.11. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a right factorization. The space $Z_{[\mathsf{M};\mathsf{N}]}$ equals $\hat{\mathscr{V}}(0)$ for all weakly right-coprime factors.*

Proof. This follows from Proposition 7.10. □

The following shows that the gap metric gives a metric on the space of holomorphic functions defined in a neighbourhood of zero that have a right factorization.

Proposition 9.12. *Let $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G}_i)$ ($i = 1, 2$). Assume that G_i has a right factorization. Then $\delta(\mathsf{G}_1, \mathsf{G}_2) = 0$ implies that $\mathsf{G}_1 = \mathsf{G}_2$ in a neighbourhood of zero.*

Proof. It follows from Proposition 7.13 that G_i has a weakly right-coprime factor $[\mathsf{M}_i; \mathsf{N}_i]$. Since $\delta(\mathsf{G}_1, \mathsf{G}_2) = 0$ it follows that $\mathscr{V}_1(0) = \mathscr{V}_2(0)$. It follows from Proposition 9.11 that the spaces $Z_{[\mathsf{M}_i;\mathsf{N}_i]}$ for $i = 1, 2$ are equal. In particular $[\mathsf{M}_1(z); \mathsf{N}_1(z)]u = [\mathsf{M}_2(z); \mathsf{N}_2(z)]u$ for all $u \in \mathscr{U}$ and $z \in \mathbb{D}$. It follows that $[\mathsf{M}_1(z); \mathsf{N}_1(z)] = [\mathsf{M}_2(z); \mathsf{N}_2(z)]$ for all $z \in \mathbb{D}$. From this we obtain $\mathsf{G}_1(z) = \mathsf{N}_1(z)\mathsf{M}_1^{-1}(z) = \mathsf{N}_2(z)\mathsf{M}_2^{-1}(z) = \mathsf{G}_2(z)$ in a neighbourhood of zero. □

The next two propositions give alternative characterizations of the directed gap.

Proposition 9.13. *Let* $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G}_i)$ *(*$i = 1, 2$*). Assume that* G_1 *and* G_2 *have normalized weakly right-coprime factors* $[\mathsf{M}_1; \mathsf{N}_1]$ *and* $[\mathsf{M}_2; \mathsf{N}_2]$*, respectively. Denote by* $T_{[\mathsf{M}_1;\mathsf{N}_1]}$ *the multiplication operator and by* $P_{Z^{\perp}_{[\mathsf{M}_2;\mathsf{N}_2]}}$ *the orthogonal projection. The directed gap equals*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) = \left\| P_{Z^{\perp}_{[\mathsf{M}_2;\mathsf{N}_2]}} T_{[\mathsf{M}_1;\mathsf{N}_1]} \right\|.$$

Proof. This follows from Lemma 9.7 and Proposition 9.11. □

Proposition 9.14. *Let* $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G}_i)$ *(*$i = 1, 2$*). Assume that* G_1 *and* G_2 *have normalized weakly right-coprime factors* $[\mathsf{M}_1; \mathsf{N}_1]$ *and* $[\mathsf{M}_2; \mathsf{N}_2]$*, respectively. Then*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} \mathsf{V} \right\|.$$

Proof. This follows from Proposition 9.13 and Corollary A.22. □

Proposition 9.15. *Let* $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G}_i)$ *(*$i = 1, 2$*). Assume that* G_1 *has a normalized weakly right-coprime factor* $[\mathsf{M}_1; \mathsf{N}_1]$ *and that* G_2 *has a right factorization. Then*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) \leq \left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} \right\|,$$

where $[\mathsf{M}_2; \mathsf{N}_2]$ *is any right factor of* G_2*.*

Proof. This follows from Propositions 7.14 and 9.14. □

The following corollary deals with ε-right factor admissible perturbations (see Definition 8.12).

Corollary 9.16. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. Assume that* G *has a normalized weakly right-coprime factorization. Let* G_Δ *be a* ε*-right factor admissible perturbation of* G*. Then* $\vec{\delta}(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon$*.*

Proof. This follows immediately from Proposition 9.15. □

Corollary 9.17. *Let $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G}_i)$ $(i = 1, 2)$. Assume that G_1 has a normalized weakly right-coprime factor $[\mathsf{M}_1; \mathsf{N}_1]$ and G_2 has a right factorization. If $\mathscr{U}$ is finite-dimensional, then*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) = \inf_{[\mathsf{M}_2;\mathsf{N}_2]} \left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} \right\|,$$

where $[\mathsf{M}_2; \mathsf{N}_2]$ is any right factor of G_2.

Proof. Let $[\mathsf{M}_2^0; \mathsf{N}_2^0]$ be a normalized weakly right-coprime factor of G_2. For each $\varepsilon > 0$ we will construct a $\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ which has a bounded inverse in zero and is such that

$$\left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \tilde{\mathsf{V}} \right\| - \vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) < \varepsilon. \tag{9.1}$$

The desired equality then immediately follows using Proposition 7.14 (and Proposition 9.15). By Proposition 9.14 there exists a $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ such that

$$\left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \mathsf{V} \right\| - \vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) < \varepsilon/2.$$

Define $\tilde{\mathsf{V}} := \mathsf{V} + \delta I$. It is then easily computed that

$$\left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \tilde{\mathsf{V}} \right\| - \vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) < \varepsilon/2 + \left\| \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \right\| \delta.$$

Define $\eta := \varepsilon/(2\|[\mathsf{M}_2^0; \mathsf{N}_2^0]\|)$. If we choose $\delta \in (0, \eta)$, then (9.1) is satisfied. Since $\mathscr{U}$ is finite-dimensional the interval $(-\eta, 0)$ must contain points that are in the resolvent set of $\mathsf{V}(0) \in \mathcal{L}(\mathscr{U})$. This implies that there exists a $\delta \in (0, \eta)$ such that $\tilde{\mathsf{V}}(0)$ has a bounded inverse. □

Proposition 9.18. *Let $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G}_i)$ $(i = 1, 2)$. Assume that G_1 and G_2 have weakly right-coprime factors $[\mathsf{M}_1; \mathsf{N}_1]$ and $[\mathsf{M}_2; \mathsf{N}_2]$, respectively. Assume further that $[\mathsf{M}_1; \mathsf{N}_1]$ is normalized and that $\|\Delta\|_\infty < 1$, where $\Delta := [\mathsf{M}_2; \mathsf{N}_2] - [\mathsf{M}_1; \mathsf{N}_1]$. Then $\delta(\mathsf{G}_1, \mathsf{G}_2) = \vec{\delta}(\mathsf{G}_1, \mathsf{G}_2) = \vec{\delta}(\mathsf{G}_2, \mathsf{G}_1) < 1$.*

Proof. Let $h \in H^2(\mathscr{U})$, we consider the projection of $[\mathsf{M}_2; \mathsf{N}_2]h$ onto $Z_{[\mathsf{M}_1;\mathsf{N}_1]}$. Since $[\mathsf{M}_1; \mathsf{N}_1]$ is inner we have, using Lemma 9.6,

$$\begin{aligned} P_{Z_{[\mathsf{M}_1;\mathsf{N}_1]}} \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} h &= \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} [\mathsf{M}_1^*, \mathsf{N}_1^*] \left(\begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} + \Delta \right) h \\ &= \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} (I + [\mathsf{M}_1^*, \mathsf{N}_1^*]\Delta) h. \end{aligned}$$

We have $\|[\mathsf{M}_1^*, \mathsf{N}_1^*]\Delta\|_\infty < 1$, which implies that $I + [\mathsf{M}_1^*, \mathsf{N}_1^*]\Delta$ is invertible in $H^\infty(\mathcal{L}(\mathscr{U}))$. It follows that $P_{Z_{[\mathsf{M}_1;\mathsf{N}_1]}}$ maps $Z_{[\mathsf{M}_2;\mathsf{N}_2]}$ onto $Z_{[\mathsf{M}_1;\mathsf{N}_1]}$ and since $[\mathsf{M}_1; \mathsf{N}_1]$ is injective, this mapping is injective. Since both $[\mathsf{M}_1; \mathsf{N}_1]$ and $[\mathsf{M}_2; \mathsf{N}_2]$ are weakly right-coprime, we have $Z_{[\mathsf{M}_i;\mathsf{N}_i]} = \hat{\mathscr{V}}_i(0)$ $(i = 1, 2)$ by Proposition 9.11. Lemma 9.5 now gives the result. □

Definition 9.19. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a right factorization. Let $\varepsilon > 0$. The **directed gap ball** with center G of radius ε is defined as

$$\vec{B}(\mathsf{G}, \varepsilon) := \{\mathsf{G}_\Delta : \mathsf{G}_\Delta \text{ has a right factorization, } \vec{\delta}_g(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon\}.$$

The **gap ball** with center G of radius ε is defined as

$$B(\mathsf{G}, \varepsilon) := \{\mathsf{G}_\Delta : \mathsf{G}_\Delta \text{ has a right factorization, } \delta_g(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon\}.$$

Proposition 9.20. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized weakly right-coprime factorization and let $\mathsf{G}_\Delta \in \vec{B}(\mathsf{G}, \varepsilon)$. Assume that $\mathscr{U}$ is finite-dimensional. Then G_Δ is an ε-right factor admissible perturbation of G.*

Proof. This is immediate from Corollary 9.17. □

Combining Corollary 9.16 and Proposition 9.20 we obtain the following.

Corollary 9.21. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized weakly right-coprime factorization and that $\mathscr{U}$ is finite-dimensional. Then $\mathsf{G}_\Delta \in \vec{B}(\mathsf{G}, \varepsilon)$ if and only if G_Δ is an ε-right factor admissible perturbation of G.*

Proof. This follows by combining Corollary 9.16 and Proposition 9.20. □

Proposition 9.22. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized strongly right-coprime factorization. Then there exists an $\eta > 0$ such that for all $\varepsilon < \eta$ we have $\vec{B}(\mathsf{G}, \varepsilon) = B(\mathsf{G}, \varepsilon)$.*

Proof. This follows from Propositions 7.26 and 9.18. □

Corollary 9.23. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ and that $\mathscr{U}$ is finite-dimensional. Denote the Hankel operator of $[\mathsf{M}; \mathsf{N}]$ by $H_{[\mathsf{M};\mathsf{N}]}$. Then η in Proposition 9.22 can be taken equal to $\sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$.*

Proof. This follows as the proof of Proposition 9.22 but using Proposition 7.32 instead of Proposition 7.26. □

The following result relates ε-right factor admissible perturbations and the gap metric.

Corollary 9.24. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. Assume that* G *has a normalized weakly right-coprime factorization, that* $\mathscr{U}$ *is finite-dimensional and that* $\varepsilon < \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$*. Then* $\mathsf{G}_\Delta \in B(\mathsf{G}, \varepsilon)$ *if and only if* G_Δ *is an* ε*-right factor admissible perturbation of* G*.*

Proof. This follows from Corollaries 9.21 and 9.23. □

Proposition 9.25. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. Suppose that* G *has a normalized doubly coprime factorization and that* $\mathscr{U}$ *is finite-dimensional. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of the normalized strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$*. Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$*. Then there exists a* ε*-robust right factor stabilizing feedback function for* G *that stabilizes all* $\mathsf{G}_\Delta \in B(\mathsf{G}, \varepsilon)$*.*

Proof. This follows from Proposition 8.17 and Corollary 9.24. □

Corollary 9.26. *Suppose that* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *with* $0 \in D(\mathsf{G})$ *is a matrix-valued rational function. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of a normalized strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$*. Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$*. Then there exists a rational* ε*-robust right factor stabilizing admissible feedback function for* G *that stabilizes all* G_Δ *with* $\delta(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon$*.*

Proof. This follows from the proof of Proposition 9.25 using Corollary 8.19. □

Notes

The gap metric as a distance between closed subspaces of a Hilbert space was first introduced in Kreĭn and Krasnosel′skiĭ [46] under the name **aperture** (see Krasnosel′skiĭ et. al. [45]). Proposition 9.14 was first proven by Georgiou [33] for rational functions. The relation between the gap metric and right factor admissible perturbation was investigated by, among others, Georgiou and Smith [34] and Sefton and Ober [87].

Chapter 10

Balanced realizations

10.1 Lyapunov-balanced realizations

In this section we collect some results on Lyapunov-balanced realizations which are available in the literature.

Definition 10.1. A discrete-time system is called **Lyapunov-balanced** if it is input and output stable and its observability and controllability gramian are equal.

The following result shows the existence and uniqueness of Lyapunov-balanced realizations.

Proposition 10.2. *Any $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function has a minimal Lyapunov-balanced realization. Minimal Lyapunov-balanced realizations are unique up to a unitary similarity transformation in the state space. Their state operator is a contraction. Both the minimal Lyapunov-balanced realization and its dual system are strongly stable.*

Proof. See Young [99] or Theorems 11.2.5 and 11.2.9 in Peller [75] for all the above statements except the ones about strong stability. The statements about strong stability can be found in Ober and Wu [63]. The complete theorem as stated above is contained in Theorem 9.5.6 of Staffans [89] (in the continuous-time version). □

Definition 10.3. A discrete-time system is called **compact Lyapunov-balanced** if it is Lyapunov-balanced and its gramian is compact.

We note that any $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function has a bounded Hankel operator (see Definition A.24). It follows from Lemma A.26 that this Hankel operator is similar, through the Z-transform, to the Hankel map of any realization of the given function. We will call this the Hankel map of the given $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function.

Proposition 10.4. *Any $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function with a compact Hankel map has a minimal compact Lyapunov-balanced realization.*

Proof. By Proposition 10.2 the given function has a minimal Lyapunov-balanced realization. Since the Hankel map is independent of the realization this Lyapunov-balanced realization has a compact Hankel map. Denote the gramian by L. It follows, using Lemma 2.4 that $L^2 = \mathcal{C}^*\mathcal{C}\mathcal{B}\mathcal{B}^* = \mathcal{C}^*\mathcal{H}\mathcal{B}^*$ is compact. From this we conclude that L is compact. □

Remark 10.5. From the proof of Propositions 10.2 and 10.4 one can obtain the following explicit form of a compact Lyapunov-balanced realization.

Assume that we are given a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function with a compact Hankel map $\mathcal{H}$. Recall the backward shift realization Σ^{rs} from Remark 2.13. Let $\mathscr{X}$ be the closure of the range of the Hankel map. Since $\mathcal{H}$ is compact there exist a nonincreasing positive sequence (σ_i) and orthonormal bases (v_i) of the closure of the range of $\mathcal{H}^*$ and (w_i) of $\mathscr{X}$ such that

$$\mathcal{H}v_i = \sigma_i w_i, \quad \mathcal{H}^* w_i = \sigma_i v_i$$

(this is known as the structure theorem for compact operators, note that the σ_i are positive since we only want the (w_i) to be a basis for $\mathscr{X}$, not for the whole of $l^2(\mathbb{Z}^+, \mathscr{Y})$). The σ_i are called the **Hankel singular values** of the system and the (v_i, w_i) the **Schmidt pairs**. We have

$$\langle A^{\mathrm{bal}} w_j, w_i \rangle_{l^2(\mathbb{Z}^+, \mathscr{Y})} = \sqrt{\frac{\sigma_j}{\sigma_i}} \langle A^{\mathrm{bs}} w_j, w_i \rangle_{l^2(\mathbb{Z}^+, \mathscr{Y})}.$$

Pick an orthonormal basis (u_i) in $\mathscr{U}$, then

$$\langle B^{\mathrm{bal}} u_j, w_i \rangle_{l^2(\mathbb{Z}^+, \mathscr{Y})} = \frac{1}{\sqrt{\sigma_i}} \langle B^{\mathrm{bs}} u_j, w_i \rangle_{l^2(\mathbb{Z}^+, \mathscr{Y})}.$$

Note that since $B^{\mathrm{bs}} u = \mathcal{H}\underline{u}$, where $\underline{u} : \mathbb{Z}^- \to \mathscr{U}$ is defined by $\underline{u}_{-1} = u$ and $\underline{u}_{-i} = 0$ if $i > 1$, we have

$$\langle B^{\mathrm{bal}} u_j, w_i \rangle_{l^2(\mathbb{Z}^+, \mathscr{Y})} = \sqrt{\sigma_i} \langle \underline{u_j}, v_i \rangle_{l^2(\mathbb{Z}^-, \mathscr{U})}.$$

Choose an orthonormal basis (y_i) in $\mathscr{Y}$, then

$$\langle C^{\mathrm{bal}} w_j, y_i \rangle_{\mathscr{Y}} = \sqrt{\sigma_j} \langle C^{\mathrm{bs}} w_j, y_i \rangle_{\mathscr{Y}}.$$

Of course D^{bal} equals the value of the given $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function at zero, as is the case with any realization. The gramians are both equal to $L^{\mathrm{bal}} := \sqrt{\mathcal{H}\mathcal{H}^*}$. So with respect to the orthonormal basis (w_i) the gramian is diagonal, we have

$$\langle L^{\mathrm{bal}} w_j, w_i \rangle = \sigma_i \delta_{ij},$$

with δ_{ij} the Kronecker delta.

Definition 10.6. Let $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ have a compact Hankel map $\mathcal{H}$. The realization from Remark 10.5 is called the compact Lyapunov-balanced realization of G with respect to the sequence of eigenvectors (w_i) and is denoted by $\Sigma^{\mathrm{bal}}_{(w_i)}$.

Note that $\Sigma^{\mathrm{bal}}_{(w_i)}$ is always approximately controllable and observable since the gramian is positive.

Definition 10.7. Let $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ have a compact Hankel map $\mathcal{H}$. Let (w_i) be an ordered sequence of eigenvectors of $\mathcal{H}\mathcal{H}^*$ (the ordering is such that the corresponding eigenvalues σ_i^2 form a nonincreasing sequence). Let $n \in \mathbb{Z}^+$ be such that $\sigma_n > \sigma_{n+1}$. The **truncated Lyapunov-balanced realization** of dimension n of G with respect to the sequence of eigenvectors (w_i) is defined as the restriction/projection of $\Sigma^{\mathrm{bal}}_{(w_i)}$ onto $\mathscr{X}_n := \{w_i : i = 1, \dots, n\}$.

Remark 10.8. Note that we used the term 'an ordered sequence of eigenvectors of $\mathcal{H}\mathcal{H}^*$' since such a sequence is not unique if $\mathcal{H}\mathcal{H}^*$ has repeated eigenvalues. As we will show in the next lemma the condition $\sigma_n > \sigma_{n+1}$ ensures that the choice of ordered sequence of eigenvectors is to a large extent unimportant. Also note that in the case that $\mathcal{H}\mathcal{H}^*$ has repeated eigenvalues the truncated Lyapunov-balanced realization of dimension n is not defined for every $n \in \mathbb{Z}^+$.

Proposition 10.9. *Let $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ have a compact Hankel map $\mathcal{H}$. Let (w_i) and $(\tilde{w}_i)$ be ordered bases of eigenvectors of $\mathcal{H}\mathcal{H}^*$. The truncated Lyapunov-balanced realization of dimension n of G with respect to the sequence of eigenvectors (w_i) and that with respect to the the sequence of eigenvectors $(\tilde{w}_i)$ are related by a unitary similarity transformation. In particular, the transfer functions are the same.*

Proof. Since $\sigma_n > \sigma_{n+1}$ we have that $\mathscr{X}_n$ is the direct sum of eigenspaces. It follows that both $(w_i)_{i=1,\dots,n}$ and $(\tilde{w}_i)_{i=1,\dots,n}$ are bases for $\mathscr{X}_n$. Define the unitary operator $U \in \mathcal{L}(\mathscr{X}_n)$ by $U w_i = \tilde{w}_i$ with $i = 1, \dots, n$. It is easily seen that this is the desired unitary similarity transformation. $\square$

We now consider the distance between a discrete-time system and its truncated Lyapunov-balanced realizations. We measure this distance by the supremum norm of the difference of the transfer functions. To formulate the conditions needed we need the concept of a nuclear operator.

Remark 10.10. The **singular values** of an operator $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ are defined as follows. The k-th singular value of T is the distance, with respect to the norm in $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$, of T from the set of operators in $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ of rank at most $k-1$. If T is compact, then the singular values are exactly the square roots of the eigenvalues of TT^*.

Remember that an operator $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ is **nuclear** if its singular values s_i satisfy $\sum_{i=1}^{\infty} s_i < \infty$. The sum of the singular values is called the nuclear norm. The set of nuclear operators is a linear subspace of $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ and the nuclear norm is a norm on this subspace. Nuclear operators are compact. The operator T is called **Hilbert-Schmidt** if $\sum_{i=1}^{\infty} s_i^2 < \infty$. Equivalently, $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ is Hilbert-Schmidt if for all orthonormal bases (e_i) of $\mathscr{H}_1$ we have $\sum_i \|Te_i\|^2 < \infty$. Hilbert-Schmidt operators are compact.

Proposition 10.11. *Assume that* $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*, with* $\mathscr{U}$ *and* $\mathscr{Y}$ *finite-dimensional, has a nuclear Hankel map* $\mathcal{H}$*. Define* G^n *as the transfer function of a truncated Lyapunov-balanced realization of dimension* n *of* G*. Then we have*

$$\|\mathsf{G} - \mathsf{G}^n\|_\infty \leq 2 \sum_{i=n+1}^{\infty} \sigma_i.$$

In particular $\mathsf{G}^n \to \mathsf{G}$ *in the* H^∞ *norm as* $n \to \infty$*.*

Proof. The proof of this proposition is on page 110. □

Remark 10.12. Note that the error-bound does not depend on the choice made in the eigenvectors w_i used to define the truncated balanced realization of dimension n. This is due to the fact that by Proposition 10.9 the transfer function G^n does not depend on the choice of eigenvectors w_i.

Remark 10.13. Note that it follows from Proposition 10.11 that, under the conditions stated in that proposition, the Hankel operator of G^n converges to the Hankel operator of G in the nuclear norm.

The following proposition identifies a fundamental limitation to approximating a system by a system with a finite-dimensional state space.

Proposition 10.14. *Assume that* $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *with* $\mathscr{U}$ *finite-dimensional. Let* G^n *be the transfer function of a discrete-time system with state space dimension* n*, input space* $\mathscr{U}$ *and output space* $\mathscr{Y}$*. Then* $\mathsf{G}^n \to \mathsf{G}$ *in the* H^∞ *norm as* $n \to \infty$ *only if* G *has a compact Hankel map.*

Proof. By assumption $\mathsf{G}^n \to \mathsf{G}$ in the H^∞ norm as $n \to \infty$. It follows that the Hankel operator of G^n converges to the Hankel operator of G in the operator norm. Since G^n is the transfer function of a discrete-time system with state space dimension n and input space dimension m its Hankel operator has rank at most mn. Since the Hankel operator of G is in the closure in the operator norm of the space of finite-rank operators it must be compact. $\square$

Most of the remainder of this section on Lyapunov-balanced realizations is devoted to a proof of Proposition 10.11. Lemmas 10.15 and 10.16 however are included for the proof of Proposition 10.17, which gives a sufficient condition for nuclearity of the Hankel map.

Lemma 10.15. *Let Σ be a discrete-time system that is exponentially stable and has a finite-dimensional output space. Then its output map is Hilbert-Schmidt.*

Proof. Since the system is exponentially stable, there exist $M \geq 0$ and $r \in [0,1)$ such that for all $x \in \mathscr{X}$ we have $\|A^n x\| \leq M r^n \|x\|$ by Proposition 3.26.

Define $p := \dim \mathscr{Y}$ and let $(y_i)_{i\in\{1,\dots p\}}$ be a basis for the output space. Define $\mathcal{C}_i : \mathscr{X} \to l^2(\mathbb{Z}^+)$ by $(\mathcal{C}_i x)_n = \langle (\mathcal{C}x)_n, y_i\rangle_{\mathscr{Y}}$, where $\mathcal{C}$ is the output map of the system. We first prove that $\mathcal{C}_i$ is Hilbert-Schmidt.

For $n \in \mathbb{Z}^+$ define $\mathcal{C}_i^n : \mathscr{X} \to \mathbb{C}$ by $\mathcal{C}_i^n x = (\mathcal{C}_i x)_n$. This mapping is a continuous linear functional. By the Riesz representation theorem there exists, for each $n \in \mathbb{Z}^+$, a $w_n \in \mathscr{X}$ such that $\mathcal{C}_i^n x = \langle x, w_n\rangle_{\mathscr{X}}$ and

$$\|w_n\| = \|\mathcal{C}_i^n\| = \sup_{\|x\|=1} |\mathcal{C}_i^n x|.$$

We have

$$|\mathcal{C}_i^n x| = |\langle (\mathcal{C}x)_n, y_i\rangle_{\mathscr{Y}}| = |\langle CA^n x, y_i\rangle_{\mathscr{Y}}| \leq \|C\|\ M\ r^n\ \|x\|,$$

where in the last step we have used exponential stability. Thus we have

$$\begin{aligned}\sum_{n=0}^{\infty} \|\mathcal{C}_i^n\|^2 &= \sum_{n=0}^{\infty} \sup_{\|x\|=1} |\mathcal{C}_i^n x|^2 \leq \sum_{n=0}^{\infty} \sup_{\|x\|=1} \|C\|^2\ M^2\ r^{2n}\ \|x\|^2 \\ &= \|C\|^2\ M^2 \sum_{n=0}^{\infty} r^{2n} = \frac{\|C\|^2\ M^2}{1-r^2} < \infty.\end{aligned}$$

Let $(x_j)_{j\in\mathbb{Z}^+}$ be an orthonormal basis for $\mathscr{X}$. We compute

$$\sum_{j=0}^{\infty} \|\mathcal{C}_i x_j\|^2 = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} |\mathcal{C}_i^n x_j|^2 = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} |\langle x_j, w_n\rangle|^2 = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} |\langle x_j, w_n\rangle|^2.$$

From the Parseval relation we obtain that this is equal to

$$\sum_{n=0}^{\infty} \|w_n\|^2 = \sum_{n=0}^{\infty} \|\mathcal{C}_i^n\|^2.$$

We already saw that the right-hand side of this equation is finite. It follows that

$$\sum_{j=0}^{\infty} \|\mathcal{C}_i x_j\|^2 < \infty,$$

for all orthonormal sequences $(x_j)_{j\in\mathbb{Z}^+}$. This shows that $\mathcal{C}_i$ is Hilbert-Schmidt. We use the fact that $\mathcal{C}_i$ is Hilbert-Schmidt to show that $\mathcal{C}$ is. We have

$$\sum_{j=0}^{\infty} \|\mathcal{C} x_j\|^2 = \sum_{j=0}^{\infty} \sum_{i=1}^{p} \|\mathcal{C}_i x_j\|^2 = \sum_{i=1}^{p} \sum_{j=0}^{\infty} \|\mathcal{C}_i x_j\|^2 < \infty.$$

This shows that $\mathcal{C}$ is Hilbert-Schmidt. □

Lemma 10.16. *Let Σ be a discrete-time system that is exponentially stable and has a finite-dimensional input space. Then its input map is Hilbert-Schmidt.*

Proof. This follows from Lemma 10.15 applied to the dual system of Σ using that the adjoint of a Hilbert-Schmidt operator is Hilbert-Schmidt. □

Proposition 10.17. *Let Σ be a discrete-time system that is exponentially stable and has finite-dimensional input and output spaces. Then its Hankel map is nuclear.*

Proof. It follows from Lemmas 10.15 and 10.16 that the input and output maps are Hilbert-Schmidt. Using that the Hankel map is the product of these two maps (Lemma 2.4) and that the product of two Hilbert-Schmidt operators is nuclear we obtain the desired result. □

All results from the next proposition to the end of this section are included as building blocks for the proof of Proposition 10.11. We first prove a property of truncated Lyapunov-balanced realizations.

Proposition 10.18. *Let $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ have a compact Hankel map with at least n nonzero singular values. Then a n-dimensional truncated Lyapunov-balanced realization is exponentially stable.*

Proof. We decompose the state space $\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2$, where $\mathscr{X}_1$ corresponds to the first n terms in the sequence (w_i). The system operator and the gramian are decomposed accordingly. The observation Lyapunov equation of $\Sigma^{\mathrm{bal}}_{(w_i)}$ gives

$$A_{11}^* L_1 A_{11} + A_{21}^* L_2 A_{21} - L_1 + C_1^* C_1 = 0.$$

Assume that $A_{11} v = \lambda v$. Using the above identity we obtain

$$(1 - |\lambda|^2)\ \|L_1^{1/2} v\|^2 = \|L_2^{1/2} A_{21} v\|^2 + \|C_1 v\|^2.$$

Since this is nonnegative, we obtain $|\lambda| \leq 1$. If $|\lambda| = 1$ then $C_1 v = 0$ and $L_2^{1/2} A_{21} v = 0$. Since $L_2 > 0$ (since the gramian of $\Sigma^{\mathrm{bal}}_{(w_i)}$ is positive), it follows that $A_{21} v = 0$. Define $V = [v; 0]$. Then (using $A_{21} v = 0$) $AV = \lambda V$, so $CA^k V = \lambda^k C_1 v = 0$. From the approximate observability of $\Sigma^{\mathrm{bal}}_{(w_i)}$ we obtain $V = 0$. We conclude that all eigenvalues of A_{11} are in the open unit disc. Since the state space is finite-dimensional, it follows that the n-dimensional truncated Lyapunov-balanced realization is exponentially stable. □

The following lemma shows continuous dependence of the eigenvalues and eigenvectors. Note that for continuity of the eigenvectors we might have to resort to a subsequence.

Lemma 10.19. *Let $T_m, T \in \mathcal{L}(\mathscr{H})$ be compact and nonnegative self-adjoint and such that $\|T_m - T\| \to 0$ as $m \to \infty$. Denote the eigenvalues of T by λ_i. Assume that the eigenvalues are ordered in decreasing magnitude and repeated according to their multiplicity. Then for the eigenvalues λ_i^m of T_m, also ordered in decreasing magnitude and repeated according to their multiplicity, we have $\lambda_i^m \to \lambda_i$. Let v_i^m be a basis of eigenvectors for T_m. There exists a subsequence T_{m_k} of T_m and a basis of eigenvectors (v_i) for T such that $\|v_i^{m_k} - v_i\| \to 0$ as $k \to \infty$.*

Proof. We recall Weyl's theorem on eigenvalues: if $A, B \in \mathcal{L}(\mathscr{H})$ are compact nonnegative self-adjoint operators, then their eigenvalues (ordered in decreasing magnitude) satisfy $\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B)$. Taking $j = 1$ we obtain $\lambda_i(A+B) \leq \lambda_i(A) + \|B\|$.

From Weyl's theorem on eigenvalues we obtain for the given situation $\lambda_i^m \leq \lambda_i + \|T - T_m\|$ and $\lambda_i \leq \lambda_i^m + \|T - T_m\|$. This shows that $\lambda_i^m \to \lambda_i$ as $m \to \infty$.

We first show the statement on the eigenvectors for the set of leading eigenvectors (i.e. the ones corresponding to the largest eigenvalue). Denote the multiplicity of the largest eigenvalue of T by N. Further denote $v^m := [v_1^m, \ldots v_N^m]$ and $v := [v_1, \ldots v_N]$, where the v_i are orthonormal eigenvectors of T corresponding to λ_1 and the v_i^m are orthonormal eigenvectors of T_m for the largest N eigenvalues (counted according to their multiplicity) λ_i^m. Decompose $v^m = R^m v + x^m$ with R^m a $N \times N$ matrix of complex numbers and $\langle x_i^m, v_j \rangle = 0$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. Denote the rows of R by R_i. Then we have $v_i^m = R_i^m v + x_i^m$. Note that by the Pythagorean Theorem $\|x_i^m\|^2 = 1 - \|R_i^m\|_{\mathbb{C}^N}^2$. We have

$$\lambda_i^m = \|T_m v_i^m\| \leq \|T v_i^m\| + \|T - T_m\| = \|\lambda_1 R_i^m v + T x_i^m\| + \|T - T_m\|.$$

Note that the v_i are orthonormal and $\langle T x_i^m, v_j \rangle = \langle x_i^m, T v_j \rangle = 0$ since $\langle x_i^m, v_j \rangle = 0$. By the Pythagorean Theorem we then have

$$\|\lambda_1 R_i^m v + T x_i^m\|^2 = \lambda_1^2 \|R_i^m\|_{\mathbb{C}^N}^2 + \|T x_i^m\|^2.$$

Denote the restriction of T to the orthogonal complement of the eigenspace corresponding to λ_1 by $\check{T}$. Then $\check{T}$ is a compact nonnegative self-adjoint operator with eigenvalues $(\lambda_i)_{i \geq N+1}$ and so its norm is λ_{N+1}. Since x_i^m is in the domain of $\check{T}$ we have $\|T x_i^m\| \leq \lambda_{N+1} \|x_i^m\|$. Combining the above we obtain

$$\lambda_i^m \leq \sqrt{\lambda_1^2 \|R_i^m\|_{\mathbb{C}^N}^2 + \lambda_{N+1}^2 \left(1 - \|R_i^m\|_{\mathbb{C}^N}^2\right)} + \|T - T_m\|.$$

This gives

$$\frac{\left(\lambda_i^m - \|T - T_m\|\right)^2 - \lambda_{N+1}^2}{\lambda_1^2 - \lambda_{N+1}^2} \leq \|R_i^m\|_{\mathbb{C}^N}^2.$$

Since we have convergence of the eigenvalues, the left-hand side converges to 1. Since we have $\|R_i^m\|_{\mathbb{C}^N}^2 \leq 1$, we must have $\|R_i^m\|_{\mathbb{C}^N}^2 \to 1$. This implies that $\|x_i^m\| \to 0$. Hence $\|x^m\| \to 0$. The sequence of matrices (R^m) is bounded, which implies that it has a convergent subsequence. Denote the limit of such a subsequence by R^∞. Since $\|x^m\| \to 0$, we then have that the corresponding subsequence of (v^m) converges to $v^\infty := R^\infty v$. Since the components of (v^m) have norm one and are orthogonal to each other, the same holds for the components of v^∞. We now replace the set of orthonormal eigenvectors $(v_i)_{i=1,\ldots,N}$ by the components of v^∞. This gives another sequence of orthonormal eigenvectors of T. For the leading eigenvalue this sequence has the desired properties.

The general result follows by induction. Assume that we have proven the assertion for the first n eigenvalues (not counting multiplicity) with respective multiplicies N_n. Denote $N := \sum_{k=1}^n N_n$. We can apply the above to the operators $\tilde{T}$ and $\tilde{T}_m$, defined as the restriction of T respectively T_m, to the orthogonal complement of the eigenspaces corresponding to the first N eigenvalues (counting multiplicity). This gives the result for the first $n+1$ eigenvalues (not counting multiplicity). □

Note that in Lemma 10.19 the eigenvectors v_i of T depend on the approximating sequence T_m: a different choice of approximating sequence may lead to a different orthonormal set of eigenvectors for T. The following lemma shows that we can obtain any desired orthonormal basis of eigenvectors for T by properly adjusting the approximating sequence. Lemma 10.20 is not needed for the proof of Proposition 10.11, but we give it for completeness.

Lemma 10.20. *Let $T_m, T \in \mathcal{L}(\mathscr{H})$ be compact and nonnegative self-adjoint and such that $\|T_m - T\| \to 0$ as $m \to \infty$. Let (v_i) be an orthonormal basis of eigenvectors of T, ordered according to decreasing magnitude of the corresponding eigenvalues. Then there exists a unitary operator $U \in \mathcal{L}(\mathscr{H})$ such that $U^*TU = T$ and there exists an orthonormal basis of eigenvectors of a subsequence of $\tilde{T}_m := U^*T_mU$ that converges to the given eigenvectors (v_i) as $m \to \infty$.*

Proof. From Lemma 10.19 we obtain an orthonormal basis of eigenvectors (w_i) of T and eigenvectors v_i^m of a subsequence of T_m such that $v_i^m \to w_i$. We define the unitary operator U by $Uv_i = w_i$. Note that since (w_i) and (v_i) are orthonormal bases this operator is indeed well-defined and unitary. Since both (w_i) and (v_i) are ordered the eigenvalues corresponding to the same index are equal. This gives $U^*TUv_i = Tv_i$, from which we obtain $U^*TU = T$ since (v_i) is a basis. Define $\tilde{v}_i^m := U^*v_i^m$. Then $\tilde{v}_i^m$ is an eigenvector of $\tilde{T}_m$ with eigenvalue λ_i^m. Since U is unitary the $\tilde{v}_i^m$ are orthonormal. We have $\tilde{v}_i^m \to U^*w_i = v_i$, since by assumption $v_i^m \to w_i$ and by definition of U we have $v_i = U^*w_i$. □

Using Lemma 10.19 we show the continuity of singular values and Schmidt pairs.

Lemma 10.21. *Let $T_m, T \in \mathcal{L}(\mathscr{H})$ be compact and such that $\|T_m - T\| \to 0$ as $m \to \infty$. Denote the singular values of T by σ_i. Assume that the singular values are ordered in decreasing magnitude and repeated according to their multiplicity. Then for the singular values σ_i^m of T_m, also ordered in decreasing magnitude and repeated according to their multiplicity, we have $\sigma_i^m \to \sigma_i$ as $m \to \infty$. Let (v_i^m, w_i^m) be Schmidt pairs for T_m. There exists a subsequence T_{m_k} of T_m and Schmidt pairs (v_i, w_i) for T such that $\|v_i^{m_k} - v_i\| \to 0$ and $\|w_i^{m_k} - w_i\| \to 0$ as $k \to \infty$. The w_i form an orthonormal basis of eigenvectors for TT^* and the v_i for T^*T.*

Proof. We first show that $T_m \to T$ implies $T_m^*T_m \to T^*T$. We have

$$\begin{aligned} \|T^*T - T_m^*T_m\| &= \|T^*T - T^*T_m + T^*T_m - T_m^*T_m\| \\ &\leq \|T\|\,\|T - T_m\| + \|T_m\|\,\|T - T_m\|. \end{aligned}$$

Since $T_m \to T$ we have $\|T_m\| \leq 2\|T\|$ for m large enough, which together with the above inequality gives the desired convergence. So we can apply Lemma 10.19 to obtain the convergence of the singular values. This lemma also gives a basis (v_i) of eigenvectors for T^*T and a basis (w_i) of eigenvectors for TT^* with the desired convergence properties. We only need to show that (v_i, w_i) is a Schmidt pair for T, i.e. $Tv_i = \sigma_i w_i$ and $T^* w_i = \sigma_i v_i$. We show this using that (v_i^m, w_i^m) is a Schmidt pair for T_m, i.e. $T_m v_i^m = \sigma_i^m w_i^m$ and $T_m^* w_i^m = \sigma_i^m v_i^m$. We have

$$\|Tv_i - \sigma_i w_i\| \leq \|T - T_m\| + \|T_m\| \ \|v_i - v_i^m\| + |\sigma_i^m - \sigma_i| + |\sigma_i| \|w_i^m - w_i\|,$$

which implies that $Tv_i = \sigma_i w_i$. The other equality is proven similarly. □

Similarly to Lemma 10.20 we can obtain any desired sequence of Schmidt pairs by changing the approximating sequence. Lemma 10.22 is not needed for the proof of Proposition 10.11, it is given for sake of completeness.

Lemma 10.22. *Let $T_m, T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ be compact and such that $\|T_m - T\| \to 0$ as $m \to \infty$. Denote the singular values of T by σ_i and corresponding Schmidt pairs by (v_i, w_i). Then for the singular values σ_i^m of T_m we have $\sigma_i^m \to \sigma_i$. Furthermore, there exist unitary operators $V \in \mathcal{L}(\mathscr{H}_1)$ and $W \in \mathcal{L}(\mathscr{H}_2)$ such that $TV = WT$ and there exist Schmidt pairs $(v_i^{m_k}, w_i^{m_k})$ for a subsequence of $\tilde{T}_m := WT_mV^*$ such that $\|v_i^{m_k} - v_i\| \to 0$ and $\|w_i^{m_k} - w_i\| \to 0$ as $k \to \infty$.*

Proof. Convergence of the singular values follows immediately from Lemma 10.19. Let $(\tilde{v}_i^m, \tilde{w}_i^m)$ be a given sequence of Schmidt pairs of T_m. By Lemma 10.19 applied to T^*T with approximating sequence $T_m^*T_m$ there exist a basis of eigenvectors $(\tilde{v}_i)$ of T^*T such that $\tilde{v}_i^m \to \tilde{v}_i$. Similarly, there exist a basis of eigenvectors $(\tilde{w}_i)$ of TT^* such that $\tilde{w}_i^m \to \tilde{w}_i$. Since $(\tilde{v}_i^m, \tilde{w}_i^m)$ is a Schmidt pair we have $T_m \tilde{v}_i^m = \sigma_i^m \tilde{w}_i^m$ and $T_m^* \tilde{w}_i^m = \sigma_i^m \tilde{v}_i^m$. Taking limits we obtain $T\tilde{v}_i = \sigma_i \tilde{w}_i$ and $T^*\tilde{w}_i = \sigma_i \tilde{v}_i$, which shows that $(\tilde{v}_i, \tilde{w}_i)$ is a Schmidt pair of T. Define $V \in \mathcal{L}(\mathscr{H}_1)$ and $W \in \mathcal{L}(\mathscr{H}_2)$ by $V\tilde{v}_i = v_i$ and $W\tilde{w}_i = w_i$, respectively. Since $(\tilde{v}_i)$, (v_i), $(\tilde{w}_i)$, (w_i) are orthonormal bases V and W are unitary. Define $\tilde{T}_m := WT_mV^*$ and $v_i^m := V\tilde{v}_i^m$, $w_i^m := W\tilde{w}_i^m$. Then (v_i^m, w_i^m) is a Schmidt pair of $\tilde{T}_m$ since

$$\tilde{T}_m v_i^m = WT_mV^* v_i^m = WT_m \tilde{v}_i^m = \sigma_i^m W \tilde{w}_i^m = \sigma_i^m w_i^m,$$

$$\tilde{T}_m^* w_i^m = VT_m^* W^* w_i^m = VT_m^* \tilde{w}_i^m = \sigma_i^m V \tilde{v}_i^m = \sigma_i^m v_i^m.$$

We have $v_i^m = V\tilde{v}_i^m \to V\tilde{v}_i = v_i$ and $w_i^m = W\tilde{w}_i^m \to W\tilde{w}_i = w_i$. We have $TV = WT$ since $W^*TV\tilde{v}_i = W^*Tv_i = \sigma_i W^* w_i = \sigma_i \tilde{w}_i = T\tilde{v}_i$. □

The following lemma gives a series expansion of a function that has a nuclear Hankel map.

Lemma 10.23. *Assume that $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional, has a nuclear Hankel map. Then there exist $c_n \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$, $\lambda_n \in \mathbb{D}$ such that*

$$\mathsf{G}(z) = \sum_{n=1}^{\infty} c_n \frac{1}{1 - \lambda_n z}.$$

If $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ then we have

$$\sum_{n=1}^{\infty} \frac{|c_n|}{1 - |\lambda_n|} < \infty. \tag{10.1}$$

Proof. The scalar statement can be found in Peller [75, page 238]. The matrix statement follows from applying the scalar statement to components. □

Definition 10.24. Assume that $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional, has a nuclear Hankel map. Let c_n and λ_n be as in Lemma 10.23. For $m \in \mathbb{Z}^+$ define

$$\mathsf{G}^m(z) := \sum_{n=1}^{m} c_n \frac{1}{1 - \lambda_n z}.$$

This is called the (m, c, λ) nuclear approximant of G.

Lemma 10.25. *With the assumptions and the notation as in Definition 10.24 we have*

$$\|\mathsf{G} - \mathsf{G}_m\|_\infty \to 0, \quad \|\mathcal{H} - \mathcal{H}_m\|_N \to 0,$$

where $\mathcal{H}$ is the Hankel map of G, $\mathcal{H}_m$ is the Hankel map of G_m and $\|\cdot\|_N$ is the nuclear norm.

Proof. We first compute the minimum of the absolute value of $z \mapsto 1 - \lambda_n z$ on the unit disc. Using the triangle inequality we have $|1 - \lambda_n z| \geq 1 - |\lambda_n|$. For $z = \bar{\lambda}_n / |\lambda_n|$ we have equality, so the minimum is $1 - |\lambda_n|$. It follows that

$$\left\| \frac{1}{1 - \lambda_n \cdot} \right\|_\infty = \frac{1}{1 - |\lambda_n|}.$$

We show convergence in the H^∞ norm for the scalar case. We have

$$\|\mathsf{G} - \mathsf{G}_m\|_\infty \leq \sum_{n=m+1}^{\infty} |c_n| \left\| \frac{1}{1 - \lambda_n \cdot} \right\|_\infty = \sum_{n=m+1}^{\infty} |c_n| \frac{1}{1 - |\lambda_n|} \to 0, \text{ as } m \to \infty$$

using Lemma 10.23. The matrix case follows from applying the above to each component of the matrix. We now turn to the case of the nuclear norm. It is easily seen that $[\lambda_n, 1; \lambda_n, 1]$ is a realization of $z \mapsto \frac{1}{1-\lambda_n z}$. The observation Lyapunov equation is $|\lambda_n|^2 L_C - L_C + |\lambda_n|^2$, which has as unique solution $L_C := \frac{|\lambda_n|^2}{1-|\lambda_n|^2}$. The control Lyapunov equation, $|\lambda_n|^2 L_B - L_B + 1$, has as unique solution $L_B := \frac{1}{1-|\lambda_n|^2}$. It follows that the unique positive Hankel singular value is $\frac{|\lambda_n|}{1-|\lambda_n|^2}$. So this is the nuclear norm of $\frac{1}{1-\lambda_n}$. Since $|\lambda_n| < 1$ we have that this nuclear norm is smaller than $\frac{1}{1-|\lambda_n|}$. Convergence in the nuclear norm is shown, using this, similarly to convergence in the H^∞ norm. □

The earlier proven continuity of singular values and Schmidt pairs applied to the nuclear approximants gives the following.

Lemma 10.26. *Assume that* $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, *with* $\mathscr{U}$ *and* $\mathscr{Y}$ *finite-dimensional, has a nuclear Hankel map. Denote its* (m, c, λ) *nuclear approximant by* G_m. *Denote the singular values of the Hankel map of* G *by* σ_i. *Assume that the singular values are ordered in decreasing magnitude and repeated according to their multiplicity. Then for the singular values* σ_i^m *of the Hankel map of* G_m, *also ordered in decreasing magnitude and repeated according to their multiplicity, we have* $\sigma_i^m \to \sigma_i$. *Let* (v_i^m, w_i^m) *be Schmidt pairs for the Hankel map of* G_m. *There exists a subsequence* G_{m_k} *of* G_m *and Schmidt pairs* (v_i, w_i) *for the Hankel map of* G *such that* $\|v_i^{m_k} - v_i\| \to 0$ *and* $\|w_i^{m_k} - w_i\| \to 0$ *as* $k \to \infty$.

Proof. Lemma 10.25 show that G_m converges to G in the infinity norm. This implies that the Hankel map of G_m converges to the Hankel map of G in the operator norm. The result then follows from Lemma 10.21. □

Remark 10.27. Lemma 10.26 implies that, for m large enough and n such that $\sigma_n > \sigma_{n+1}$, we have $\sigma_n^m > \sigma_{n+1}^m$. So if the n-dimensional Lyapunov-balanced trucation of G is well-defined, then so is the n-dimensional Lyapunov-balanced trucation of G_m for m large enough. By Lemma 10.9 the transfer function of this n-dimensional Lyapunov-balanced trucation of G_m does not depend on the sequence of eigenvectors chosen. Denote this transfer function by G_m^n.

Lemma 10.28. *With the assumptions and the notation as in Definition 10.24 and Remark 10.27 we have (if* G *has at least* n *nonzero Hankel singular values)*

$$\|\mathsf{G}^n - \mathsf{G}_{m_k}^n\|_\infty \to 0 \text{ as } k \to \infty.$$

Proof. For notational convenience we will assume that G_m^n has been replaced by the subsequence $\mathsf{G}_{m_k}^n$. By Lemma 10.26 we have convergence of the singular values as $m \to \infty$. Given Schmidt pairs (v_i^m, w_i^m) of the Hankel map of G_m converge to certain Schmidt pairs (v_i, w_i) of the Hankel map of G by the same lemma. Denote by Σ_m the Lyapunov-balanced realization of G_m with respect to the eigenvectors (w_i^m) and by Σ_m^n its n-dimensional Lyapunov-balanced truncation. It follows from Remark 10.27 that the transfer function of Σ_m^n equals G_m^n. Denote by Σ the Lyapunov-balanced realization of G with respect to the sequence of eigenvectors (w_i) and by Σ^n its n-dimensional Lyapunov-balanced truncation. Using Proposition 10.9 it follows that the transfer function of Σ^n equals G^n. Note that we have

$$A_m^n = P_{\mathscr{X}_m^n} A_m|_{\mathscr{X}_m^n}, \quad A^n = P_{\mathscr{X}^n} A|_{\mathscr{X}^n}.$$

Let $x \in \mathscr{X}$. Then we have $x = \sum_{j=1}^{\infty} \langle x, w_j^m \rangle w_j^m$. It follows that

$$\langle A_m x, w_i^m \rangle = \sum_{j=1}^{\infty} \langle x, w_j^m \rangle \langle A_m w_j^m, w_i^m \rangle.$$

Using the explicit description from Remark 10.5 we obtain

$$\langle A_m x, w_i^m \rangle = \sum_{j=1}^{\infty} \langle x, w_j^m \rangle \sqrt{\frac{\sigma_j^m}{\sigma_i^m}} \langle A^{\mathrm{bs}} w_j^m, w_i^m \rangle.$$

Since we have $A_m^n x = \sum_{i=1}^{n} \langle A_m x, w_i^m \rangle w_i^m$ it follows that

$$A_m^n x = \sum_{i=1}^{n} \sum_{j=1}^{\infty} \langle x, w_j^m \rangle \sqrt{\frac{\sigma_j^m}{\sigma_i^m}} \langle A^{\mathrm{bs}} w_j^m, w_i^m \rangle w_i^m.$$

Similarly we have

$$A^n x = \sum_{i=1}^{n} \sum_{j=1}^{\infty} \langle x, w_j \rangle \sqrt{\frac{\sigma_j}{\sigma_i}} \langle A^{\mathrm{bs}} w_j, w_i \rangle w_i.$$

Along similar lines it follows that

$$B_m^n u = \sum_{k=1}^{n} \langle B_m^n, w_k^m \rangle w_k^m = \sum_{k=1}^{n} \sqrt{\sigma_k^m} \langle \underline{u}, v_k^m \rangle w_k^m,$$

$$B^n u = \sum_{k=1}^{n} \langle B^n, w_k \rangle w_k = \sum_{k=1}^{n} \sqrt{\sigma_k} \langle \underline{u}, v_k \rangle w_k,$$

where $\underline{u} : \mathbb{Z}^- \to \mathscr{U}$ is defined by $\underline{u}_{-1} = u$ and otherwise zero. It also follows that

$$C_m^n x = \sum_{i=1}^{p} \sum_{k=1}^{\infty} \sqrt{\sigma_k^m} \langle x, w_k^m \rangle \langle C^{\mathrm{bs}} w_k^m, y_i \rangle y_i,$$

$$C_m^n x = \sum_{i=1}^{p} \sum_{k=1}^{\infty} \sqrt{\sigma_k} \langle x, w_k \rangle \langle C^{\mathrm{bs}} w_k, y_i \rangle y_i,$$

where $p = \dim \mathscr{Y}$. From the above we obtain that

$$C_m^n A_m^n B_m^n u = \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_j^m \langle \underline{u}, v_j^m \rangle \langle A^{\mathrm{bs}} w_j^m, w_i^m \rangle \langle C^{\mathrm{bs}} w_k^m, y_k \rangle y_k,$$

and

$$C^n A^n B^n u = \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_j \langle \underline{u}, v_j \rangle \langle A^{\mathrm{bs}} w_j, w_i \rangle \langle C^{\mathrm{bs}} w_k, y_k \rangle y_k.$$

It follows that $C_m^n A_m^n B_m^n \to C^n A^n B^n$ as $m \to \infty$. Similarly we obtain for all $k \in \mathbb{Z}^+$ that $C_m^n (A_m^n)^k B_m^n \to C^n (A^n)^k B^n$ as $m \to \infty$. It follows that we have convergence of the Taylor coefficients of G_m^n to those of G^n.

Proposition 10.18 shows that Σ_m^n (for m large enough) and Σ^n are exponentially stable (note that since the Hankel singular values of Σ_m converge to those of Σ it follows that Σ_m has at least n nonzero Hankel singular values for m large enough). By dominated convergence it now follows that the transfer function G_m^n converges to G^n pointwise in $\overline{\mathbb{D}}$. Since $\overline{\mathbb{D}}$ is compact this is equivalent to uniform convergence and so we have $\|\mathsf{G}_m^n - \mathsf{G}^n\|_{H^\infty(\mathcal{L}(\mathscr{U},\mathscr{Y}))} \to 0$ as desired. □

Proof of Proposition 10.11. We have

$$\|\mathsf{G} - \mathsf{G}^n\|_\infty \le \|\mathsf{G} - \mathsf{G}_m\|_\infty + \|\mathsf{G}_m - \mathsf{G}_m^n\|_\infty + \|\mathsf{G}_m^n - \mathsf{G}^n\|_\infty. \tag{10.2}$$

Let $\varepsilon > 0$. From Lemma 10.25 we have $\|\mathsf{G} - \mathsf{G}_m\|_\infty \to 0$ as $m \to \infty$ and so there exists a M_1 such that if $m \ge M_1$ then $\|\mathsf{G} - \mathsf{G}_m\|_\infty < \varepsilon$. From Lemma 10.28 we have $\|\mathsf{G}^n - \mathsf{G}_m^n\|_\infty \to 0$ as $m \to \infty$. Hence there exists a M_2 such that if $m \ge M_2$ then $\|\mathsf{G}_m^n - \mathsf{G}^n\|_\infty < \varepsilon$. We now consider the second term on the right-hand side of (10.2). From finite-dimensional theory (see Zhou, Doyle and Glover [103, page 566]) we obtain

$$\|\mathsf{G}_m - \mathsf{G}_m^n\|_\infty \le 2 \sum_{i=n+1}^{m} \sigma_i^m.$$

Since $\sigma_i^m \to \sigma_i$ by Lemma 10.26 there exists an M_3 such that if $m \geq M_3$, then $|\sigma_i^m - \sigma_i| < \varepsilon/n$ from which it follows that

$$\sum_{i=1}^{n} \sigma_i^m > \sum_{i=1}^{n} \sigma_i - \varepsilon.$$

By the convergence in the nuclear norm of G_m to G (Lemma 10.25) we have $\|\mathsf{G}_m\|_N \to \|\mathsf{G}\|_N$, which implies the existence of a M_4 such that if $m \geq M_4$, then

$$\sum_{i=1}^{m} \sigma_i^m \leq \sum_{i=1}^{\infty} \sigma_i + \varepsilon.$$

Combining the above three inequalities we see that if $m \geq M_3$ and $m \geq M_4$, then

$$\|\mathsf{G}_m - \mathsf{G}_m^n\|_\infty \leq 2 \sum_{i=n+1}^{m} \sigma_i^m = 2\sum_{i=1}^{m} \sigma_i^m - 2\sum_{i=1}^{n} \sigma_i^m$$

$$\leq 2\sum_{i=1}^{\infty} \sigma_i + 2\varepsilon - 2\sum_{i=1}^{n} \sigma_i + 2\varepsilon = 2\sum_{i=n+1}^{\infty} \sigma_i + 4\varepsilon.$$

Define $M = \max\{M_1, M_2, M_3, M_4\}$. Then for $m \geq M$

$$\|\mathsf{G} - \mathsf{G}^n\|_\infty \leq 2 \sum_{i=n+1}^{\infty} \sigma_i + 6\varepsilon.$$

Since this holds for all $\varepsilon > 0$ we obtain

$$\|\mathsf{G} - \mathsf{G}^n\|_\infty \leq 2 \sum_{i=n+1}^{\infty} \sigma_i.$$

□

10.2 LQG-balanced realizations

In this section we study LQG-balanced realizations. The optimal closed-loop system from Definition 6.32 plays a crucial role in relating LQG-balanced realizations and Lyapunov-balanced realizations.

Proposition 10.29. *Let Σ be an input and output stabilizable discrete-time system. Let $Q^{\min}$ and $P^{\min}$ denote the optimal cost operators of the system and of its dual system, respectively, and let L_B and L_C denote the gramians of the optimal closed-loop system. Then $\lambda \in \sigma(P^{\min}Q^{\min})$ if and only if $\lambda/(1+\lambda) \in \sigma(L_B L_C)$.*

Proof. From Propositions 6.35 and 6.43 we obtain the equality $L_B L_C = (I + P^{\min}Q^{\min})^{-1}P^{\min}Q^{\min}$, from which it follows that $I - L_B L_C = (I + P^{\min}Q^{\min})^{-1}$. So $1 \in \rho(L_B L_C)$, and $P^{\min}Q^{\min} = (I - L_B L_C)^{-1}L_B L_C$. Let $\lambda \in \mathbb{C} - \{-1\}$ and define $\mu := \lambda/(1+\lambda)$, then $\lambda = \mu/(1-\mu)$. We have

$$\begin{aligned}\lambda I - P^{\min}Q^{\min} &= \frac{\mu}{1-\mu}I - L_B L_C(I - L_B L_C)^{-1}\\ &= \frac{1}{1-\mu}\left[\mu I - (1-\mu)L_B L_C(I - L_B L_C)^{-1}\right]\\ &= \frac{1}{1-\mu}\left[\mu(I - L_B L_C) - (1-\mu)L_B L_C\right](I - L_B L_C)^{-1}\\ &= \frac{1}{1-\mu}(\mu I - L_B L_C)(I - L_B L_C)^{-1}.\end{aligned}$$

This shows that $\lambda \in \sigma(P^{\min}Q^{\min})$ if and only if $\mu = \lambda/(1+\lambda) \in \sigma(L_B L_C)$. □

Proposition 10.30. *Let Σ_i with $i = 1, 2$ be two input and output stabilizable discrete-time systems. Let $Q_i^{\min}$ and $P_i^{\min}$ denote the optimal cost operators of the system and of its dual system, respectively. If the two systems have the same transfer function then, with the possible exception of zero, the spectra of $P_1^{\min}Q_1^{\min}$ and $P_2^{\min}Q_2^{\min}$ are equal.*

Proof. Denote the gramians of the optimal closed-loop system of Σ_i by L_{B_i} and L_{C_i}. Then according to Proposition 10.29, the proposition would be proved if the nonzero elements in the spectrum of $L_{B_1}L_{C_1}$ equal the nonzero elements in the spectrum of $L_{B_2}L_{C_2}$. Since the transfer function of both optimal closed-loop systems is a normalized weakly right-coprime factor of the transfer function of both the Σ_i by Proposition 7.11, there exists by Proposition 7.15 a unitary $V \in \mathcal{L}(\mathscr{U})$ such that $[\mathsf{M}_2; \mathsf{N}_2] = [\mathsf{M}_1; \mathsf{N}_1]V$. For the Hankel maps of the optimal closed-loop systems this implies $\mathcal{H}_2 = \mathcal{H}_1 V$, which implies that $\mathcal{H}_2\mathcal{H}_2^* = \mathcal{H}_1\mathcal{H}_1^*$. Since for arbitrary bounded operators S and T we have that the nonzero elements in the spectrum of ST equal the nonzero elements in the spectrum of TS (Lemma 3.16), we have that the nonzero elements in the spectrum of $L_B L_C = \mathcal{B}\mathcal{B}^*\mathcal{C}^*\mathcal{C}$ equal the nonzero elements in the spectrum of $\mathcal{H}\mathcal{H}^* = \mathcal{C}\mathcal{B}\mathcal{B}^*\mathcal{C}^*$. This shows that the nonzero elements in the spectrum of $L_{B_1}L_{C_1}$ equal the nonzero elements in the spectrum of $L_{B_2}L_{C_2}$. □

Definition 10.31. Let Σ be an input and output stabilizable system. Denote the optimal cost operator by $Q^{\min}$ and the optimal cost operator of the dual system by $P^{\min}$. The square roots of the points in the spectrum of $P^{\min}Q^{\min}$, with the exception of zero, are called the **LQG-characteristic values** of Σ.

Note that Proposition 10.30 shows that the LQG-characteristic values only depend on the transfer function, not on the particular realization.

Corollary 10.32. *Let Σ be an input and output stabilizable system. Denote the Hankel map of a realization of a normalized weakly right-coprime factor of the transfer function of Σ by $\mathcal{H}$. Then μ is a LQG-characteristic value of Σ if and only if $\mu \neq 0$ and $\mu^2/(1+\mu^2) \in \sigma(\mathcal{H}\mathcal{H}^*)$. In particular we have $\mu_1^2/(1+\mu_1^2) = \|\mathcal{H}\|^2$ for the largest LQG-characteristic value μ_1.*

Proof. The relationship between the LQG-characteristic values and the spectrum of $\mathcal{H}^*\mathcal{H}$ was proven in the proof of Proposition 10.30. The formula for the largest LQG-characteristic value follows using that since $\mathcal{H}\mathcal{H}^*$ is nonnegative self-adjoint its norm equals the largest eigenvalue and also equals $\|\mathcal{H}\|^2$. □

Definition 10.33. A discrete-time system is called **LQG-balanced** if it is input and output stabilizable and the optimal cost operator of the system and that of its dual system are equal.

The following result shows the existence and uniqueness of LQG-balanced realizations.

Proposition 10.34. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$ and assume that G has a strongly right-coprime factorization. Then G has a minimal LQG-balanced realization. Conversely, the transfer function of a LQG-balanced realization has a strongly right coprime-factorization. Minimal LQG-balanced realizations are unique up to a unitary similarity transformation in the state space.*

Proof. Using Proposition 7.24 it follows from the assumption that G has a strongly right-coprime factorization that G has a normalized strongly right-coprime factorization. Denote a normalized strongly right-coprime factor by $[\mathsf{M}; \mathsf{N}]$. From Proposition 10.2 we obtain that $[\mathsf{M}; \mathsf{N}]$ has a minimal Lyapunov-balanced realization $\check{\Sigma}$. It follows from Proposition 5.7 that $\check{\Sigma}$ is energy preserving with storage operator L equal to the gramian. Define the discrete-time system Σ as in Proposition 2.23. Propositions 6.45 and 6.46 show that L is a solution of the control algebraic Riccati equation and $(I-L^2)^{-1}L$ is a solution of the filter algebraic Riccati equation of Σ. Obviously $\check{\Sigma}$ is the

Riccati closed-loop system of Σ corresponding to the solution $Q = L$. Since $\check{\Sigma}$ is strongly stable by Proposition 10.2 it follows from Proposition 6.38 that L is the unique solution of the control algebraic Riccati equation of Σ. We will show that $(I-L^2)^{-1}L$ is the unique solution of the filter algebraic Riccati equation of Σ. Suppose that there are two solutions P_1 and P_2. Then, by Proposition 6.39, $(I + P_iL)^{-1}P_i$, $i = 1, 2$, are both solutions of the control Lypunov equation of $\check{\Sigma}$. Since the dual system of the Lyapunov-balanced realization $\check{\Sigma}$ is strongly stable by Proposition 10.2 it follows from Proposition 3.14 that the control Lypunov equation of $\check{\Sigma}$ has a unique solution. This implies that $(I + P_1L)^{-1}P_1 = (I + P_2L)^{-1}P_2$, from which $P_1 = P_2$ easily follows. Now apply the similarity transformation $(I - L^2)^{1/4}$ to Σ to obtain a system Σ_{LQG}. It is easily seen that this system has $L(I - L^2)^{1/2}$ as the unique solution to both its control and filter algebraic Riccati equation. Hence Σ_{LQG} is LQG-balanced.

We now show that since $\check{\Sigma}$ is minimal, so is Σ_{LQG}. Since $\check{\Sigma}$ is minimal its gramian L is positive by Proposition 3.11. This implies that the optimal cost operator $L(I - L^2)^{1/2}$ of Σ_{LQG} is positive. It follows using Proposition 6.12 that Σ_{LQG} is approximately observable. Since the optimal cost operator of the dual system of Σ_{LQG} is positive it follows that Σ_{LQG} is approximately controllable.

We now show the uniqueness of minimal LQG-balanced realizations. Assume that Σ_i, $i = 1, 2$ are both minimal LQG-balanced realizations of the same transfer function. Denote the optimal cost operator of Σ_i by $Q_i^{\min}$. First apply the similarity transformation $(I + (Q_i^{\min})^2)^{1/4}$ to Σ_i and then construct the optimal closed-loop systems $\check{\Sigma}_i$. Using Propositions 6.35 and 6.43 it follows that $\check{\Sigma}_i$ is Lyapunov-balanced. The optimal control operator of Σ_i is positive by minimality (using Proposition 6.12), which implies that the gramian of $\check{\Sigma}_i$ is positive, which implies that $\check{\Sigma}_i$ is minimal using Propositions 3.11 and 3.15. Denote the transfer function of $\check{\Sigma}_i$ by $[\mathsf{M}_i; \mathsf{N}_i]$. It follows from Proposition 7.11 that these are normalized weakly right-coprime factors of the transfer function of the Σ_i. From Proposition 7.15 it follows that there exists a unitary $V \in \mathcal{L}(\mathscr{U})$ such that $[\mathsf{M}_1; \mathsf{N}_1] = [\mathsf{M}_2; \mathsf{N}_2]V$. It is easily seen that if we apply the input-space transformation V to $\check{\Sigma}_2$, then we obtain a minimal Lyapunov-balanced realization of $[\mathsf{M}_1; \mathsf{N}_1]$. Since $\check{\Sigma}_1$ is also a minimal Lyapunov-balanced realization of $[\mathsf{M}_1; \mathsf{N}_1]$ they are related by a unitary similarity transformation $U \in \mathcal{L}(\mathscr{X})$ by Proposition 10.2. From this it follows, using (2.5) which gives the system operator of Σ_i in terms of that of $\check{\Sigma}_i$, that Σ_1 and Σ_2 are related by the same unitary similarity transformation. Note that the operator V cancels when we apply (2.5). □

Corollary 10.35. *Let Σ be LQG-balanced with optimal cost operator $Q^{\min}$.*

Apply the similarity transformation $(I + (Q^{\min})^2)^{1/4}$ *to* Σ. *Denote by* Σ_{LYAP} *the optimal closed-loop system of this transformed system. Then* Σ_{LYAP} *is Lyapunov-balanced and its transfer function is a normalized strongly right-coprime factor of the transfer function of* Σ.

Proof. This follows from the proof of Proposition 10.34. □

Definition 10.36. The discrete-time system Σ_{LYAP} is called the Lyapunov-balanced system corresponding to the LQG-balanced discrete-time system Σ.

Definition 10.37. A discrete-time system is called **compact LQG-balanced** if it is LQG-balanced and its optimal cost operator is compact.

Definition 10.38. Given a compact LQG-balanced realization Σ, let (w_i) be an ordered sequence of eigenvectors of the optimal cost operator $Q^{\min}$ (the ordering is such that the corresponding eigenvalues μ_i form a non-increasing sequence). Let $n \in \mathbb{Z}^+$ be such that $\mu_n > \mu_{n+1}$. The **truncated LQG-balanced realization** of dimension n with respect to the sequence of eigenvectors (w_i) is defined as the restriction/projection of Σ onto $\mathscr{X}_n := \{w_i : i = 1, \ldots, n\}$.

Remark 10.39. The sequence (w_i) from Definition 10.38 is also an ordered basis of eigenvectors for the gramian L of the corresponding compact Lyapunov-balanced realization Σ_{LYAP}. Indeed, since $L = (I + (Q^{\min})^2)^{-1/2} Q^{\min}$ we have $\sigma_i = \mu_i / \sqrt{1 + \mu_i^2}$ for the corresponding eigenvalues.

Lemma 10.40. *Let* Σ *be a discrete-time system, let* $[F, G]$ *be an admissible feedback pair and denote by* $\Sigma_{[F,G]}$ *the corresponding closed-loop system. Let* $\tilde{\mathscr{X}} \subset \mathscr{X}$ *be a subspace. Let* $\tilde{\Sigma}$ *be the projection/restriction of* Σ *onto* $\tilde{\mathscr{X}}$. *Then* $[F|_{\tilde{\mathscr{X}}}, G]$ *is an admissible feedback pair for* $\tilde{\Sigma}$ *and the corresponding closed-loop system equals the projection/restriction of* $\Sigma_{[F,G]}$ *onto* $\tilde{\mathscr{X}}$.

Proof. This is easily seen from the definitions. □

Lemma 10.41. *Let* Σ *be a discrete-time system and* $\tilde{\mathscr{X}} \subset \mathscr{X}$ *a subspace. Let* $\tilde{\Sigma}$ *be the projection/restriction of* Σ *onto* $\tilde{\mathscr{X}}$. *Let* $T \in \mathcal{L}(\mathscr{X})$ *have a bounded inverse and map* $\tilde{\mathscr{X}}$ *onto itself. Denote the discrete-time system obtained from* Σ *by applying the similarity transformation* T *by* Σ_T. *Denote the discrete-time system obtained from* $\tilde{\Sigma}$ *by applying the similarity transformation* $T|_{\tilde{\mathscr{X}}}$ *by* $\tilde{\Sigma}_T$. *Then* $\tilde{\Sigma}_T$ *is the projection/restriction of* Σ_T *onto* $\tilde{\mathscr{X}}$.

Proof. This follows easily. □

Proposition 10.42. *Let Σ be a compact LQG-balanced realization. Let (w_i) be an ordered sequence of eigenvectors of the optimal cost operator $Q^{\min}$ (the ordering is such that the corresponding eigenvalues μ_i form a nonincreasing sequence). Denote the optimal feedback pair for Σ as given in Proposition 6.33 by $[F^{\min}, G^{\min}]$. Denote the truncated LQG-balanced realization with respect to the sequence (w_i) by Σ_n. Define Σ_n^{cl} as the closed-loop system of Σ_n with the feedback pair $[F^{\min}|_{\mathscr{X}_n}, G^{\min}]$. Apply the similarity transformation $T := (I + Q^{\min}|^2_{\mathscr{X}_n})^{1/4}$ to Σ_n^{cl} to obtain $\Sigma_{n,\mathrm{LYAP}}$. Define $\Sigma_{\mathrm{LYAP},n}$ as the truncation of the Lyapunov-balanced realization Σ_{LYAP} corresponding to the LQG-balanced discrete-time system Σ. Then $\Sigma_{n,\mathrm{LYAP}} = \Sigma_{\mathrm{LYAP},n}$.*

Proof. This follows using Lemmas 10.40 and 10.41. □

Corollary 10.43. *Using the assumptions and notation of Proposition 10.42 we have that the transfer function of $\Sigma_{\mathrm{LYAP},n}$ is a right factor of the transfer function of Σ_n.*

Proof. The discrete-time system $\Sigma_{\mathrm{LYAP},n}$ is input-output stable by Proposition 10.18 combined with Proposition 3.28. Since $\Sigma_{n,\mathrm{LYAP}}$ is obtained from Σ_n by feedback and a similarity transformation we have the relation in (2.5) (up to the similarity transformation) between their system operators. The relationship between their transfer functions as in Proposition 2.23 follows. Since $\Sigma_{n,L} = \Sigma_{L,n}$ by Proposition 10.42 we obtain the desired result. □

Remark 10.44. We use the notation of Proposition 10.42. The feedback pair $[F^{\min}|_{\mathscr{X}_n}, G^{\min}]$ is in general not the optimal feedback pair for Σ_n. It follows from Proposition 10.42 that Σ_n^{cl} is input stable, output stable and input-output stable. It follows that Σ_n is output stabilizable. Hence it has an optimal cost operator $Q_n^{\min}$. We have for every $x_0 \in \mathscr{X}_n$ that $\langle Q_n^{\min} x_0, x_0\rangle \leq \langle Q^{\min}|_{\mathscr{X}_n} x_0, x_0\rangle$, where $Q^{\min}$ is the optimal cost operator of Σ. Since $\mu_1 = \|Q^{\min}\| = \|Q^{\min}|_{\mathscr{X}_n}\|$ and for μ_1^n, the largest LQG characteristic value of Σ_n, we have $\mu_1^n = \|Q_n^{\min}\|$ we obtain $\mu_1^n \leq \mu_1$.

Proposition 10.45. *Let Σ be a compact LQG-balanced realization with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Define G^n as the transfer function of a truncated LQG-balanced realization of dimension n of Σ. Then we have*

$$\vec{\delta}_g(\mathsf{G}, \mathsf{G}^n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}.$$

Proof. Denote, as in Proposition 10.42, by Σ_{LYAP} the Lyapunov-balanced realization corresponding to the LQG-balanced realization Σ and by $\Sigma_{\mathrm{LYAP},n}$

its truncation. It follows from Corollary 10.43 that the transfer function $[\mathsf{M}^n; \mathsf{N}^n]$ of $\Sigma_{\text{LYAP},n}$ is a right factor of G^n. Let $[\mathsf{M}; \mathsf{N}]$ denote the transfer function of Σ_{LYAP}. It follows from Corollary 10.35 that $[\mathsf{M}; \mathsf{N}]$ is a normalized strongly right-coprime factor of G. Proposition 9.15 shows that

$$\vec{\delta}_g(\mathsf{G}, \mathsf{G}^n) \leq \left\| \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} - \begin{bmatrix} \mathsf{M}^n \\ \mathsf{N}^n \end{bmatrix} \right\|.$$

Using the relation between the LQG-characteristic values of Σ and the Hankel singular values of Σ_{LYAP} from Remark 10.39 we see that Σ_{LYAP} has a nuclear Hankel map. Proposition 10.11 shows that

$$\left\| \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} - \begin{bmatrix} \mathsf{M}^n \\ \mathsf{N}^n \end{bmatrix} \right\| \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}.$$

The desired result follows. □

Proposition 10.46. *Let Σ be a compact LQG-balanced realization with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Define G^n as the transfer function of a truncated LQG-balanced realization of dimension n of Σ. Then there exists a $N \in \mathbb{Z}^+$ such that*

$$2 \sum_{i=N+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < \frac{1}{\sqrt{1+\mu_1^2}}. \tag{10.3}$$

For $n \geq N$ we have

$$\delta_g(\mathsf{G}, \mathsf{G}^n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}.$$

Proof. The existence of N such that (10.3) holds follows from the assumption that (μ_i) is a summable sequence.

From the proof of Proposition 10.45 we obtain that the function G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ and G^n has a right factor $[\mathsf{M}^n; \mathsf{N}^n]$ such that

$$\left\| \begin{bmatrix} \mathsf{M} - \mathsf{M}_n \\ \mathsf{N} - \mathsf{N}_n \end{bmatrix} \right\| \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}.$$

It follows from Proposition 7.32 that since $n \geq N$ we have that $[\mathsf{M}^n; \mathsf{N}^n]$ is strongly right-coprime. Here we have used that the right-hand side of

(10.3) is exactly $\sqrt{1-\|\mathcal{H}\|^2}$, where $\mathcal{H}$ is the Hankel map corresponding to $[\mathsf{M};\mathsf{N}]$, a formula that follows from the one given in Corollary 10.32 for the largest LQG-characteristic value. It follows using Proposition 9.18 that for all $n \geq N$ we have $\delta(\mathsf{G},\mathsf{G}^n) = \vec{\delta}(\mathsf{G},\mathsf{G}^n) = \vec{\delta}(\mathsf{G}^n,\mathsf{G})$. The result now follows using Proposition 10.45. $\square$

Proposition 10.47. *Let Σ be a compact LQG-balanced realization with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Define G^n as the transfer function of a truncated LQG-balanced realization of dimension n of Σ. Then there exists a $N \in \mathbb{Z}^+$ such that for all $n \geq N$*

$$2\sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < \frac{1}{\sqrt{1+(\mu_1^n)^2}}, \tag{10.4}$$

where μ_1^n is the largest LQG-characteristic value of G^n. For given $n \geq N$ choose ε_n such that

$$2\sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < \varepsilon_n < \frac{1}{\sqrt{1+(\mu_1^n)^2}}.$$

Then the ε_n-robust right factor stabilizing feedback function for G^n stabilizes G.

Proof. We have $\mu_1^n \leq \mu_1$ by Remark 10.44. This implies that the right hand side of (10.4) is bounded from below by $1/\sqrt{1+\mu_1^2}$. Formula (10.4) then follows from the fact that (μ_i) forms a summable sequence. With the indicated choice of ε_n we obtain from Proposition 10.46 that $\delta_g(\mathsf{G},\mathsf{G}^n) < \varepsilon_n$. The result then follows from Proposition 9.25 using that the right hand side of (10.4) equals $\sqrt{1-\|\mathcal{H}_n\|^2}$, where $\mathcal{H}_n$ is the Hankel map of a normalized strongly right-coprime factor of G^n. $\square$

Remark 10.48. Note that since G^n is rational it has a finite-dimensional state space realization. Consequently, Corollary 8.19 implies that the ε_n-robust right factor stabilizing feedback function mentioned in Proposition 10.47 can be chosen to be rational.

Remark 10.49. Consider the situation as in Proposition 10.47. Let G_Δ be such that $\delta_g(\mathsf{G},\mathsf{G}_\Delta) < 1/\sqrt{1+\mu_1^2}$, where μ_1 is the largest LQG-characteristic value of G. Then there exists a $N \in \mathbb{Z}^+$ such that for $n \geq N$ the ε_n-robust right factor stabilizing feedback function for G^n, with ε_n chosen sufficiently close to $1/\sqrt{1+(\mu_1^n)^2}$, stabilizes G_Δ. This follows using the triangle inequality.

We give a sufficient condition for the LQG-characteristic values (μ_i) to form a summable sequence.

Proposition 10.50. *Let Σ be an exponentially stabilizable and detectable system with finite-dimensional input and output spaces. Then its LQG-characteristic values (μ_i) form a summable sequence.*

Proof. It follows from Corollary 4.13 that the optimal closed-loop system Σ^{opt} of Σ is exponentially stable. Proposition 10.17 then shows that Σ^{opt} has a nuclear Hankel map. It follows from Remark 10.39 that the Hankel singular values of Σ^{opt} equal $\mu_i/\sqrt{1+\mu_i}$. So $(\mu_i/\sqrt{1+\mu_i})$ forms a summable sequence. It is easily seen that this is equivalent to (μ_i) being a summable sequence. $\square$

Notes

Lyapunov-balanced realization were introduced by Moore [57] for finite dimensional systems. LQG-balanced realizations were introduced by Verriest [93], also in the context of finite dimensional systems. See also Jonckheere and Silverman [41] for LQG-balanced realizations for finite-dimensional systems. Propositions 10.2 and 10.4 are due to Young [99], except for the statement on strong stability in Proposition 10.2, which is due to Ober and Wu [63]. Proposition 10.11 is due to Glover, Curtain and Partington [35] in the continuous-time case and based on these ideas by Bonnet [5] in the discrete-time case. Both references treat only the case with nonrepeating eigenvalues, but the general case considered here follows along the same lines as was already indicated in [35]. Proposition 10.17 is based on Curtain and Sasane [9]. The results on LQG-balanced realizations are based on Opmeer and Curtain [71] and Opmeer [68].

Part II

Continuous-time systems

Chapter 11

Basic objects

In this chapter we provide a new framework for continuous-time systems.

11.1 Resolvent linear systems

A finite-dimensional linear system is usually described by specifying four matrices A, B, C, D and defining for a given initial state x_0 and an input function $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{\mathrm{u}})$ the state $x \in C(0,\infty;\mathbb{C}^{\mathrm{x}})$ and the output $y \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{\mathrm{y}})$ as the unique solutions of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t). \tag{11.1}$$

As is well-known, these unique solutions are given explicitly by

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,ds, \tag{11.2}$$

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-s)}Bu(s)\,ds + Du(t).$$

If we Laplace transform the equations (11.1) and solve for x and y we obtain

$$\hat{x}(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}B\hat{u}(s) \tag{11.3}$$

$$\hat{y}(s) = C(sI-A)^{-1}x_0 + \big(C(sI-A)^{-1}B + D\big)\,\hat{u}(s).$$

Our approach to continuous-time infinite-dimensional systems will be to generalize the situation (11.3) rather than the situation (11.1) or (11.2).

In this section we study the generalizations of the matrix-valued functions $(sI-A)^{-1}$, $(sI-A)^{-1}B$, $C(sI-A)^{-1}$ and $C(sI-A)^{-1}B+D$. The generalization of the dynamical system (11.3) will be considered in Section 11.2. We first consider the generalization of the resolvent.

Definition 11.1. Let $\mathscr{X}$ be a Hilbert space and Λ a nonempty subset of the complex plane. A function $\mathfrak{a} : \Lambda \to \mathcal{L}(\mathscr{X})$ that satisfies the following **resolvent equation**

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda$$

is called a **pseudoresolvent**. A pseudoresolvent $\mathfrak{a}_{\max}$ is called a **maximal pseudoresolvent** if there is no pseudoresolvent that is a proper extension of $\mathfrak{a}_{\max}$.

Lemma 11.2. *Every pseudoresolvent has a unique extension to a maximal pseudoresolvent $\mathfrak{a}_{\max} : \Lambda_{\max} \to \mathcal{L}(\mathscr{X})$. The set $\Lambda_{\max}$ is open and $\mathfrak{a}_{\max}$ is holomorphic.*

Proof. This is contained in Hille and Phillips [38, Chapter 5.2]. □

We now consider the generalization of all the indicated matrix-valued functions.

Definition 11.3. A **resolvent linear system** on a triple of Hilbert spaces $(\mathscr{U}, \mathscr{X}, \mathscr{Y})$ consists of a nonempty subset Λ of the complex plane and four operator valued function $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ satisfying
$\mathfrak{a} : \Lambda \to \mathcal{L}(\mathscr{X})$ satisfies

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.4}$$

$\mathfrak{b} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{X})$ satisfies

$$\mathfrak{b}(\beta) - \mathfrak{b}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.5}$$

$\mathfrak{c} : \Lambda \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$ satisfies

$$\mathfrak{c}(\beta) - \mathfrak{c}(\alpha) = (\alpha - \beta)\mathfrak{c}(\alpha)\mathfrak{a}(\beta) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.6}$$

$\mathfrak{d} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ satisfies

$$\mathfrak{d}(\beta) - \mathfrak{d}(\alpha) = (\alpha - \beta)\mathfrak{c}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.7}$$

The function $\mathfrak{a}$ is called the **pseudoresolvent**, $\mathfrak{b}$ the **incoming wave function**, $\mathfrak{c}$ the **outgoing wave function** and $\mathfrak{d}$ the **characteristic function** of the resolvent linear system. The pseudoresolvent is assumed to be maximal.

Proposition 11.4. *The pseudoresolvent, the wave functions and the characteristic function of a resolvent linear system are holomorphic.*

Proof. For the pseudoresolvent this was already stated in Lemma 11.2. For the other three functions it follows from the functional equations using that the pseudoresolvent is holomorphic. For example to prove that $\mathfrak{b}$ is holomorphic in a point β first fix a point α and note that the term on the right-hand side of (11.5) is holomorphic in β. It follows that the term on the left-hand side of the equation is and since $\mathfrak{b}(\alpha)$ is constant it follows that $\mathfrak{b}$ is holomorphic in β. □

A resolvent linear system is completely determined by the values of the pseudoresolvent, the wave functions and the characteristic function at one point in the following sense.

Proposition 11.5. *For $a \in \mathcal{L}(\mathscr{X})$, $b \in \mathcal{L}(\mathscr{U}, \mathscr{X})$, $c \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$, $d \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$ and $\alpha \in \mathbb{C}$ there exists a unique resolvent linear system with $\alpha \in \Lambda$ and $\mathfrak{a}(\alpha) = a$, $\mathfrak{b}(\alpha) = b$, $\mathfrak{c}(\alpha) = c$, $\mathfrak{d}(\alpha) = d$.*

Proof. The function $\tilde{\mathfrak{a}} : \{\alpha\} \to \mathcal{L}(\mathscr{X})$ defined by $\tilde{\mathfrak{a}}(\alpha) = a$ defines a pseudoresolvent. By Lemma 11.2 it has a maximal extension which we denote by $\mathfrak{a}$ and whose domain we denote by Λ. Define the operator-valued functions $\mathfrak{b} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{X})$, $\mathfrak{c} : \Lambda \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$, $\mathfrak{d} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by $\mathfrak{b}(s) := b + (\alpha - s)\mathfrak{a}(s)b$, $\mathfrak{c}(s) := c + (\alpha - s)c\mathfrak{a}(s)$, $\mathfrak{d}(s) := d + (\alpha - s)\mathfrak{c}(s)b$. It is easily seen that his gives a resolvent linear system. The desired uniqueness follows from the uniqueness of the maximal pseudoresolvent. □

We now show how unbounded operators A, B, C can be constructed that generalize the matrices considered earlier in this section. Assume that $\mathfrak{a}$ is the resolvent of a densely defined closed operator A with nonempty resolvent set. A necessary and sufficient condition for such an A to exist is that there exists an $\alpha \in \Lambda$ such that $\mathfrak{a}(\alpha)$ is one-to-one and has dense range. We now introduce two spaces. Let $\mathscr{X}_1$ be $D(A)$ with the norm $\|x\|_1 := \|(\alpha - A)x\|$. For every $\alpha \in \rho(A)$ this is a Hilbert space with norm equivalent to the graph norm. Let $\mathscr{X}_{-1}$ be the completion of $\mathscr{X}$ with respect to the norm $\|x\|_{-1} := \|\mathfrak{a}(\alpha)x\|$. The operator A has an extension $A_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}_{-1}$. Define $B : \mathscr{U} \to \mathscr{X}_{-1}$ by $B := (\alpha - A_{\mathscr{X}})\mathfrak{b}(\alpha)$, it follows from the functional equation (11.5) that B does not depend on α. Define the operator $C : \mathscr{X}_1 \to \mathscr{Y}$ by $C := \mathfrak{c}(\alpha)(\alpha - A)$, it follows from the functional equation (11.6) that C does not depend on α. A meaningful generalization of the matrix D is not always possible.

We make the following definition.

Definition 11.6. An **operator node** is a resolvent linear system for which the pseudoresolvent is the resolvent of a densely defined closed operator with nonempty resolvent set.

Remark 11.7. One can define an operator node through four generating operators. An operator A on the state space $\mathscr{X}$ which is densely defined and has nonempty resolvent set. An operator $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_{-1})$, an operator $C \in \mathcal{L}(\mathscr{X}_1, \mathscr{Y})$ and an operator $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$. The corresponding resolvent linear system is then defined as follows. The pseudoresolvent is the resolvent of A. The incoming wave function is defined as $\mathfrak{b}(s) := (sI - A_{\mathscr{X}})^{-1}B$, the outgoing wave function by $\mathfrak{c}(s) := C(sI - A)^{-1}$ and the charcteristic function by fixing $\alpha \in \rho(A)$, defining $\mathfrak{d}(\alpha) = D$ and extending this to the whole of $\rho(A)$ by using (11.7).

11.2 Distributional resolvent linear systems

In this section we define a subclass of the set of resolvent linear systems for which the dynamical system (11.3) has a meaningful generalization.

Definition 11.8. A **distributional resolvent linear system** is a resolvent linear system with the additional propery that there exist constants $\alpha > 0, \beta \in \mathbb{R}$ and a polynomial p such that

$$\Lambda_E(\alpha, \beta) := \{s \in \mathbb{C} : \text{Re } s \geq \beta, \quad |\text{Im } s| \leq \mathrm{e}^{\alpha \text{Re } s}\} \subset \Lambda \tag{11.8}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \forall\ s \in \Lambda_E. \tag{11.9}$$

A region Λ_E as above is called an exponential region (see Arendt, El-Mennaoui and Kéyantuo [2]). Note that the wave functions and characteristic function of a distributional resolvent linear system are also polynomially bounded on Λ_E (this follows from the functional equations in Definition 11.3).

Equivalently we could assume that the pseudoresolvent is polynomially bounded on a logarithmic region. A logarithmic region is a region of the form

$$\Lambda_L(a, b, c) := \{s \in \mathbb{C} : \text{Re } s \geq c, \text{Re } s \geq \frac{1}{a} \log |s| + b\} \tag{11.10}$$

with $a > 0$ and $b, c \in \mathbb{R}$. This is true since one can show that an exponential region is contained in a logarithmic region is contained in an exponential region (see Arendt, El-Mennaoui and Kéyantuo [2]).

We also define the following subclass of distributional resolvent linear systems where we do not work on an exponential region, but on a half-plane.

Definition 11.9. A distributional resolvent linear system is called **exponentially bounded** if there exists a $\gamma \in \mathbb{R}$ and a polynomial p such that

$$\Lambda_H(\gamma) := \{s \in \mathbb{C} : \text{Re } s \geq \gamma\} \subset \Lambda \tag{11.11}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \forall\, s \in \Lambda_H. \tag{11.12}$$

Remark 11.10. In the sequel we will need the following well-known characterization of Laplace transformable Banach space valued distributions by Schwartz. The image of the Schwartz-Laplace transformable Banach-space valued distributions is exactly the set of polynomially bounded holomorphic functions defined on some right half-plane. For details see Schwartz [86]. A generalization of this characterization is due to Kunstmann [49]. He defined a space of Banach space valued distributions that can be Laplace transformed and whose image under the Laplace transform is exactly the set of polynomially bounded holomorphic functions defined on some exponential region.

Using Remark 11.10 we are now in a position to generalize the dynamical system (11.3). Let u be a $\mathscr{U}$-valued Kunstmann-Laplace transformable distribution. For a distributional resolvent linear system $\mathfrak{a}(s)x_0 + \mathfrak{b}(s)\hat{u}(s)$ is holomorphic and polynomially bounded on some exponential region and therefore it is the Kunstmann-Laplace transform of some $\mathscr{X}$-valued Kunstmann-Laplace transformable distribution. Similar arguments apply to $\mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s)$. This leads to the following definition.

Definition 11.11. The state x and output y of a distributional resolvent linear system corresponding to the initial state $x_0 \in \mathscr{X}$ and the input u (a $\mathscr{U}$-valued Kunstmann-Laplace transformable distribution) are defined through their Kunstmann-Laplace transforms by

$$\hat{x}(s) := \mathfrak{a}(s)x_0 + \mathfrak{b}(s)\hat{u}(s), \quad \hat{y}(s) := \mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s), \tag{11.13}$$

where s is restricted to the intersection of Λ_E and the exponential region on which $\hat{u}$ is holomorphic and polynomially bounded.

Remark 11.12. If the distributional resolvent linear system in Definition 11.11 is assumed to be exponentially bounded and the input u is assumed to be a Schwartz-Laplace transformable distribution, then the state and output of the system are Schwartz-Laplace transformable distributions.

We recall the concept of a system node. See Staffans [89, Section 4.7].

Definition 11.13. A **system node** is an operator node for which A is the generator of a strongly continuous semigroup.

Remark 11.14. Since the resolvent of the generator of a strongly continuous semigroup is uniformly bounded on a right half-plane by the Hille-Yosida conditions, a system node defines an exponentially bounded distributional resolvent linear system.

The concept of a well-posed system as given below is equivalent to the usual one as can be found in Staffans [89].

Definition 11.15. A distributional resolvent linear system is called **well-posed** if there exists a $\sigma \in \mathbb{R}$ such that

- the pseudoresolvent is the resolvent of the generator of a strongly continuous semigroup,
- the restriction of $\mathfrak{b}(\cdot)^{\dagger} x_0$ to the right half-plane $\mathbb{C}_{\sigma}^{+}$ is an element of $H^2(\mathbb{C}_{\sigma}^{+}, \mathscr{U})$ for all $x_0 \in \mathscr{X}$,
- the restriction of $\mathfrak{c}(\cdot) x_0$ to the right half-plane $\mathbb{C}_{\sigma}^{+}$ is an element of $H^2(\mathbb{C}_{\sigma}^{+}, \mathscr{Y})$ for all $x_0 \in \mathscr{X}$,
- the restriction of the characteristic function to the right half-plane $\mathbb{C}_{\sigma}^{+}$ is an element of $H^{\infty}(\mathbb{C}_{\sigma}^{+}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.

Figure 11.1 gives a picture of the inclusion relationships between the different classes of systems we have encountered.

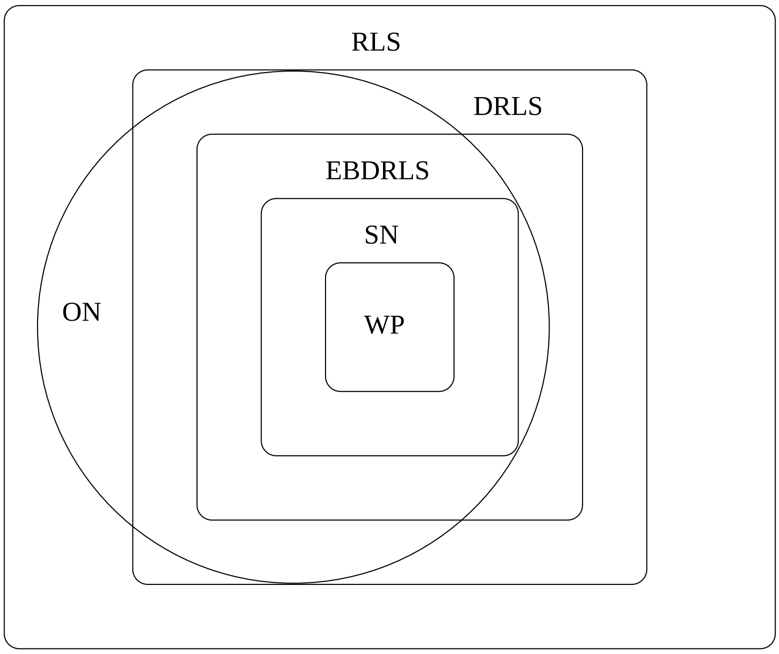

Figure 11.1: Classes of systems. WP=well-posed, SN=system nodes, EBDRLS=exponentially bounded distributional resolvent linear systems, DRLS=distributional resolvent linear systems, ON=operator nodes, RLS= resolvent linear systems.

Notes

The definition of resolvent linear system is taken from Opmeer [66], where also the subclasses of distributional resolvent linear systems and exponentially bounded distributional resolvent linear systems were introduced (the last class under the name integrated resolvent linear systems). See Opmeer [67] for the corresponding time-domain definitions.

The set of operator nodes is implicitly present in Salamon [85]. It is the set of systems that satisfy his assumption (S0) on page 385, but not necessarily his assumptions (S1) to (S4). We refer to Staffans [89, Section 4.7] for alternative characterizations of operator nodes and historical remarks.

Our assumption on the pseudoresolvent in the case of distributional resolvent linear systems (exponentially bounded or not) is much weaker than assumption (S1) of Salamon [85] (the system node assumption). Moreover, we drop assumptions (S2-S4) of Salamon. Hence we obtain a much larger class of systems than the well-posed linear systems introduced by Salamon in [85]. This class of well-posed linear systems has been the state-of-the-art for the last two decades (see Staffans [89]).

The concept of a distributional resolvent linear system is the natural generalization of the concept of distribution semigroup from systems with only a state to input/state/output systems. Distribution semigroups were introduced by Lions [52]. Important contributions were made by Chazarain [7]. The case of not necessarily densely defined generators A is treated in Kunstmann [48] and Wang [95]. The general case (including the degenerate case where the pseudoresolvent is not a resolvent) is treated in Kisyński [44]. See Fattorini [26] for further information on distribution semigroups.

Chapter 12

Partial differential equations

In this chapter we illustrate how partial differential equations with boundary control and observation fit into the framework presented in Chapter 11. We emphasize that the examples given in this chapter are certainly not the only ones that can be formulated in that framework.

In Section 12.1 we recall the concept of an abstract boundary control systems as studied in Salamon [85, Section 2.2] and show that in this setting our wave functions and characteristic function are solution operators of certain elliptic problems. In Section 12.2 we review some results on elliptic differential operators. In Section 12.3 we study partial differential equations which are first order in time (in particular the heat equation) and in Section 12.4 partial differential equations which are second order in time (in particular the wave equation).

12.1 Abstract boundary control systems

We review the concept of an abstract boundary control system.

Definition 12.1. An abstract boundary control system on a quadruple of Hilbert spaces $(\mathscr{U}, \mathscr{K}, \mathscr{X}, \mathscr{Y})$ where $\mathscr{K} \subset \mathscr{X}$ with a continuous and dense injection consists of three operators: $\Delta \in \mathcal{L}(\mathscr{K}, \mathscr{X})$, $\Gamma \in \mathcal{L}(\mathscr{K}, \mathscr{U})$, $K \in \mathcal{L}(\mathscr{K}, \mathscr{Y})$ that satisfy: Γ is onto, $\ker \Gamma$ is dense in $\mathscr{X}$, there exists a $\mu \in \mathbb{R}$ such that $\ker \mu I - \Delta \cap \ker \Gamma = \{0\}$ and $\mu I - \Delta$ is onto.

Let A be the restriction of Δ to $\ker \Gamma$, let C be the restriction of K to $\ker \Gamma$, and given $u \in U$, choose $x \in \mathscr{K}$ such that $\Gamma x = u$ and define

$$Bu = \Delta x - Ax, \quad \mathfrak{d}(\mu) = Kx - C(\mu I - A)^{-1}(\mu x - \Delta x).$$

(note that the A in the definition of B and $\mathfrak{d}$ above is the extension to an operator in $\mathcal{L}(\mathscr{X}, \mathscr{X}_{-1})$ as studied in Section 11.1 and that the definitions are independent of the particular x that is chosen). Then it follows as in Salamon [85, Proposition 2.8] that $A, B, C, \mathfrak{d}(\mu)$ determine an operator node (and hence a resolvent linear system).

It is interesting to note (see Salamon [85, p 391]) that for $\mu \in \rho(A)$ the operator $\mathfrak{b}(\mu)$ is the solution operator for the abstract elliptic problem

$$(\mu - \Delta)x = 0, \quad \Gamma x = u, \tag{12.1}$$

in the sense that for $u \in \mathscr{U}$ the solution is given by $x = \mathfrak{b}(\mu)u$. Similarly, $\mathfrak{a}(\mu)$ is the solution operator of the abstract elliptic problem

$$(\mu - \Delta)x = x_0, \quad \Gamma x = 0, \tag{12.2}$$

$\mathfrak{c}(\mu)$ is the solution operator of the abstract elliptic problem

$$(\mu - \Delta)x = x_0, \quad \Gamma x = 0, \quad Kx = y, \tag{12.3}$$

and $\mathfrak{d}(\mu)$ is the solution operator of the abstract elliptic problem

$$(\mu - \Delta)x = 0, \quad \Gamma x = u, \quad Kx = y. \tag{12.4}$$

Since it is not always easy to see what the space $\mathscr{K}$ should be, we will work with the abstract elliptic problems (12.1-12.4) and not directly with abstract boundary control systems.

With an abstract boundary control system the following dynamical system is associated

$$\begin{aligned} \dot{x}(t) = \Delta x(t), \quad & x(0) = x_0, \\ \Gamma x(t) &= u(t), \\ y(t) &= Kx(t). \end{aligned}$$

We refer to Salamon [85, Section 2.2] and Staffans [89, Section 5.2] for more on abstract boundary control systems.

12.2 An elliptic differential operator

In this section we review some results from the literature on elliptic differential operators. In this section $\Omega \subset \mathbb{R}^n$ is a bounded open domain whose boundary $\partial\Omega$ is a compact orientable C^∞-manifold. We denote the standard Sobolev spaces by $H^s(\Omega)$. The space of infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. The space $H_0^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $H^s(\Omega)$ norm.

An n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called a multi-index. We define

$$\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}, \qquad |\alpha| = \sum_{i=1}^n \alpha_i, \qquad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}.$$

We consider the differential operator L from $H^{2m}(\Omega)$ to $L^2(\Omega)$ defined by

$$L\varphi := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha \varphi,$$

with complex-valued coefficients a_α in $C^\infty(\overline{\Omega})$. The operator L is called **strongly elliptic** if there exists a constant $c > 0$ such that

$$\text{Re } (-1)^m \sum_{|\alpha|=2m} a_\alpha(\xi)\zeta^\alpha \geq c|\zeta|^{2m} \quad \xi \in \overline{\Omega}, \zeta \in \mathbb{R}^n.$$

The formal adjoint of L is the differential operator

$$L^*\psi := \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha \left(\overline{a_\alpha}\psi\right),$$

which is strongly elliptic if and only if L is.

A **Dirichlet form** is a sesquilinear form d on $H^m(\Omega)$ defined by

$$d(\varphi, \psi) := \sum_{|\rho|,|\sigma| \leq m} \langle D^\rho \varphi, a_{\rho\sigma} D^\sigma \psi \rangle_{L^2(\Omega)},$$

here $a_{\rho\sigma}$ are complex-valued functions in $C^\infty(\overline{\Omega})$. A Dirichlet form is called strongly elliptic if

$$\sum_{|\rho|,|\sigma|=m} a_{\rho\sigma}(\xi)\zeta^\rho\zeta^\sigma \geq c|\zeta|^{2m} \quad \xi \in \overline{\Omega}, \zeta \in \mathbb{R}^n,$$

for some constant $c > 0$. The adjoint of the Dirichlet form d is the Dirichlet form d^* defined by $d^*(\psi, \varphi) = \overline{d(\varphi, \psi)}$. A Dirichlet form d is a Dirichlet form for the operator L if

$$d(\varphi, \psi) = \langle \varphi, L\psi \rangle_{L^2(\Omega)} \quad \text{for all } \varphi, \psi \in C_0^\infty(\Omega).$$

Every differential operator as above has an associated Dirichlet form (this follows from integration by parts), however different Dirichlet forms can correspond to the same operator. This nonuniqueness will not be a problem for us. The differential operator L is strongly elliptic if and only if every Dirichlet form for L is strongly elliptic. If $d = d^*$, then $L = L^*$ and if $L = L^*$ then we can choose an associated Dirichlet form such that $d = d^*$.

The above can be found in Folland [27]. See also Agmon [1], Friedman [29] and Bers at al. [4].

12.3 First order equations

We consider the first order (in time) PDE with Dirichlet boundary control described by the equations

$$\frac{\partial x}{\partial t}(\xi,t) + Lx(\xi,t) = 0, \quad \xi \in \Omega, t > 0, \tag{12.5}$$

$$D_\nu^j x(\xi,t) = u_j(\xi,t), \quad \xi \in \partial\Omega, t > 0, j = 0, \dots m-1, \tag{12.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain whose boundary $\partial\Omega$ is a compact orientable C^∞-manifold, L is a strongly elliptic differential operator of order $2m$ (as defined in Section 12.2) and D_ν the normal derivative at $\partial\Omega$ directed towards the exterior of Ω.

We define the observation

$$y_j(\xi,t) = D_\nu^j x(\xi,t), \quad \xi \in \partial\Omega, t > 0, j = m, \dots 2m-1. \tag{12.7}$$

This system can be written as an abstract boundary control system with the formal operators

$$\Delta = -L$$

$$\Gamma x = \begin{bmatrix} D_\nu^0 x|_{\partial\Omega} \\ \vdots \\ D_\nu^{m-1} x|_{\partial\Omega} \end{bmatrix}, K x = \begin{bmatrix} D_\nu^m x|_{\partial\Omega} \\ \vdots \\ D_\nu^{2m-1} x|_{\partial\Omega} \end{bmatrix}.$$

However, the spaces $\mathscr{U}$, $\mathscr{K}$, $\mathscr{X}$, $\mathscr{Y}$ on which these formal operators have the desired properties are not obvious. To obtain these spaces we study the elliptic problems (12.1)-(12.4) with the operators Δ, Γ, K as above.

The pseudoresolvent

We first study the partial differential equation (12.5) with zero Dirichlet boundary conditions. This is a well-studied problem and we recall its solution. Define $A\varphi = -L\varphi$ on $D(A) := H^{2m}(\Omega) \cap H_0^m(\Omega)$. It follows as in Pazy [74, Section 7.2] that A generates an analytic semigroup on $L^2(\Omega)$.

Some spaces

We introduce some spaces needed in the sequel. The Hilbert space $\Xi^r(\Omega)$ for $r \in \mathbb{R}$ is defined as in Lions and Magenes [53, Section 2.6.3 p170]. We need these spaces for $r \in [-2m, 0]$. The only properties of these spaces that we need are

$$\Xi^0(\Omega) = L^2(\Omega), \quad L^2(\Omega) \subset \Xi^r(\Omega),$$

with a continuous injection for $r \leq 0$. Fix $\mu \in \rho(A) \cap \mathbb{R}$ and define the space $D^r_{L+\mu}(\Omega)$ for $r \in [0, 2m]$ as in [53, Section 2.7.2 p 186]

$$D^r_{L+\mu}(\Omega) := \{x \in H^r(\Omega) : (L+\mu)x \in \Xi^{r-2m}(\Omega)\},$$

provided with the graph norm

$$\|x\|_{D^r_{L+\mu}(\Omega)} := \sqrt{\|x\|^2_{H^r(\Omega)} + \|(L+\mu)x\|^2_{\Xi^{r-2m}(\Omega)}},$$

which makes $D^r_{L+\mu}(\Omega)$ a Hilbert space. Note that for $r \in [0, 2m]$ we have $D^r_{L+\mu}(\Omega) \subset L^2(\Omega)$ with a continuous injection.

The incoming wave function

We study the incoming wave function. That is, we study the solution operator of the elliptic problem

$$\begin{aligned} (L+\mu)x &= 0 \quad \text{on } \Omega, \\ \Gamma x &= u \quad \text{on } \partial\Omega, \end{aligned}$$

where $\mu \in \rho(A)$ and L, Γ as above.

Define for $r \in [0, 2m]$ the space

$$\mathscr{U}^r := \Pi_{j=0}^{m-1} H^{r-j-1/2}(\partial\Omega).$$

It follows from [53, Theorem 7.4 p 188] that for all $r \in [0, 2m]$ the map $u \mapsto x$ from $\mathscr{U}^r$ to $D^r_{L+\mu}(\Omega)$ is bounded. It follows that the map $u \mapsto x$ from $\mathscr{U}^r$ to $L^2(\Omega)$ is bounded for all $r \in [0, 2m]$. Hence $\mathfrak{b}(\mu) \in \mathcal{L}(\mathscr{U}^r, L^2(\Omega))$.

The outgoing wave function

We study the outgoing wave function. We consider the problem

$$\begin{aligned} (L+\mu)x &= x_0 \quad \text{on } \Omega, \\ \Gamma x &= 0 \quad \text{on } \partial\Omega, \\ y &= Kx \quad \text{on } \partial\Omega, \end{aligned} \tag{12.8}$$

where $\mu \in \rho(A)$ and L, Γ, K are as above.

Define for $r \in [0, 2m]$ the space

$$\mathscr{Y}^r := \Pi_{j=0}^{m-1} H^{r-m-j-1/2}.$$

It follows from [53, Theorem 7.4 p 188] that for all $r \in [0, 2m]$ the map $x_0 \mapsto x$, defined by the first two equations of (12.8), from $\Xi^{r-2m}(\Omega)$ to $D^r_{L+\mu}(\Omega)$ is bounded. It follows from [53, Theorem 7.3 p 187] that for all $r \in [0, 2m]$ the operator $K : D^r_{L+\mu} \to \mathscr{Y}^r$ is bounded. It follows that the map $x_0 \mapsto y$ from $L^2(\Omega)$ to $\mathscr{Y}^r$ is bounded for all $r \in [0, 2m]$. Hence $\mathfrak{c}(\mu) \in \mathcal{L}(L^2(\Omega), \mathscr{Y}^r)$.

The characteristic function

We study the characteristic function. In order to do so we consider the elliptic problem

$$\begin{aligned}(L+\mu)x &= 0 \quad \text{on } \Omega,\\ \Gamma x &= u \quad \text{on } \partial\Omega,\\ y &= Kx \quad \text{on } \partial\Omega,\end{aligned} \tag{12.9}$$

where $\mu \in \rho(A)$ and L, Γ, K are as above.

It follows as in the case of the incoming wave function that for all $r \in [0, 2m]$ the map $u \mapsto x$, defined by the first two equations of (12.9), from $\mathscr{U}^r$ to $D^r_{L+\mu}(\Omega)$ is bounded. Combined with the result mentioned above on the operator K we obtain that for all $r \in [0, 2m]$ the map $u \mapsto y$ from $\mathscr{U}^r$ to $\mathscr{Y}^r$ is bounded.

First order equations: conclusion

The results obtained show that the PDE (12.5-12.7) can be formulated as a distributional resolvent linear system (even as a system node) on the state space $\mathscr{X} = L^2(\Omega)$ with possible choices of input and output spaces

$$\mathscr{U}^r := \Pi_{j=0}^{m-1} H^{r-j-1/2}(\partial\Omega), \quad \mathscr{Y}^r := \Pi_{j=0}^{m-1} H^{r-m-j-1/2},$$

for $r \in [0, 2m]$.

12.4 Second order equations

We consider the following second order (in time) PDE with Dirichlet boundary control and boundary observation

$$\frac{\partial^2 x}{\partial t^2}(\xi,t) + Lx(\xi,t) = 0 \quad \xi \in \Omega, t > 0, \tag{12.10}$$

$$D^j_\nu x(\xi,t) = u_j(\xi,t), \quad \xi \in \partial\Omega, t > 0, j = 0, \dots m-1, \tag{12.11}$$

$$y_j(\xi,t) = D^j_\nu x(\xi,t), \quad \xi \in \partial\Omega, t > 0, j = m, \dots 2m-1. \tag{12.12}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open domain whose boundary $\partial\Omega$ is a compact orientable C^∞-manifold and $L = L^*$ is a self-adjoint strongly elliptic differential operator (see Section 12.2).

As in section 12.3 the formal differential operator, formal boundary control operator and formal boundary observation operator are obvious:

$$\tilde{\Delta} = \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix}, \quad \tilde{\Gamma} := [\Gamma \;\; 0], \quad \tilde{K} := [K \;\; 0],$$

where Γ and K are as in Section 12.3. We use the theory of cosine functions and that of elliptic problems to determine the spaces $\mathscr{U}$, $\mathscr{K}$, $\mathscr{X}$, $\mathscr{Y}$ on which these formal operators have the desired properties.

The pseudoresolvent

We first study the operator A as defined in Section 12.3 further for the case $L = L^*$ as considered here. It follows as in Fattorini [25, Section IV.8] that A generates a cosine function on $L^2(\Omega)$ (note that the arguments in [25] only make use of the fact that $d = d^*$). This implies that

$$\tilde{A} := \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

with domain $H^{2m}(\Omega) \cap H_0^m(\Omega) \times L^2(\Omega)$ generates an exponentially bounded integrated semigroup on $L^2(\Omega) \times L^2(\Omega)$ (see Arendt et al. [3, Theorem 3.14.7]).

The incoming wave function

We see that the elliptic problem (12.1) is equivalent to

$$(L + \mu^2)x_1 = 0, \quad \Gamma x_1 = u, \quad x_2 = \mu x_1,$$

so it follows as in the case of the incoming wave function for first order equations that the map $u \mapsto x = [x_1; x_2]$ is bounded from $\mathscr{U}^r$ to $D^r_{L+\mu^2}(\Omega) \times \mathscr{H}$ for any Hilbert space $\mathscr{H}$ such that $D^r_{L+\mu^2}(\Omega) \subset \mathscr{H}$ continuously for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. it follows that the map $u \mapsto x = [x_1; x_2]$ is bounded from $\mathscr{U}^r$ to $L^2(\Omega) \times L^2(\Omega)$ for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. Hence $\mathfrak{b}(\mu) \in \mathcal{L}(\mathscr{U}^r, L^2(\Omega) \times L^2(\Omega))$.

The outgoing wave function

We see that the elliptic problem (12.3) is equivalent to

$$(L + \mu^2)x_1 = x_2^0 + \mu x_1^0, \quad x_2 = \mu x_1 - x_1^0, \quad \Gamma x_1 = 0, \quad y = K x_1,$$

so it follows as in the case of the incoming wave function for first order equations that the map $x^0 = [x_1^0; x_2^0] \mapsto y$ is bounded from $\Xi^{r-2m}(\Omega) \times \Xi^{r-2m}(\Omega)$ to $\mathscr{Y}^r$ for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. Hence we obtain that the map $x^0 = [x_1^0; x_2^0] \mapsto y$ is bounded from $L^2(\Omega) \times L^2(\Omega)$ to $\mathscr{Y}^r$ for all $r \in [0, 2m]$. Hence $\mathfrak{c}(\mu) \in \mathcal{L}(L^2(\Omega) \times L^2(\Omega), \mathscr{Y}^r)$.

The characteristic function

We see that the elliptic problem (12.4) is equivalent to

$$(L+\mu^2)x_1 = 0, \quad x_2 = \mu x_1 - x_1^0, \quad \Gamma x_1 = u, \quad y = Kx_1,$$

so it follows as in the case of the characteristic function for first order equations that the map $u \mapsto y$ is bounded from $\mathscr{U}^r$ to $\mathscr{Y}^r$ for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. Hence $\mathfrak{d}(\mu) \in \mathcal{L}(\mathscr{U}^r, \mathscr{Y}^r)$.

Second order equations: conclusion

The results obtained in this section show that the PDE (12.10-12.12) can be formulated as a distributional resolvent linear system on the state space $\mathscr{X} = L^2(\Omega) \times L^2(\Omega)$ with possible choices of input and output spaces

$$\mathscr{U}^r := \Pi_{j=0}^{m-1} H^{r-j-1/2}(\partial\Omega), \quad \mathscr{Y}^r := \Pi_{j=0}^{m-1} H^{r-m-j-1/2},$$

for $r \in [0, 2m]$. Note that since $\tilde{A}$ does not generate a strongly continuous semigroup on $L^2(\Omega) \times L^2(\Omega)$, this distributional resolvent linear system is not a system node.

Notes

The content of this chapter appeared before in Opmeer [67]. Virtually all results depend on the study of non-homogeneous boundary value problems performed in Lions and Magenes [53].

Chapter 13

The Cayley transform

In this section we investigate the relationship between the class of resolvent linear systems and the class of discrete-time systems. This is the tool we shall use to deduce many properties of resolvent linear systems from the corresponding ones for discrete-time systems.

We first define the Cayley transforms of a resolvent linear system. Note that in the literature usually the Cayley transform with parameter $\alpha = 1$ is used.

Definition 13.1. Let $\alpha > 0$. The Cayley transform with parameter α of a resolvent linear system with $\alpha \in \Lambda$ is the discrete-time system with generating operators

$$\begin{aligned} A_d &:= -I + 2\alpha\ \mathfrak{a}(\alpha), & B_d &:= \sqrt{2\alpha}\ \mathfrak{b}(\alpha), \\ C_d &:= \sqrt{2\alpha}\ \mathfrak{c}(\alpha), & D_d &:= \mathfrak{d}(\alpha). \end{aligned} \tag{13.1}$$

Proposition 13.2. *The Cayley transform with parameter α gives a one-to-one correspondence between the set of resolvent linear systems with $\alpha \in \Lambda$ and the set of discrete-time systems.*

Proof. This follows from Proposition 11.5. $\square$

Remark 13.3. The pseudoresolvent of a resolvent linear system is a resolvent if and only if -1 is not in the point spectrum of the state operator of its Cayley transform. A resolvent linear system is an operator node if and only if -1 is not in the point spectrum and not in the residual spectrum of the state operator of its Cayley transform. These conditions are very often hard, if not impossible to check. This was one of the reasons for introducing the class of resolvent linear systems instead of working with the class of operator nodes.

In the following remark we recall some facts about linear fractional transformations.

Remark 13.4. Let $\alpha \in \mathbb{C}$ be nonzero. The map $s \mapsto z = (\alpha - s)/(\alpha + s)$, with inverse $z \mapsto s = \alpha(1-z)/(1+z)$, maps $\mathbb{C} - \{-\alpha\}$ one-to-one onto $\mathbb{C} - \{-1\}$. The unit circle in the z-plane is the image of the line $\{s \in \mathbb{C} : \mathrm{Im}\alpha \ \mathrm{Im}s + \mathrm{Re}\alpha \ \mathrm{Re}s = 0\}$ in the s-plane. In particular, whenever α is real, the unit circle is the image of the imaginary axis. If $\alpha > 0$ the unit disc is the image of the right half-plane. For $\alpha > 0$ the map $M : H^2(\mathbb{C}_0^+, \mathscr{H}) \to H^2(\mathbb{D}, \mathscr{H})$ given by

$$(Mg)(z) = \frac{\sqrt{2\alpha}}{1+z} \ g\left(\alpha \ \frac{1-z}{1+z}\right), \tag{13.2}$$

is unitary with inverse

$$(M^{-1}f)(s) = \frac{\sqrt{2\alpha}}{\alpha+s} \ f\left(\frac{\alpha-s}{\alpha+s}\right). \tag{13.3}$$

The operator M is called the **Möbius operator**. With some abuse of notation we will denote the unitary operator $L^2(\mathbb{R}^+, \mathscr{H}) \to l^2(\mathbb{Z}^+, \mathscr{H})$ induced by M using the Z-transform and the Laplace transform by the same letter.

The above indicates that the above linear fractional transformation has nice mapping properties between the unit disc and the right half-plane. The situation is however drastically different when we look at arbitrary right half-planes, exponential regions and arbitrary discs centered at zero.

The line $\mathrm{Re}s = x$ in the s-plane is mapped onto the circle with center $-x/(\alpha + x)$ and radius $\alpha/(\alpha + x)$. Note that this circle contains the point -1, which is the image of the point at infinity. If α is chosen in the right half-plane $\mathrm{Re}s > x$, then the circle has zero in its interior.

An exponential region $\Lambda_E(a,b) := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq b, \quad |\mathrm{Im}\ s| \leq \mathrm{e}^{a\mathrm{Re}\ s}\}$ that contains α in its interior is mapped onto a subset of the above indicated disc (where x is replaced by b is the formulas) since it is contained in the right half-plane $\mathrm{Re}s \geq b$. Also here -1 is on the boundary of the image since it is the image of the point at infinity.

A disc in the z-plane centered at zero with radius strictly smaller than one can never be mapped onto an exponential region or a right half-plane. This follows since the indicated disc does not have -1 on its boundary whereas the images of exponential regions and right half-planes do. Actually, the image of the indicated disc is a bounded region in the s-plane.

We have the following relation between a resolvent linear system and the resolvent, the wave functions and the characteristic function of its Cayley transform.

Proposition 13.5. *Let Σ be a resolvent linear system with $\alpha \in \Lambda$ where $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α as defined in Definition 13.1. Denote the resolvent of Σ_d by $\mathfrak{A}$, its incoming wave function by $\mathfrak{B}$, its outgoing wave function by $\mathfrak{C}$ and its characteristic function by $\mathfrak{D}$. Let $s \in \Lambda$ and define $z := (\alpha - s)/\alpha + s)$. If $z \in 1/\rho(A_d)$, then*

$$\mathfrak{a}(s) = (1+z)\mathfrak{A}(z)\mathfrak{a}(\alpha), \qquad \mathfrak{b}(s) = \frac{1+z}{z\sqrt{2\alpha}}\,\mathfrak{B}(z), \tag{13.4}$$
$$\mathfrak{c}(s) = \frac{1+z}{\sqrt{2\alpha}}\,\mathfrak{C}(z), \qquad \mathfrak{d}(s) = \mathfrak{D}(z).$$

Proof. We first show that the equation

$$(I - zA_d)\mathfrak{a}(s) = (1+z)\mathfrak{a}(\alpha) \tag{13.5}$$

is equivalent to the functional equation (11.4). Substituting for A_d from (13.1) we see that (13.5) is equivalent to

$$(I - z\,[-I + 2\alpha\ \mathfrak{a}(\alpha)])\mathfrak{a}(s) = (1+z)\mathfrak{a}(\alpha).$$

Simplyfying the left-hand side shows that this is equivalent to

$$(1+z)\left[I - \frac{2\alpha}{1+z}\mathfrak{a}(\alpha)\right]\mathfrak{a}(s) = (1+z)\mathfrak{a}(\alpha).$$

Noting that $2\alpha z/(1+z) = \alpha - s$ and cancelling $1+z$ on both sides shows that this is equivalent to

$$[I - (\alpha - s)\mathfrak{a}(\alpha)]\,\mathfrak{a}(s) = \mathfrak{a}(\alpha),$$

and this is obviously equivalent to (11.4). Since $s \in \Lambda$ we have that (11.4) and therefore (13.5) holds. Since by assumption $z \in 1/\rho(A_d)$ it follows from (13.5) that $\mathfrak{a}(s) = (1+z)\mathfrak{A}(z)\mathfrak{a}(\alpha)$.

We now turn to the equation relating the incoming wave functions. We first note that the equation

$$(I - zA_d)\mathfrak{b}(s) = (1+z)\mathfrak{b}(\alpha)$$

is equivalent to the functional equation (11.5). The proof is almost exactly the same as the equivalence of (13.5) and (11.4) proven above and is left to the reader. The given equation for the incoming wave function follows easily. The argument for the equation relating the outgoing wave functions is entirely similar.

We prove the equation relating the characteristic functions. We have

$$\mathfrak{D}(z) = D_d + C_d \mathfrak{B}(z).$$

Using the relation between the incoming wave functions and substituting for D_d and C_d from (13.1) we obtain

$$\mathfrak{D}(z) = \mathfrak{d}(\alpha) + \frac{2\alpha z}{1+z} \mathfrak{c}(\alpha)\mathfrak{b}(s).$$

Since $2\alpha z/(1+z) = \alpha - s$ this is equivalent to

$$\mathfrak{D}(z) = \mathfrak{d}(\alpha) + (\alpha - s)\mathfrak{c}(\alpha)\mathfrak{b}(s).$$

Using the functional equation (11.7) it follows that $\mathfrak{D}(z) = \mathfrak{d}(s)$ as desired. □

Analogous to the discrete-time case for a distributional resolvent linear system we define the set of **stable input-output pairs**

$$\mathscr{V}(x_0) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} L^2(\mathbb{R}^+; \mathscr{U}) \\ L^2(\mathbb{R}^+; \mathscr{Y}) \end{bmatrix} : y \text{ satisfies } (11.13) \right\}.$$

The following theorem shows that, for a suitably chosen parameter α, there is a one-to-one relationship between the stable input-output pairs of a distributional resolvent linear system and those of its Cayley transform.

Proposition 13.6. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ and $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α. Then $[u; y] \in \mathscr{V}(x_0)$ if and only if $[Mu; My] \in \mathscr{V}_d(x_0)$.*

Proof. Let $[u; y] \in \mathscr{V}(x_0)$. From (11.13) we obtain

$$\hat{y}(s) := \mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s). \tag{13.6}$$

Since $\hat{u} \in H^2(\mathbb{C}_0^+, \mathscr{U})$ the above holds for $s \in \Lambda_E \cap \mathbb{C}_0^+$. Define $\Lambda_\alpha := \Lambda_E \cap \mathbb{C}_0^+$. Then Λ_α is an exponential region and it contains α. With $z := (\alpha - s)/\alpha + s)$ and using Proposition 13.5 we obtain from (13.6) that for $s \in \Lambda_\alpha$

$$\hat{y}(s) = \frac{1+z}{\sqrt{2\alpha}} \mathfrak{C}(z)x_0 + \mathfrak{D}(z)\hat{u}(s).$$

It follows that for z in a neighbourhood of zero (we use here that $\alpha \in \Lambda_\alpha$ is mapped to zero)

$$\frac{\sqrt{2\alpha}}{1+z}\hat{y}\left(\alpha\, \frac{1-z}{1+z}\right) = \mathfrak{C}(z)x_0 + \mathfrak{D}(z)\frac{\sqrt{2\alpha}}{1+z}\hat{u}\left(\alpha\, \frac{1-z}{1+z}\right). \tag{13.7}$$

But the right-hand side of (13.7) is the Z-transform of the output of Σ_d for initial state x_0 and input $M\hat{u}$ and the left-hand side equals $M\hat{y}$. Using the identity theorem for holomorphic functions we obtain that the output of Σ_d for initial state x_0 and input $M\hat{u}$ is $M\hat{y}$.

That $[u; y] \in \mathscr{V}_d(x_0)$ implies $[M^{-1}u; M^{-1}y] \in \mathscr{V}(x_0)$ follows in the same way. $\square$

Proposition 13.6 is the key connection between continuous-time systems and their Cayley transforms. It is this result that will allow us to translate many results from discrete-time to continuous-time.

Notes

The definition of the Cayley transform presented here is inspired by and generalizes the one in Staffans [89, Section 12.3]. The main idea presented in this chapter, Proposition 13.6, was first put forward in Opmeer and Curtain [70] for well-posed linear systems.

Chapter 14

Basic results

In this chapter we translate the main results obtained in part I of this thesis to continuous-time.

We first define the dual system of a resolvent linear system

Definition 14.1. The **dual system** of a resolvent linear system is the resolvent linear system on the set $\overline{\Lambda}$ of complex conjugates with pseudoresolvent $\mathfrak{a}_{\text{dual}}(s) := \mathfrak{a}(\bar{s})^*$, incoming wave function $\mathfrak{b}_{\text{dual}}(s) := \mathfrak{c}(\bar{s})^*$, outgoing wave function $\mathfrak{c}_{\text{dual}}(s) := \mathfrak{b}(\bar{s})^*$ and characteristic function $\mathfrak{d}_{\text{dual}}(s) := \mathfrak{d}(\bar{s})^*$.

It is easily seen that the Cayley transform with parameter α of the dual system is the dual system of the Cayley transform with parameter $\bar{\alpha}$.

Definition 14.2. A distributional resolvent linear system is called **approximately observable** if for input zero the output is only zero if the initial state is zero. It is called **approximately controllable** if its dual system is approximately observable and it is called **minimal** if it is both approximately controllable and approximately observable.

It is easily seen that approximate observability translates under the Cayley transform. It follows that approximate controllability and minimality do also.

14.1 Stability

Definition 14.3. A distributional resolvent linear system is called

- **exponentially stable** if for all $x_0 \in \mathscr{X}$ the state x defined by (11.13) with $u = 0$ is in $L^2(\mathbb{R}^+, \mathscr{X})$.

- **output stable** if for all $x_0 \in \mathscr{X}$ the output y defined by (11.13) with $u = 0$ is in $L^2(\mathbb{R}^+, \mathscr{Y})$.
- **input stable** if the dual system is output stable.
- **input-output stable** if for all $u \in L^2(\mathbb{R}^+, \mathscr{U})$ the output y defined by (11.13) with $x_0 = 0$ is in $L^2(\mathbb{R}^+, \mathscr{Y})$.

Proposition 14.4. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ and $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α. Then Σ is output stable if and only if Σ_d is, Σ is input stable if and only if Σ_d is, Σ is input-output stable if and only if Σ_d is.*

Proof. This follows easily using Proposition 13.6. □

Remark 14.5. Note that exponential stability almost never translates under the Cayley transform. It is easily seen that -1 is in the resolvent set of the state operator of the Cayley transform if and only if $\mathfrak{a}(\alpha)$ has a bounded inverse. It follows that if $\mathfrak{a}(\alpha)$ does not have a bounded inverse, then the Cayley transform is not exponentially stable.

Definition 14.6. Let Σ be an output stable distributional resolvent linear system. Let $y_w \in L^2(\mathbb{R}^+, \mathscr{Y})$ be the output for initial state $w \in \mathscr{X}$ and zero input. The **observability gramian** $L_C \in \mathcal{L}(\mathscr{X})$ is defined by $\langle L_C w, w\rangle = \|y_w\|^2_{L^2(\mathbb{R}^+, \mathscr{Y})}$.

Proposition 14.7. *Let Σ be an output stable distributional resolvent linear system and $\alpha \in \Lambda_E$ with $\alpha > 0$. The observability gramian of Σ equals the observability gramian of the Cayley transform with parameter α of Σ.*

Proof. This is easily seen using that the Möbius operator, which relates the output of the system and its Cayley transform, is unitary. □

Definition 14.8. Let Σ be an input stable distributional resolvent linear system. The **controllability gramian** $L_B \in \mathcal{L}(\mathscr{X})$ is defined as the observability gramian of the dual system.

Remark 14.9. In the special case that Σ is a system node one can show that the observability gramian is the minimal nonnegative self-adjoint solution of the observation Lyapunov equation

$$\langle Lw, Aw\rangle + \langle Aw, Lw\rangle = \|Cw\|^2, \quad w \in D(A).$$

A dual statement holds for the controllability gramian.

14.2 Stabilizability

Definition 14.10. An **admissible feedback pair** for a resolvent linear system is a pair $[\mathfrak{f}, \mathfrak{g}] : \Lambda \to \mathcal{L}(\mathscr{X} \times \mathscr{U}, \mathscr{U})$ that satisfies

$$\begin{aligned}
\mathfrak{f}(\beta) - \mathfrak{f}(\alpha) &= (\alpha - \beta)\mathfrak{f}(\alpha)\mathfrak{a}(\beta), \\
\mathfrak{g}(\beta) - \mathfrak{g}(\alpha) &= (\alpha - \beta)\mathfrak{f}(\alpha)\mathfrak{b}(\beta),
\end{aligned} \tag{14.1}$$

and such that $I - \mathfrak{g}(s)$ has a bounded inverse for some $s \in \Lambda$.

The **closed-loop system** of a resolvent linear system with an admissible feedback pair is the resolvent linear system

$$\mathfrak{a}^{\mathrm{cl}} := \mathfrak{a} + \mathfrak{b}(I - \mathfrak{g})^{-1}\mathfrak{f}, \quad \mathfrak{b}^{\mathrm{cl}} := \mathfrak{b}(I - \mathfrak{g})^{-1},$$

$$\mathfrak{c}^{\mathrm{cl}} := \begin{bmatrix} (I - \mathfrak{g})^{-1}\mathfrak{f} \\ \mathfrak{c} + \mathfrak{d}(I - \mathfrak{g})^{-1}\mathfrak{f} \end{bmatrix}, \quad \mathfrak{d}^{\mathrm{cl}} := \begin{bmatrix} (I - \mathfrak{g})^{-1} \\ \mathfrak{d}(I - \mathfrak{g})^{-1} \end{bmatrix}.$$

Remark 14.11. Note that Λ^{cl}, the domain of definition of the closed-loop system, consists of those $s \in \Lambda$ for which $I - \mathfrak{g}(s)$ has a bounded inverse.

Lemma 14.12. *Let Σ be a resolvent linear system and $[\mathfrak{f}, \mathfrak{g}]$ and admissible feedback pair. Assume that there exists an $\alpha \in \Lambda$ with $\alpha > 0$ such that $I - \mathfrak{g}(\alpha)$ has a bounded inverse. Denote the Cayley transform with parameter α of Σ by Σ_d. Then $[\sqrt{2\alpha}\mathfrak{f}(\alpha), \mathfrak{g}(\alpha)]$ is an admissible feedback pair for Σ_d. Moreover, the corresponding closed-loop system equals the Cayley transform with parameter α of the closed-loop system of Σ with the admissible feedback pair $[\mathfrak{f}, \mathfrak{g}]$.*

Proof. That $[\sqrt{2\alpha}\mathfrak{f}(\alpha), \mathfrak{g}(\alpha)]$ is an admissible feedback pair for Σ_d is immediate. The indicated equality of systems also follows immediately from the definitions. □

Definition 14.13. An **exponential region admissible feedback pair** for a distributional resolvent linear system is an admissible feedback pair $[\mathfrak{f}, \mathfrak{g}]$ such that $I - \mathfrak{g}(s)$ has a bounded inverse for all s in some exponential region and $(I - \mathfrak{g}(s))^{-1}$ is polynomially bounded in this exponential region.

A **half-plane admissible feedback pair** for an exponentially bounded distributional resolvent linear system is an admissible feedback pair $[\mathfrak{f}, \mathfrak{g}]$ such that $I - \mathfrak{g}(s)$ has a bounded inverse for all s in some right half-plane and $(I - \mathfrak{g}(s))^{-1}$ is polynomially bounded in this right half-plane.

Proposition 14.14. *The closed-loop system of a distributional resolvent linear system with an exponential region admissible feedback pair is a distributional resolvent linear system.*

The closed-loop system of an exponentially bounded distributional resolvent linear system with a half-plane admissible feedback pair is an exponentially bounded distributional resolvent linear system.

Proof. This is easily checked. □

Remark 14.15. If Σ is a system node and $[\mathfrak{f}, \mathfrak{g}]$ is a half-plane admissible feedback pair, then the closed-loop system will in general not be a system node. A concept of feedback under which the closed-loop system is again a system node is given in Staffans [89, Section 7.4]. This concept is very complicated and seems to be impossible to check.

Remark 14.16. If Σ is well-posed and $[\mathfrak{f}, \mathfrak{g}]$ is a half-plane admissible feedback pair that is uniformly bounded on the indicated right half-plane, then the closed-loop system is again a well-posed system.

Definition 14.17. A distributional resolvent linear system is called **output stabilizable** if there exists an exponential region admissible feedback pair such that the corresponding closed-loop system is output stable.

Remark 14.18. The other stabilizability notions introduced earlier for discrete-time systems also have (now hopefully obvious) counterparts in continuous-time.

Remark 14.19. It follows using Lemma 14.12 and Proposition 14.4 that, with the right choice of α, output stabilizability translates under the Cayley transform.

14.3 The LQ optimal control problem

Definition 14.20. We say that a distributional resolvent linear system satisfies the **finite cost condition** if for every $x_0 \in \mathscr{X}$ the set $\mathscr{V}(x_0)$ is nonempty.

Proposition 14.21. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ and $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α. Σ satisfies the finite cost condition if and only if Σ_d does. In this case the optimal cost operators are equal.*

Proof. This follows immediately from Proposition 13.6. □

Proposition 14.22. *Let Σ be a distributional resolvent linear system that satisfies the finite cost condition and let $\alpha > 0$ be such that $\alpha \in \Lambda_E$. Let Σ_d be the Cayley transform of Σ with parameter α. Denote the admissible feedback pair from Proposition 6.33 that gives the optimal closed-loop system of Σ_d*

by $[F_d, G_d]$*. Define* $[\mathfrak{f}, \mathfrak{g}]$ *by* $[\mathfrak{f}(\alpha), \mathfrak{g}(\alpha)] = [F_d/\sqrt{2\alpha}, G_d]$ *and extending to* Λ *using the functional equations* (14.1)*. Then* $[\mathfrak{f}, \mathfrak{g}]$ *is an exponential region admissible feedback pair for* Σ*. Denote the closed-loop system of* Σ *and this admissible feedback pair by* $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$*. Then the optimal closed-loop system of* Σ_d *is the Cayley transform with parameter* α *of* $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$*. The set* Λ *on which* $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$ *is defined and polynomially bounded contains* $\Lambda_E \cap \mathbb{C}_0^+$*.*

Proof. The equations (14.1) are satisfied by definition. Since $I - \mathfrak{g}(\alpha) = I - G_d$, it follows that $[\mathfrak{f}, \mathfrak{g}]$ is an admissible feedback pair. We will show that $I - \mathfrak{g}(s)$ has a bounded inverse on the exponential region $\Lambda_E^0 := \Lambda_E \cap \mathbb{C}_0^+$. Note that under the map $z = (\alpha - s)/(\alpha + s)$ the region Λ_E^0 is mapped into the connected component of $1/\rho(A_d) \cap \overline{\mathbb{D}}$ that contains zero. Define the function $\mathfrak{G}(z) := \mathfrak{g}(s)$, where $z = (\alpha - s)/(\alpha + s)$. It follows that $\mathfrak{G}(z) = G_d + F_d z(I - zA_d)^{-1}B_d$. We have that $I - \mathfrak{G}(z)$ is invertible on $\rho(A_Q)$, where A_Q is the state operator of the optimal closed-loop system of Σ_d. Lemma B.8 shows that the connected component of $1/\rho(A_Q) \cap \overline{\mathbb{D}}$ that contains zero contains the connected component of $1/\rho(A_d) \cap \overline{\mathbb{D}}$ that contains zero. It follows that it contains the image of Λ_E^0. From this we see that indeed $I - \mathfrak{g}(s)$ has a bounded inverse on the exponential region Λ_E^0. Lemma 14.12 shows that the optimal closed-loop system of Σ_d is the Cayley transform with parameter α of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. Since the optimal closed-loop system of Σ_d is input-output stable it follows that $I - \mathfrak{g}$ has an inverse that extends to a function in $H^\infty(\mathbb{C}_0^+, \mathscr{U})$. So $(I - \mathfrak{g})^{-1}$ is uniformly bounded on Λ_E^0. It follows that $[\mathfrak{f}, \mathfrak{g}]$ is an exponential region admissible feedback pair. The other statements follow easily from the above. $\square$

Definition 14.23. Let Σ be a distributional resolvent linear system that satisfies the finite cost condition and let $\alpha > 0$ be such that $\alpha \in \Lambda_E$. The exponential region admissible feedback pair $[\mathfrak{f}, \mathfrak{g}]$ from Lemma 14.22 is called the **α-optimal feedback pair**. The corresponding closed-loop system is called the **α-optimal closed-loop system**.

Note that it easily follows from the proof of Proposition 14.22 that an α-optimal feedback pair for an exponentially bounded distributional resolvent linear system is a half-plane admissible feedback pair.

Proposition 14.24. *Let* Σ *be a distributional resolvent linear system that satisfies the finite cost condition and let* $\alpha > 0$ *be such that* $\alpha \in \Lambda_E$*. Then the* α*-optimal closed-loop system is output stable and input-output stable.*

Proof. This follows from the corresponsing discrete-time results and Proposition 14.4 which shows that output stability and input-output stability translate under the Cayley transform. $\square$

Proposition 14.25. *Let Σ be a distributional resolvent linear that satisifies the finite cost condition. Denote the element of minimal norm in $\mathscr{V}(x_0)$ by $[u^{\min}_{x_0}; y^{\min}_{x_0}]$. Let $\alpha \in \Lambda_E$ with $\alpha > 0$ and let $[\mathfrak{f}, \mathfrak{g}]$ be an α-optimal feedback pair. Then $\hat{u}^{\min}_{x_0}(s) = (I - \mathfrak{g}(s))^{-1}\mathfrak{f}(s)x_0$ for $s \in \Lambda_E \cap \mathbb{C}^+_0$.*

Proof. Denote the Cayley transform of Σ with parameter α by Σ_d. Note that $\mathscr{V}(x_0)$ has a unique element of minimal norm by Proposition 13.6 and the fact that $\mathscr{V}_d(x_0)$ has a unique element of minimal norm $[u^d; y^d]$. Proposition 14.22 shows that the optimal closed-loop system of Σ_d is the Cayley transform with parameter α of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. The first component of the output for initial state x_0 and input zero of the optimal closed-loop system of Σ_d is easily seen to be the optimal input u^d. Since this optimal closed-loop system of Σ_d is the Cayley transform with parameter α of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$, it follows that the first component of the output for initial state x_0 and input zero of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$ equals $M^{-1}u_d$. But $M^{-1}u_d$ also equals $\hat{u}^{\min}_{x_0}$, the first component of the element of minimal norm in $\mathscr{V}(x_0)$. It follows that $\hat{u}^{\min}_{x_0}$ is the first component of the output for initial state x_0 and input zero of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. This gives the desired formula $\hat{u}^{\min}_{x_0}(s) = (I - \mathfrak{g}(s))^{-1}\mathfrak{f}(s)x_0$ for s in all exponential regions on which $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$ has the polynomial boundedness property. That $\Lambda_E \cap \mathbb{C}^+_0$ is such a region follows from Proposition 14.22. □

14.4 Coprime factorization

We will focus exclusively on strongly right-coprime factorizations. The other cases treated in Chapter 7 can be treated analogously.

Definition 14.26. Let $\mathsf{M} \in H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$, $\mathsf{N} \in H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_3))$.

The functions M and N are called **strongly right-coprime** if $[\mathsf{M}; \mathsf{N}]$ has a left-inverse in $H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{H}_2 \times \mathscr{H}_3, \mathscr{H}_1))$, i.e. if there exist $\tilde{\mathsf{X}} \in H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ and $\tilde{\mathsf{Y}} \in H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{H}_3, \mathscr{H}_1))$ such that

$$\tilde{\mathsf{X}}(s)\mathsf{M}(s) - \tilde{\mathsf{Y}}(s)\mathsf{N}(s) = I_{\mathscr{H}_1} \quad \forall s \in \mathbb{C}^+_0. \tag{14.2}$$

The functions $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ are called **right Bezout factors** for the pair (M, N).

Definition 14.27. Let $\mathsf{G} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$, with Λ_E an exponential region, be holomorphic and polynomially bounded.

G has a **right factorization** if there exist $\mathsf{M} \in H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{U}))$ and $\mathsf{N} \in H^\infty(\mathbb{C}^+_0; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ such that $\mathsf{M}(s)$ is invertible for s in some exponential region, M^{-1} is polynomially bounded on an exponential region, and $\mathsf{G}(s) = \mathsf{N}(s)\mathsf{M}(s)^{-1}$ for s in some exponential region. The factor $[\mathsf{M}; \mathsf{N}]$ provides a **strongly right-coprime factorization** if M and N are strongly right-coprime. The right factor $[\mathsf{M};\mathsf{N}]$ is called **normalized** when multiplication with $[\mathsf{M};\mathsf{N}]$ is an isometry from $H^2(\mathbb{C}_0^+,\mathscr{U})$ into $H^2(\mathbb{C}_0^+,\mathscr{U}\times\mathscr{Y})$.

Note that characteristic functions of distributional resolvent linear systems belong to the class of functions to which the above definitions of factorization apply. The following result shows under which conditions the characteristic function of a distributional resolvent linear system has a strongly right-coprime factorization.

Proposition 14.28. *Assume that the distributional resolvent linear system Σ and its dual system both satisfy the finite cost condition and let $\alpha > 0$ be such that $\alpha \in \Lambda_E$. Then the characteristic function of the α-optimal closed-loop system has a holomorphic extension to $\mathbb{C}_0^+$ and this extension provides a normalized strongly right-coprime factorization of the characteristic function of Σ.*

Proof. Let $\alpha \in \Lambda_E$ and $\alpha > 0$. Denote the Cayley transform of Σ with parameter α by Σ_d. It follows from Proposition 7.20 that the transfer function of Σ_d has a normalized strongly right-coprime factor $[\mathsf{M}_d;\mathsf{N}_d]$ with Bezout pair $[\tilde{\mathsf{X}}_d,\tilde{\mathsf{Y}}_d]$. The function $[\mathsf{M}_d;\mathsf{N}_d]$ is the transfer function of the optimal closed-loop system of Σ_d. Define $[\mathsf{M}(s);\mathsf{N}(s)] := [\mathsf{M}_d(z);\mathsf{N}_d(z)]$, $[\tilde{\mathsf{X}}(s),\tilde{\mathsf{Y}}(s)] := [\tilde{\mathsf{X}}_d(z),\tilde{\mathsf{Y}}_d(z)]$, where z and s are related by $z = (\alpha - s)/(\alpha + s)$. It easily follows that the indicated functions are in H^∞ and that (14.2) holds. It is also easily seen that M^{-1} equals $(I-\mathfrak{g}(s))^{-1}\mathfrak{f}(s)$ on $\Lambda_E\cap\mathbb{C}_0^+$, where $[\mathfrak{f},\mathfrak{g}]$ is the α-optimal feedback pair. It follows that M is invertible on $\Lambda_E\cap\mathbb{C}_0^+$ and that M^{-1} is polynomially bounded on $\Lambda_E\cap\mathbb{C}_0^+$. The equality $\mathfrak{d}(s) = \mathsf{N}(s)\mathsf{M}(s)^{-1}$ on $\Lambda_E\cap\mathbb{C}_0^+$ also follows. □

Remark 14.29. In Section 14.7 we will see that any function that has a strongly right-coprime factorization coincides on some exponential region with the characteristic function of some distributional resolvent linear system that satisfies the finite cost condition and whose dual system satisfies the finite cost condition.

Remark 14.30. It is easily shown that, as in the discrete-time case, the existence of a strongly left-coprime factorization, the existence of a normalized strongly left-coprime factorization, the existence of a strongly right-coprime factorization, the existence of a normalized strongly right-coprime factorization, the existence of a doubly coprime factorization and the existence of a normalized doubly coprime factorization are all equivalent.

Remark 14.31. We consider the special case that G is holomorphic and polynomially bounded on a right half-plane and not just on an exponential region.

The Bezout equation gives $\tilde{\mathsf{X}} - \tilde{\mathsf{Y}}(s)\mathsf{G}(s) = \mathsf{M}(s)^{-1}$ for s in an exponential region. Using that the left-hand side is a holomorphic and polynomially bounded function on a right half-plane it is not difficult to see that $\mathsf{M}(s)$ is invertible for s in a right half-plane and that the inverse function is polynomially bounded on a right half-plane.

It follows similarly that if G is holomorphic and uniformly bounded on a right half-plane, then so is M^{-1}.

14.5 The gap metric

The gap between distributional resolvent linear systems is defined similarly as for discrete-time systems using the gap metric on subspaces of a given Hilbert space (see Definition 9.1).

Definition 14.32. Let Σ_i $(i = 1, 2)$ be distributional resolvent linear systems with the same input and output spaces. The **gap** $\delta(\Sigma_1, \Sigma_2)$ is defined to be $\delta(\mathscr{V}_1(0), \mathscr{V}_2(0))$.

Proposition 14.33. *Let Σ_i $(i = 1, 2)$ be distributional resolvent linear systems with the same input and output spaces, $\alpha \in \Lambda_E$ for both systems, and $\alpha > 0$. Let Σ_d^i be the respective Cayley transforms with parameter α. Then $\delta(\Sigma_1, \Sigma_2) = \delta(\Sigma_d^1, \Sigma_d^2)$.*

Proof. This follows easily using that $\mathscr{V}_i(0)$ is isometrically isomorphic to $\mathscr{V}_d^i(0)$ $(i = 1, 2)$ under the Möbius operator by Proposition 13.6. □

14.6 Stabilization

Definition 14.34. Let $\mathsf{G} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ with Λ_E an exponential region be holomorphic and polynomially bounded. We say that K is an **admissible feedback function** for G if $\mathsf{K} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ is holomorphic and polynomially bounded and $I - \mathsf{KG}$ has a bounded inverse on an exponential region.

Definition 14.35. An admissible feedback function K for G is called **stabilizing** if

$$\begin{bmatrix} (I - \mathsf{KG})^{-1} & \mathsf{K}(I - \mathsf{GK})^{-1} \\ \mathsf{G}(I - \mathsf{KG})^{-1} & (I - \mathsf{GK})^{-1} \end{bmatrix}$$

extends to a function in $H^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$.

We discuss the continuous-time analogue of the discrete-time robust right factor stabilizing feedback function from Definition 8.13. We only do this for the finite-dimensional case as this is all we need in the sequel and a full discussion of the general case would take us too far afield (see Curtain [11] for this case). Assume that $\mathscr{U}, \mathscr{X}$ and $\mathscr{Y}$ are finite-dimensional and the system Σ is described by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t). \tag{14.3}$$

Further assume that this system has solutions Q and P to its (continuous-time) control and filter algebraic Riccati equation, respectively. Here the (continuous-time) control algebraic Riccati equation is

$$A^*Q + QA + C^*C = (QB + C^*D)(I + D^*D)^{-1}(B^*Q + D^*C),$$

and the (continuous-time) filter algebraic Riccati equation is

$$AP + PA^* + BB^* = (PC^* + BD^*)(I + DD^*)^{-1}(CP + DB^*).$$

Let $\varepsilon < 1/\sqrt{1+\mu_1^2}$, where μ_1 is the largest LQG-characteristic value of Σ. Define the controller by its system operator

$$\left[\begin{array}{c|c} A + BF + WPC^*(C + DF) & WPC^* \\ \hline B^*Q & -D^* \end{array}\right],$$

where $F := -(I + D^*D)^{-1}(D^*C + B^*Q)$ and $W := ((1 - \varepsilon^2)I + \varepsilon^2 PQ)^{-1}$. Denote the transfer function of the controller by K and the transfer function of Σ by G. Then K is an admissible feedback function and it is stabilizing for all G_Δ with $\delta_g(\mathsf{G}, \mathsf{G}_\Delta) \leq \varepsilon$. The above follows from McFarlane and Glover [54].

14.7 Balanced realizations

Definition 14.36. An input and output stable distributional resolvent linear system is called **Lyapunov-balanced** if its controllability and observability gramian are equal.

Proposition 14.37. *Any function in $H^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ has a minimal Lyapunov-balanced realization. Minimal Lyapunov-balanced realizations are unique up to a unitary similarity transformation in the state space. The pseudoresolvent is the resolvent of the generator of a strongly continuous contraction semigroup. Both the minimal Lyapunov-balanced realization and its dual system are strongly stable.*

Proof. See Theorem 9.5.6 of Staffans [89]. □

LQG-characteristic values are defined as in discrete-time.

Definition 14.38. Let Σ be a distributional resolvent linear system that satisfies the finite cost condition and whose dual system satisfies the finite cost condition. Denote the optimal cost operator by $Q^{\min}$ and the optimal cost operator of the dual system by $P^{\min}$. The square roots of the points in the spectrum of $P^{\min}Q^{\min}$, with the exception of zero, are called the **LQG-characteristic values** of Σ.

Note that, using the Cayley transform and the corresponding discrete-time result, it is easily seen that two distributional resolvent linear systems whose characteristic functions coincide on an exponential region have the same LQG-characteristic values.

Definition 14.39. A distributional resolvent linear system is called **LQG-balanced** if it and its dual system satisfy the finite cost condition and the optimal cost operator of the system and of its dual system are equal. It is called **compact LQG-balanced** if it is LQG-balanced and the optimal cost operator is compact.

Proposition 14.40. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ with $\alpha > 0$. Σ is LQG-balanced if and only if its Cayley transform with parameter α is. It is compact LQG-balanced if and only if its Cayley transform with parameter α is.*

Proof. This easily follows using Proposition 14.21. □

Proposition 14.41. *Let $\mathsf{G} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$, with Λ_E an exponential region, be holomorphic and polynomially bounded. Assume that G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Then there exists a minimal LQG-balanced distributional resolvent linear system whose characteristic function coincides with G on some exponential region. Such a system is unique up to a unitary transformation in the state space.*

Proof. By Proposition 14.37 $[\mathsf{M}; \mathsf{N}]$ has a minimal Lyapunov-balanced realization $\check{\Sigma}_{\text{LYAP}}$. Note that since a minimal Lyapunov-balanced realization is strongly stable the set $\check{\Lambda}$ of $\check{\Sigma}_{\text{LYAP}}$ contains the whole right half-plane. Since $\check{\mathfrak{d}}_1$, the first component of the characteristic function of $\check{\Sigma}_{\text{LYAP}}$, coincides with M on the right half-plane we have that $\check{\mathfrak{d}}_1(s)$ has a bounded inverse for s in some exponential region and that $\check{\mathfrak{d}}_1^{-1}$ is polynomially bounded on some exponential region. Define the distributional resolvent linear system Σ by

$$\left[\begin{array}{c|c} \mathfrak{a} & \mathfrak{b} \\ \hline \mathfrak{c} & \mathfrak{d} \end{array}\right] := \left[\begin{array}{c|c} \check{\mathfrak{a}} - \check{\mathfrak{b}}\check{\mathfrak{d}}_1^{-1}\check{\mathfrak{c}}_1 & \check{\mathfrak{b}}\check{\mathfrak{d}}_1^{-1} \\ \hline \check{\mathfrak{c}}_2 - \check{\mathfrak{d}}_2\check{\mathfrak{d}}_1^{-1}\check{\mathfrak{c}}_1 & \check{\mathfrak{d}}_2\check{\mathfrak{d}}_1^{-1} \end{array}\right].$$

It follows from the above that this is indeed a distributional resolvent linear system. Apply the similarity transformation $(I - L^2)^{1/4}$ to Σ to obtain a distributional resolvent linear system Σ_{LQG}. Here L is the gramian of the minimal Lyapunov-balanced realization $\check{\Sigma}_{\mathrm{LYAP}}$. Now let $\alpha > 0$ be in the exponential region of all the above constructed systems. Applying the Cayley transform with parameter α to the above systems and comparing with the proof of Proposition 10.34 shows that the Cayley transform of Σ_{LQG} is minimal and LQG-balanced. It follows using Proposition 14.40 that Σ_{LQG} is. Its characteristic function $\check{\mathfrak{d}}_2\check{\mathfrak{d}}_1^{-1}$ equals $\mathsf{N}\mathsf{M}^{-1}$ on an exponential region and this equals G. The desired result follows. $\square$

Remark 14.42. It is easily seen from the proof of Proposition 14.41 and remark 14.31 that if G is holomorphic and polynomially bounded on a right half-plane instead of only on an exponential region, then a minimal LQG-balanced realization is an exponentially bounded distributional resolvent linear system. Similarly it follows that a minimal LQG-balanced realization is well-posed if G is holomorphic and uniformly bounded on a right half-plane.

Definition 14.43. Given a compact LQG-balanced distributional resolvent linear system Σ, let (w_i) be an ordered sequence of eigenvectors of the optimal cost operator $Q^{\min}$ (the ordering is such that the corresponding eigenvalues μ_i form a nonincreasing sequence). Let $\alpha \in \Lambda_E$ with $\alpha > 0$. Let $n \in \mathbb{Z}^+$ be such that $\mu_n > \mu_{n+1}$. The α-**truncated LQG-balanced realization** of dimension n with respect to the sequence of eigenvectors (w_i) is defined as the restriction/projection of the operator

$$\begin{bmatrix} \mathfrak{a}(\alpha) & \mathfrak{b}(\alpha) \\ \mathfrak{c}(\alpha) & \mathfrak{d}(\alpha) \end{bmatrix}$$

onto $\mathscr{X}_n := \{w_i : i = 1, \ldots, n\}$.

The following result shows that the Cayley transform and α-LQG-balanced truncation commute.

Proposition 14.44. *Let Σ be a compact LQG-balanced distributional resolvent linear system and $\alpha \in \Lambda_E$ with $\alpha > 0$. The Cayley transform with parameter α of the α-truncated LQG-balanced realization of dimension n equals the truncated LQG-balanced realization of the Cayley transform with parameter α of Σ.*

Proof. This follows trivially from the definitions. $\square$

From the above we immediately obtain the following analogues of Propositions 10.45, 10.46 and 10.47.

Proposition 14.45. *Let Σ be a compact LQG-balanced distributional resolvent linear system with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Let $\alpha \in \Lambda_E$ with $\alpha > 0$. Denote by Σ_n the α-truncated LQG-balanced system of dimension n of Σ. Then we have*

$$\vec{\delta}_g(\Sigma, \Sigma_n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}.$$

Proposition 14.46. *Let Σ be a compact LQG-balanced distributional resolvent linear system with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Let $\alpha \in \Lambda_E$ with $\alpha > 0$. Denote by Σ_n the α-truncated LQG-balanced system of dimension n of Σ. Then there exists a $N \in \mathbb{Z}^+$ such that*

$$2 \sum_{i=N+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < \frac{1}{\sqrt{1+\mu_1^2}}. \tag{14.4}$$

For $n \geq N$ we have

$$\delta_g(\Sigma, \Sigma_n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}. \tag{14.5}$$

Remark 14.47. The condition that the LQG-characteristic values are summable is satisfied by many systems, but there are also many systems for which it is not satisfied. In Chapter 15 we consider a typical case in which the condition is satisfied. In that example it is crucial that the damping parameter β is positive, for $\beta = 0$ the LQG-characteristic values are not summable.

Remark 14.48. It follows from Proposition 14.46 and the results collected in Section 14.6 that the robust right factor stabilizing feedback function reviewed in Section 14.6 designed for the α-truncated LQG-balanced system of dimension n for n large enough stabilizes Σ. In fact, it also stabilizes all systems close to Σ in the gap metric and by choosing n large enough the robustness radius converges to the optimal robustness radius $1/\sqrt{1+\mu_1^2}$.

Notes

The definition of admissible feedback pair as given in this chapter are taken from Opmeer [66]. The solution of the linear quadratic optimal control problem as given here were also presented earlier in [66]. The results on coprime factorization are slight modifications of those in Curtain and Opmeer [16]. The results on LQG-balanced realizations are from Opmeer [68]. Earlier results on Lyapunov-balanced realizations in continuous-time for infinite-dimensional systems are among other Glover, Curtain and Partington [35] and Ober and Montgomery-Smith [62]. To obtain a complete theory of admissible feedback functions one needs to consider **controllers with internal loop** as in Weiss and Curtain [97] and Curtain, Weiss and Weiss [17].

Chapter 15

An example

We consider the robust stabilization of a beam. In Section 15.2 we show that the conditions under which we have convergence of LQG-balanced truncations are satisfied for this example. Section 15.3 contains numerical results for this example.

15.1 The model

The system we consider is a one-dimensional Euler-Bernoulli beam with Voigt-damping and with free ends. The measurements are the displacement and the angle of rotation of the middle of the beam. As actuators we choose a force and a moment at the middle of the beam.

We obtain the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + \beta \frac{\partial^5 w}{\partial x^4 \partial t} + \alpha \frac{\partial^4 w}{\partial x^4} = \frac{u_1 \delta - u_2 \delta'}{\rho a},$$

$$\alpha \frac{\partial^2 w}{\partial x^2}(-1,t) + \beta \frac{\partial^3 w}{\partial x^2 \partial t}(-1,t) = 0, \quad \alpha \frac{\partial^2 w}{\partial x^2}(1,t) + \beta \frac{\partial^3 w}{\partial x^2 \partial t}(1,t) = 0,$$

$$\alpha \frac{\partial^3 w}{\partial x^3}(-1,t) + \beta \frac{\partial^4 w}{\partial x^3 \partial t}(-1,t) = 0, \quad \alpha \frac{\partial^3 w}{\partial x^3}(1,t) + \beta \frac{\partial^4 w}{\partial x^3 \partial t}(1,t) = 0,$$

$$y(t) = \begin{bmatrix} w(0,t) \\ \frac{\partial w}{\partial x}(0,t) \end{bmatrix},$$

where $w(t,x)$ is the displacement of the beam at position $x \in (-1,1)$ at time t, $u_1(t)$ is the force applied and $u_2(t)$ the moment applied to the middle ($x = 0$) of the beam, $y(t)$ holds the measurements, ρ, a, α and β are (positive) physical parameters and δ is the Dirac delta distribution and δ' is its distributional derivative. A derivation of this model from physical considerations is given in Bontsema [6].

We put the above partial differential equation in an abstract operator-theoretic framework. We note that the spaces H^s used in this section are Sobolev spaces. Define the operator $L : D(L) \subset L^2(-1,1) \to L^2(-1,1)$ as

$$L := \frac{d^4}{d\xi^4},$$

$$D(L) := \left\{ w \in H^4(-1,1) : \frac{d^2 w}{d\xi^2}(-1) = \frac{d^2 w}{d\xi^2}(1) = \frac{d^3 w}{d\xi^3}(-1) = \frac{d^3 w}{d\xi^3}(1) = 0 \right\}.$$

It is elementary to show that L is a densely defined nonnegative operator. We define the spaces $\mathscr{X} := D(L^{1/2}) \times L^2(-1,1)$, $\mathscr{U} := \mathbb{C}^2$, $\mathscr{Y} = \mathbb{C}^2$ and the following operators. The operator $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ is defined by

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} x_2 \\ -\alpha L \left(x_1 + \frac{\beta}{\alpha} x_2 \right) \end{bmatrix},$$

$$D(A) := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathscr{X} : x_2 \in D(L^{1/2}), x_1 + \frac{\beta}{\alpha} x_2 \in D(L) \right\}.$$

The operator B is defined through its adjoint $B^* : D(A) \subset \mathscr{X} \to \mathscr{U}$

$$B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \frac{1}{\rho a} \begin{bmatrix} x_2(0) \\ x_2'(0) \end{bmatrix}.$$

The operator $C : D(A) \subset \mathscr{X} \to \mathscr{Y}$ is defined by

$$C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \frac{1}{\rho a} \begin{bmatrix} x_1(0) \\ x_1'(0) \end{bmatrix}.$$

The feedthrough operator D is taken equal to zero. In Bontsema [6] it is shown that this is indeed a representation of the partial differential equation obtained earlier.

15.2 Theoretical results

The next proposition shows that our beam system has a compact LQG-balanced realization and the error bound (14.5) holds.

Proposition 15.1. *The system considered satisfies all the assumptions of Proposition 14.46.*

Proof. If follows from Bontsema [6, Lemma 2.13] that the system under consideration is a well-posed linear system, which implies that it is an exponentially bounded distributional resolvent linear system. The input and output space are both two-dimensional. It remains to show that the finite cost condition and the dual finite cost condition are satisfied, and that the LQG-characteristic values form a summable sequence.

A spectral decomposition of the main operator A as performed in Bontsema [6] shows that A has α/β in its continuous spectrum, the other spectral points are eigenvalues and these are either located on a circle with center $-\alpha/\beta$ and radius α/β or on the negative part of the real line (see figure 15.1).

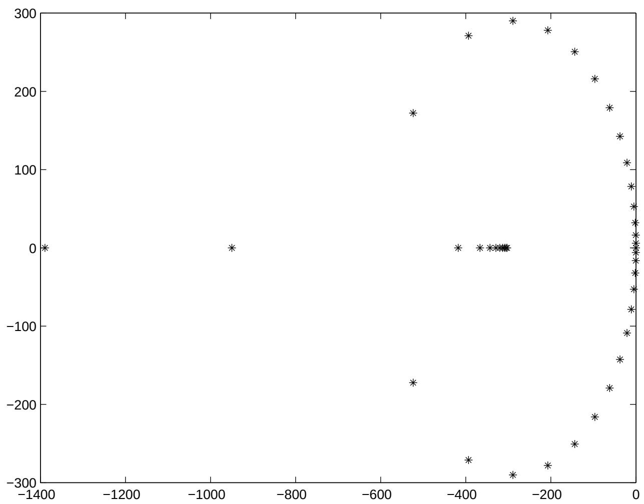

Figure 15.1: Eigenvalues of the A operator of the beam

All spectral points are in the open left half-plane, except for a quadruple eigenvalue at zero. From the above spectral decomposition one can conclude that the operator A generates an analytic semigroup (this follows as in the appendix of Chen and Triggiani [8]). It is shown in Bontsema [6] that the control operator B is unbounded, but not maximally unbounded and that the observation operator C is bounded. Using the spectral decomposition of A we can split the system into a stable part and an unstable part as in Curtain and Zwart [18, Section 5.2]. Since the unstable part is controllable we conclude that the system is exponentially stabilizable, which implies that it satisfies the finite cost condition. That the system satisfies the dual finite cost condition follows similarly. From the fact that the semigroup is analytic and the control operator not maximally unbounded we conclude that the optimal state feedback is bounded (see Lasiecka and Triggiani [50]). From this it follows that the optimal closed-loop system has an analytic semigroup, a control operator that is not maximally unbounded and a bounded observation operator. We invoke Curtain and Sasane [9, Theorem 6] to show that the Hankel operator of this closed-loop system is nuclear. This shows that the LQG-characteristic values of the original system are summable (the relation between the Hankel singular values of the closed-loop system and the LQG-characteristic values of the original system are the same as given in Corollary 10.32 for discrete-time systems). □

15.3 Numerical results

For the purpose of numerical investigations we choose the physical parameters in accordance with De Silva [21]. These parameter values are

$$\rho a = 47.2, \quad \alpha = 1.129, \quad \beta = 3.89 \times 10^{-4}.$$

We analyze different approximation techniques using LQG-singular values and Bode diagrams. We only show the Bode diagrams from the first input to the first output, the response from the second input to the second output is similar and the other two responses are zero. Also, we only show the Bode magnitude diagram.

15.3.1 Modal approximation

It is relatively easy to obtain a modal approximation of our model based on the eigenvectors of the fourth derivative operator with boundary conditions as above. For more complicated models of physical systems it will not be easy (or even possible) to obtain a modal approximation. In figure 15.2 the solid line is a Bode-diagram of the 30 dimensional modal approximation. Table 15.1 shows the largest ten LQG-characteristic values for modal approximations.

If we construct the controller mentioned in Remark 14.48 based on a 4 mode approximation it stabilizes the 30 mode approximation, for a design based on a lower order approximation this is no longer the case. Since the unstable subspace is four-dimensional this is of course not very surprising.

15.3.2 Finite-difference approximation

We have obtained finite-difference approximations of our model. In figure 15.2 the dashed line is a 30 dimensional finite-difference approximation and in figure 15.3 the dashed line is a 6 dimensional finite-difference approximation.

Table 15.1: largest 10 LQG-characteristic values for modal approximations

6 modes	10 modes	14 modes	22 modes	30 modes
2.4142	2.4134	2.4134	2.4134	2.4134
2.4135	2.4123	2.4116	2.4111	2.4109
0.4143	0.4146	0.4147	0.4147	0.4148
0.4142	0.4144	0.4144	0.4144	0.4144
0.1071	0.1071	0.1071	0.1071	0.1071
0.1068	0.1068	0.1068	0.1068	0.1068
-	0.1010	0.1010	0.1010	0.1010
-	0.1004	0.1004	0.1004	0.1004
-	0.0009	0.0104	0.0104	0.0104
-	0.0009	0.0102	0.0102	0.0102

From this and the ‘intermediate’ Bode diagrams not shown it can be seen that the resonance peaks are at too low a frequency and this error converges slowly to zero. The 6 dimensional finite-difference approximation also has an incorrect slope for low frequencies. In table 15.2 the LQG-characteristic values for finite difference approximations are given.

Table 15.2: largest 10 LQG-characteristic values for finite difference approximations

6 dim f-d	10 dim f-d	14 dim f-d	22 dim f-d	30 dim f-d
2.4142	2.4129	2.4129	2.4131	2.4132
0.9964	2.4125	2.4122	2.4116	2.4113
0.9799	0.4146	0.4146	0.4147	0.4147
0.6408	0.4144	0.4145	0.4144	0.4144
0.6394	0.3189	0.2286	0.1711	0.1503
0.4142	0.3183	0.2282	0.1708	0.1500
-	0.1133	0.1255	0.1225	0.1183
-	0.1129	0.1250	0.1219	0.1177
-	0.0089	0.0073	0.0109	0.0114
-	0.0088	0.0072	0.0108	0.0112

We can see here also that the 6 dimensional finite-difference approximation is not good and that convergence is slower then in the modal approximation. However, from the 10 dimensional finite-difference approximation

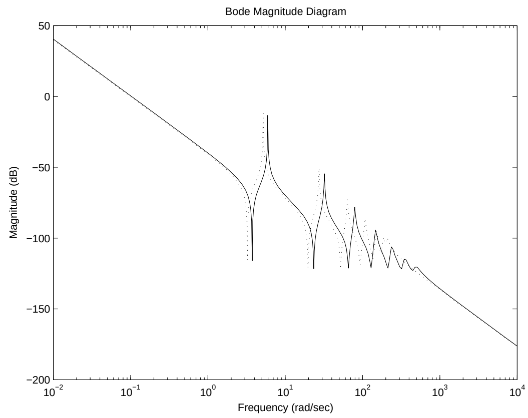

Figure 15.2: 30 mode approximation (-) and 30 dimensional finite difference approximation (:)

on the first four LQG-characteristic values are fairly accurate and the other LQG-characteristic values seem to converge to their correct values. It turns out that the controller mentioned in Remark 14.48 when based on a 6 dimensional finite difference approximation is not stabilizing and that the one based on a 10 dimensional finite difference approximation is. We conclude that controller-design using finite-difference approximations leads to a controller of more than 6 dimensions.

15.3.3 LQG-balanced approximation

We have shown that our model has a compact LQG-balanced realization. Computing this realization exactly is however impossible. The method of LQG-balancing can however be used to obtain good low-order approximations of good high-order approximations. We compute a LQG-balanced realization for the 30 dimensional finite-difference approximation of the beam (this is finite-dimensional LQG-balancing, so it can be done using an algorithm from finite-dimensional theory). The Bode diagram of a 14 dimensional LQG-balanced truncation of the 30 dimensional finite-difference approximation of the beam is shown in figure 15.4 and that of a 4 dimensional LQG-balanced truncation of the 30 dimensional finite-difference approximation of the beam is shown in figure 15.5.

As can be seen the approximation is about as good as can be expected given the order of the approximation. The controller mentioned in Remark 14.48 when based on a 4 dimensional LQG-balanced truncation of a 30 dimensional finite difference approximation stabilizes the 30 dimensional modal approximation. Thus it can be expected that it will stabilize the beam.

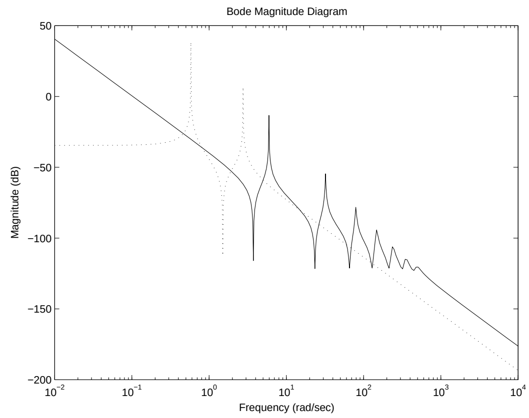

Figure 15.3: 30 mode approximation (-) and 6 dimensional finite difference approximation (:)

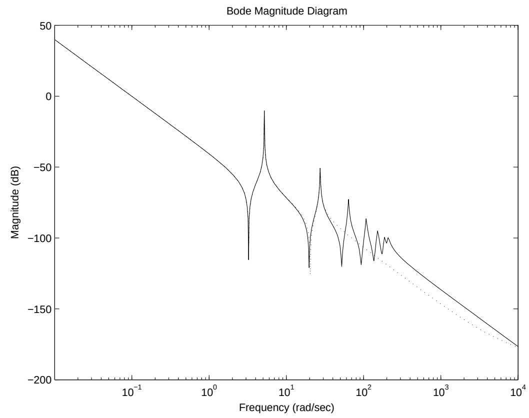

Figure 15.4: 30 dimensional finite difference approximation (-) and its 14 dimensional LQG-balanced truncation

15.4 Conclusion

We showed that a finite difference approximation followed by a LQG-balanced truncation gives a stabilizing 4 dimensional controller. This is as good as can be obtained using a modal approximation. A stabilizing controller based only on a finite difference approximation must have more then 6 states. This shows that the combination of a finite difference approximation and LQG-balancing is better than a finite difference approximation alone.

We note that it is crucial for the analysis presented here that the damping parameter β is positive. If $\beta = 0$, then one can show that the LQG-characteristic values do not form a summable sequence. Controllers designed

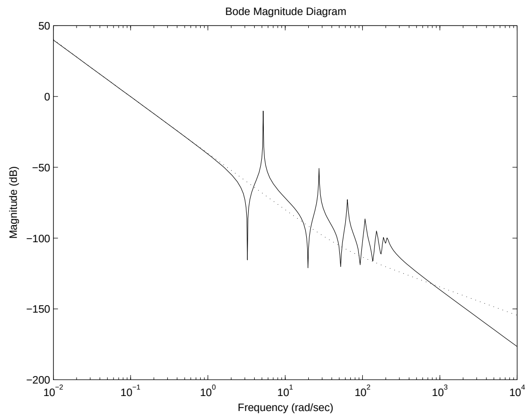

Figure 15.5: 30 dimensional finite difference approximation (-) and its 4 dimensional LQG-balanced truncation (:)

based on approximations in this case do not give a satisfactory performance in numerical simulations.

Notes

The beam model presented here was thoroughly analyzed in Bontsema [6]. Proposition 15.1 and the numerical results presented here were reported earlier in Opmeer, Wubs and Van Mourik [73]. The numerical results are based on Van Mourik [92], where many more numerical results concerning our example can be found. Some more numerical work on LQG-balanced realizations can be found in Evans [24].

Chapter 16

Concluding remarks

By collecting various results from the previous chapters we obtain the following theorem.

Theorem 16.1. *Let H_i $(i = 1, 2, 3, 4)$ be*

1. *The set of functions $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ that are holomorphic and with $0 \in D(\mathsf{G})$.*

2. *The set of functions $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ defined on an exponential region that are holomorphic and bounded in norm by a polynomial.*

3. *The set of functions $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ defined on a right half-plane that are holomorphic and bounded in norm by a polynomial.*

4. *The set of functions $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ defined on a right half-plane that are holomorphic and uniformly bounded in norm.*

Let S_i $(i = 1, 2, 3, 4)$ be

1. *The class of discrete-time systems.*

2. *The class of distributional resolvent linear systems.*

3. *The class of exponentially bounded distributional resolvent linear systems.*

4. *The class of well-posed linear systems.*

Let $\mathsf{G} \in H_i$. Then the following are equivalent.

- *G has a strongly right-coprime factorization.*

- *G has normalized strongly right-coprime factorization.*

- G *has a strongly left-coprime factorization.*
- G *has normalized strongly left-coprime factorization.*
- G *has a doubly coprime factorization.*
- G *has a normalized doubly coprime factorization.*
- G *has a input and output stabilizable realization in* S_i*.*
- G *has a minimal input and output stabilizable realization in* S_i*.*
- G *has a realization in* S_i *that satisfies the finite cost condition and whose dual system also satisfies the finite cost condition.*
- G *has a minimal realization in* S_i *that satisfies the finite cost condition and whose dual system also satisfies the finite cost condition.*
- G *has a LQG-balanced realization in* S_i*.*
- G *has a minimal LQG-balanced realization in* S_i*.*

In the case that $i = 1$ *(the discrete-time case) the above is also equivalent with*

- *There exists a stabilizing admissible feedback function for* G*.*
- G *has a realization in* S_i *that has bounded nonnegative self-adjoint solutions to both its control and its filter algebraic Riccati equation.*
- G *has a minimal realization in* S_i *that has bounded nonnegative self-adjoint solutions to both its control and its filter algebraic Riccati equation.*

In the other cases ($i = 2, 3, 4$*) this is also true, but one should use controllers with internal loop and a more general form of the algebraic Riccati equations than the usual continuous-time ones.*

In the case that G has a compact LQG-balanced realization, the input and output space are finite-dimensional, and the LQG characteristic values are summable we obtained that LQG-balanced truncations converge in the gap metric, or equivalently, we have convergence of normalized strongly right-coprime factors in H^∞. Using a controller design that is robust with respect to right factor perturbations in this case the plant with transfer function G can be stabilized by a finite-dimensional controller. The performance of this controller approaches the performance of the corresponding infinite-dimensional controller as the state space dimension of the approximation goes to infinity.

Appendix A

Hardy spaces

In this appendix we give some basic definitions and results on Hardy spaces that are needed in this thesis.

Definition A.1. Let $\mathscr{H}$ be a Hilbert space. The Hardy space $H^2(\mathbb{D};\mathscr{H})$ is defined as follows: $\mathsf{F} \in H^2(\mathbb{D};\mathscr{H})$ if $\mathsf{F} : \mathbb{D} \to \mathscr{H}$ is holomorphic and

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|\mathsf{F}(re^{i\theta})\|^2_{\mathscr{H}}\, d\theta < \infty. \tag{A.1}$$

Lemma A.2. *1. The space $H^2(\mathbb{D};\mathscr{H})$ is a Hilbert space with as norm the square root of the expression on the left-hand side of* (A.1)*.*

2. There is an isometric isomorphism between $l^2(\mathbb{Z}^+;\mathscr{H})$ and $H^2(\mathbb{D};\mathscr{H})$ given by $(a_n)_{n\in\mathbb{Z}^+} \mapsto \sum_{n=0}^{\infty} a_n z^n$.

Definition A.3. Let $\mathscr{B}$ be a Banach space. The Hardy space $H^\infty(\mathbb{D};\mathscr{B})$ is defined as follows: $\mathsf{F} \in H^\infty(\mathbb{D};\mathscr{B})$ if $\mathsf{F} : \mathbb{D} \to \mathscr{B}$ is holomorphic and

$$\sup_{|z|<1} \|\mathsf{F}(z)\|_{\mathscr{B}} < \infty. \tag{A.2}$$

Lemma A.4. *1. The space $H^\infty(\mathbb{D};\mathscr{B})$ is a Banach space with norm the expression on the left-hand side of* (A.2)*.*

2. There is an isometric isomorphism between $H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ and the set of bounded linear maps from $H^2(\mathbb{D};\mathscr{U})$ to $H^2(\mathbb{D};\mathscr{Y})$ that commute with multiplication by z. The latter are all of the form $h \mapsto \mathsf{F}h$ with $\mathsf{F} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$.

For $\delta \in (0,1)$ and $\zeta \in \mathbb{T}$ define Ω^δ_ζ as the convex hull of $\{z \in \mathbb{C} : |z| \leq \delta\} \cup \{\zeta\}$. The sets Ω^δ_ζ are used to define the following **nontangential limits**.

Lemma A.5. *If* $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{H})$ *or* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*, then for almost all* $\zeta \in \mathbb{T}$ *and all* $\delta \in (0,1)$

$$\lim_{z \to \zeta, z \in \Omega_\zeta^\delta} \mathsf{F}(z)$$

exists and is independent of δ*. The limit is taken in the strong topology in the case that* $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{H})$ *and in the strong operator topology in the case that* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.

Proof. This follows from Rosenblum and Rovnyak [84, Theorem 4.6.A]. □

In the cases considered in Lemma A.5 a **boundary function** F_b of F is defined almost everywhere on $\mathbb{T}$ by

$$\mathsf{F}_b(\zeta) := \lim_{z \to \zeta, z \in \Omega_\zeta^\delta} \mathsf{F}(z).$$

Definition A.6. Let $\mathscr{H}$ be a Hilbert space. The Hardy space $H^2(\mathbb{T}; \mathscr{H})$ is defined as follows: $\mathsf{F} \in H^2(\mathbb{T}; \mathscr{H})$ if $\mathsf{F} \in L^2(\mathbb{T}, \mathscr{H})$ and for all integers $n \geq 1$

$$\int_{\mathbb{T}} \mathsf{F}(\zeta)\zeta^n d\zeta = 0.$$

Definition A.7. The Hardy space $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is defined as follows: $\mathsf{F} \in H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ if $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ and for all integers $n \geq 1$

$$\int_{\mathbb{T}} \mathsf{F}(\zeta)\zeta^n d\zeta = 0.$$

Lemma A.8. *The Hardy space* $H^2(\mathbb{T}; \mathscr{H})$ *is a closed subspace of* $L^2(\mathbb{T}, \mathscr{H})$*. It follows that with the induced norm it is a Hilbert space.*

Lemma A.9. *The Hardy space* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *is a closed subspace of* $L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*. It follows that with the induced norm it is a Banach space.*

Lemma A.10. *The boundary function of a* $H^2(\mathbb{D}; \mathscr{H})$ *function is an element of* $H^2(\mathbb{T}; \mathscr{H})$*. This mapping is a unitary operator between these two Hardy spaces.*

Lemma A.11. *The boundary function of a* $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *function is in* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*. This mapping is an isometric operator onto the space* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.

Due to Lemmas A.10 and A.11 we do not have to be careful about the distinction between Hardy spaces on the unit disc and on the unit circle.

Lemma A.12. *The space $l^2(\mathbb{Z}, \mathscr{H})$ is isometrically isomorphic to the space $L^2(\mathbb{T}, \mathscr{H})$. This isomorphism is given by*

$$(a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n=-\infty}^{\infty} a_n \zeta^n.$$

The subspace $l^2(\mathbb{Z}^+, \mathscr{H})$ is mapped onto $H^2(\mathbb{T}, \mathscr{H})$.

The transformation from Lemma A.12 is called the **Z-transform**.

The space $H^\infty(\mathbb{D}^+; \mathscr{B})$ can be defined analogously to $H^\infty(\mathbb{D}; \mathscr{B})$. Here $\mathbb{D}^+$ is the (open) exterior of the closed unit disc. Define for $\mathsf{F} : \mathbb{D} \to \mathscr{B}$ the function $\mathsf{F}^- : \mathbb{D}^+ \to \mathscr{B}$ is by $\mathsf{F}^-(z) := \mathsf{F}(1/z)$. It is easily seen that $\mathsf{F} \mapsto \mathsf{F}^-$ is an isometric isomorphism from $H^\infty(\mathbb{D}; \mathscr{B})$ onto $H^\infty(\mathbb{D}^+; \mathscr{B})$. Functions in $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ also have nontangential limits almost everywhere and we obtain a norm-preserving injection $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y})) \to L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Similar comments apply to the Hardy space $H^2(\mathbb{D}^+, \mathscr{H})$.

Lemma A.13. *If $\mathsf{F} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y})) \cap H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, then F is constant.*

Proof. Define for $u \in \mathscr{U}$ the function $\mathsf{F}_u : \mathbb{D} \to \mathscr{Y}$ by $\mathsf{F}_u(z) := \mathsf{F}(z)u$. It easily follows that $\mathsf{F}_u \in H^2(\mathbb{D}, \mathscr{Y}) \cap H^2(\mathbb{D}^+, \mathscr{Y})$. Using Lemma A.12 we can write

$$\mathsf{F}(z)u = \sum_{k=-\infty}^{\infty} a_k(u) z^k.$$

Since $\mathsf{F}_u \in H^2(\mathbb{D}, \mathscr{Y})$ it follows that $a_k(u) = 0$ for $k < 0$ for all $u \in \mathscr{U}$. From $\mathsf{F}_u \in H^2(\mathbb{D}^+, \mathscr{Y})$ we obtain that $a_k(u) = 0$ for $k > 0$ for all $u \in \mathscr{U}$. We conclude that

$$\mathsf{F}(z)u = a_0(u).$$

Hence F is a constant operator. □

Corollary A.14. *If $\mathsf{H} \in L^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is the boundary function of both a function H_1 in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ and a function H_2 in $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, then H is constant.*

Proof. Define the function $\mathsf{F} : \mathbb{D} \cap \mathbb{D}^+ \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by $\mathsf{F}(z) := \mathsf{H}_1(z)$ if $z \in \mathbb{D}$ and $\mathsf{F}(z) := \mathsf{H}_2(z)$ if $z \in \mathbb{D}^+$. Apply Lemma A.13 to this function. We conclude that F is constant. It follows that H_1 is constant. The boundary function of a constant function is obvously constant, so H is constant. □

Definition A.15. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we define the operator $L_\mathsf{F} : L^2(\mathbb{T}, \mathscr{U}) \to L^2(\mathbb{T}, \mathscr{Y})$ by $L_\mathsf{F}\mathsf{H} = \mathsf{F}\mathsf{H}$. This operator is called the **Laurent operator** of F.

Definition A.16. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we define the operator $T_\mathsf{F} : H^2(\mathbb{T}, \mathscr{U}) \to H^2(\mathbb{T}, \mathscr{Y})$ by $T_\mathsf{F} = P_{H^2(\mathbb{T},\mathscr{Y})} L_\mathsf{F}|_{H^2(\mathbb{T},\mathscr{U})}$. This operator is called the **Toeplitz operator** of F.

Note that we can identify the Toeplitz operator T_F with an operator from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$ using the identification of functions on the unit disc with their boundary functions discussed earlier.

Lemma A.17. *We have $L_\mathsf{F}^* = L_{\mathsf{F}^*}$ and $T_\mathsf{F}^* = T_{\mathsf{F}^*}$, where the function $\mathsf{F}^* \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ is defined by $\mathsf{F}^*(z) = \mathsf{F}(z)^*$.*

Lemma A.18. *Let $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. The Laurent operator L_F is isometric if and only if $\mathsf{F}(\zeta)$ is isometric for almost all $\zeta \in \mathbb{T}$.*

Definition A.19. A function $\mathsf{F} \in H^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is called **inner** if its Laurent operator L_F is an isometry.

Lemma A.20. *Let $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. The Toeplitz operator T_F is isometric if and only if the Laurent operator L_F is.*

The following is known as Sarason's theorem.

Lemma A.21. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ an inner function and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2))$. Let $\operatorname{Im}(T_\mathsf{H})$ denote the image of the Toeplitz operator of H, $\operatorname{Im}(T_\mathsf{H})^\perp$ its orthogonal complement in $H^2(\mathbb{D}, \mathscr{H}_2)$ and $P_{\operatorname{Im}(T_\mathsf{H})^\perp} \in \mathcal{L}(H^2(\mathbb{D}, \mathscr{H}_2))$ the orthogonal projection onto this ortogonal complement. Then*

$$\|P_{\operatorname{Im}(T_\mathsf{H})^\perp} T_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2))}.$$

Proof. See for example Nikol′skiĭ [59, p191]. □

We need the following two corollaries of Sarason's theorem.

Corollary A.22. *Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, $\mathsf{H}, \mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ with H an inner function. Then*

$$\|P_{\operatorname{Im}(T_\mathsf{H})^\perp} T_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))}.$$

Proof. We apply Lemma A.21 and denote the operators there with tildes to distinguish them from the operators used in the statement of this lemma. Apply Lemma A.21 with $\mathscr{H}_1 = \mathscr{U}$, $\mathscr{H}_2 = \mathscr{U} \times \mathscr{Y}$, $\tilde{\mathsf{H}} = \mathsf{H}$ and $\tilde{\mathsf{F}} = [\mathsf{F}, 0]$. It is easily seen from the form of $\tilde{\mathsf{F}}$ that

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^\perp} T_{\mathsf{F}}\| = \|P_{\mathrm{Im}(T_{\tilde{\mathsf{H}}})^\perp} T_{\tilde{\mathsf{F}}}\|.$$

The parameter $\tilde{\mathsf{V}}$ from Lemma A.21 can be decomposed as $\tilde{\mathsf{V}} = [\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2]$. We have

$$\begin{aligned}
&\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|\tilde{\mathsf{F}} - \tilde{\mathsf{H}}\tilde{\mathsf{V}}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\tilde{\mathsf{V}}_1, -\mathsf{H}\tilde{\mathsf{V}}_2]\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{V \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))},
\end{aligned}$$

where we have used that the infimum over $\tilde{\mathsf{V}}_2$ is reached for $\tilde{\mathsf{V}}_2 = 0$. From Lemma A.21 we now obtain the desired equality. □

Corollary A.23. *Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ an inner function and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$. Then*

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^\perp} T_{\mathsf{F}}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))}.$$

Proof. We apply Lemma A.21 and denote the operators there with tildes to distinguish them from the operators used in the statement of this lemma. Apply Lemma A.21 with $\mathscr{H}_1 = \mathscr{U}$, $\mathscr{H}_2 = \mathscr{U} \times \mathscr{Y}$, $\tilde{\mathsf{H}} = \mathsf{H}$ and $\tilde{\mathsf{F}} = [0, \mathsf{F}]$. It is easily seen from the form of $\tilde{\mathsf{F}}$ that

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^\perp} T_{\mathsf{F}}\| = \|P_{\mathrm{Im}(T_{\tilde{\mathsf{H}}})^\perp} T_{\tilde{\mathsf{F}}}\|.$$

The parameter $\tilde{\mathsf{V}}$ from Lemma A.21 can be decomposed as $\tilde{\mathsf{V}} = [\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2]$. We have

$$\begin{aligned}
&\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|\tilde{\mathsf{F}} - \tilde{\mathsf{H}}\tilde{\mathsf{V}}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|[-\mathsf{H}\tilde{\mathsf{V}}_1, \mathsf{F} - \mathsf{H}\tilde{\mathsf{V}}_2]\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))},
\end{aligned}$$

where we have used that the infimum over $\tilde{\mathsf{V}}_1$ is reached for $\tilde{\mathsf{V}}_1 = 0$. From Lemma A.21 we now obtain the desired equality. □

Definition A.24. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ we define the operator $H_\mathsf{F} : L^2(\mathbb{T}, \mathscr{H}_1) \to L^2(\mathbb{T}, \mathscr{H}_2)$ by $H_\mathsf{F} := P_{H^2(\mathbb{D},\mathscr{H}_2)} L_\mathsf{F} P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp}$. This operator is called the **Hankel operator** of F.

Remark A.25. We warn the reader that in the literature there are several different definitions of the concept of the Hankel operator of a function.

The next result relates the Hankel operator of a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function with the Hankel map of an input-output stable discrete-time system (see the definition on page 9, Definition 3.1 and Proposition 3.22).

Lemma A.26. *Let Σ be an input-output stable discrete-time system with transfer function D and Hankel map $\mathcal{H}$. Denote the Hankel operator of $\mathsf{D} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ by H_D and the Z-transform from $l^2(\mathbb{Z}, \mathscr{H})$ to $L^2(\mathbb{T}, \mathscr{H})$ by $Z_\mathscr{H}$. Then $H_\mathsf{D} Z_\mathscr{U} = Z_\mathscr{Y} \mathcal{H}$. In particular, $\|H_\mathsf{D}\| = \|\mathcal{H}\|$.*

Proof. This follows easily from the definitions. □

The following result is known as Nehari's theorem (or sometimes Page's theorem).

Lemma A.27. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $\mathsf{F} \in L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. Then*

$$\|H_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F}^* - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

Proof. It is shown in e.g. Peller [75, page 68], Nikol′skiĭ [59, p191] that for $\mathsf{H} \in L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$

$$\|P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp} L_\mathsf{H} P_{H^2(\mathbb{D},\mathscr{H}_2)}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{H} - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

Applying this with $H = F^*$ gives

$$\|P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp} L_\mathsf{F}^* P_{H^2(\mathbb{D},\mathscr{H}_2)}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F}^* - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

The desired result now follows from noting that the left-hand side of this last expression is the norm of the adjoint of the Hankel operator of F and that an operator and its adjoint have the same norm. □

Lemma A.28. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be finite-dimensional Hilbert spaces and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ a rational function. Then, for each $\sigma > \|H_\mathsf{F}\|$ there exists a rational $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ such that*

$$\|\mathsf{F}^* - \mathsf{H}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))} \le \sigma.$$

Proof. This follows from the explicit state space formulas given in McFarlane and Glover [54, Appendix B]. □

The second statement in the following result is known as the Corona theorem.

Lemma A.29. *Let* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. *If there exists a function* $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *such that* $\mathsf{H}(s)\mathsf{F}(s) = I$ *for all* $s \in \mathbb{D}$, *then there exists a* $\varepsilon > 0$ *such that for all* $s \in \mathbb{D}$ *and all* $h \in \mathscr{H}_1$

$$\|\mathsf{F}(s)h\|_{\mathscr{H}_2} \geq \varepsilon \|h\|_{\mathscr{H}_1}. \tag{A.3}$$

If $\mathscr{H}_1$ *is finite-dimensional then the converse is also true, i.e., if there exists a* $\varepsilon > 0$ *such that for all* $s \in \mathbb{D}$ *and all* $h \in \mathscr{H}_1$ *we have* (A.3), *then there exists a* $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *such that* $\mathsf{H}(s)\mathsf{F}(s) = I$ *for all* $s \in \mathbb{D}$.

Notes

General references on Hardy space theory are Duren [23], Garnett [32] and Hoffman [39]. The vector-valued case can be found in Nikol′skiĭ [61], [59], [60], Peller [75] and Rosenblum and Rovnyak [84].

Appendix B

Algebraic Riccati equations

In this appendix we prove some simple algebraic results concerning algebraic Riccati equations.

Lemma B.1. *Let P and Q be nonnegative self-adjoint operators. Define*

$$A_P := A - (BD^* + APC^*)(\underline{R} + CPC^*)^{-1}C, \tag{B.1}$$

$$A_Q := A - B(\underline{S} + B^*QB)^{-1}(D^*C + B^*QA), \tag{B.2}$$

$$\underline{A} := A - B\underline{S}^{-1}D^*C. \tag{B.3}$$

where $\underline{S} := I + D^*D$ and $\underline{R} := I + DD^*$. Then

$$A_P(I + PC^*\underline{R}^{-1}C) = \underline{A} = (I + B\underline{S}^{-1}B^*Q)A_Q, \tag{B.4}$$

$$A_Q = (I + B\underline{S}^{-1}B^*Q)^{-1}A_P(I + PC^*\underline{R}^{-1}C), \tag{B.5}$$

$$A_P = (I + B\underline{S}^{-1}B^*Q)A_Q(I + PC^*\underline{R}^{-1}C)^{-1}. \tag{B.6}$$

Proof. We prove that $A_P(I + PC^*\underline{R}^{-1}C) = \underline{A}$. The equality $\underline{A} = (I + B\underline{S}^{-1}B^*Q)A_Q$ is proved similarly. By writing out A_P in full we have

$$\begin{aligned}
&A_P(I + PC^*\underline{R}^{-1}C) \\
&\quad = A(I + PC^*\underline{R}^{-1}C) - (BD^* + APC^*)(\underline{R} + CPC^*)^{-1}C(I + PC^*\underline{R}^{-1}C) \\
&\quad = A(I + PC^*\underline{R}^{-1}C) - (BD^* + APC^*)(\underline{R} + CPC^*)^{-1}(\underline{R} + CPC^*)\underline{R}^{-1}C \\
&\quad = A + APC^*\underline{R}^{-1}C - (BD^* + APC^*)\underline{R}^{-1}C = A - BD^*\underline{R}^{-1}C \\
&\quad = A - B\underline{S}^{-1}D^*C,
\end{aligned}$$

since $D^*\underline{R}^{-1} = \underline{S}^{-1}D^*$. This completes the proof of (B.4). Equations (B.5) and (B.6) easily follow from (B.4). □

Remark B.2. Note that in the above lemma we have not assumed that P and Q are solutions of the Riccati equations.

We now prove that the Riccati equations can be written in several different but equivalent versions.

Lemma B.3. *Let P and Q be nonnegative self-adjoint operators.*

1. *P satisfies*

$$A_P P(I + C^* \underline{R}^{-1} CP) A_P^* - P + B\underline{S}^{-1}B^* = 0, \tag{B.7}$$

where A_P is defined by (B.1)*, if and only if it satisfies*

$$\underline{A}P(I + C^* \underline{R}^{-1} CP)^{-1} \underline{A}^* - P + B\underline{S}^{-1}B^* = 0, \tag{B.8}$$

where $\underline{A}$ is defined by (B.3).

2. *P satisfies* (B.7) *if and only if it satisifies the filter algebraic Riccati equation.*

3. *Q satisfies*

$$A_Q^*(I + QB\underline{S}^{-1}B^*)QA_Q - Q + C^* \underline{R}^{-1} C = 0, \tag{B.9}$$

where A_Q is defined by (B.2)*, if and only if it satisfies*

$$\underline{A}^* Q(I + B\underline{S}^{-1}B^* Q)^{-1} \underline{A} - Q + C^* \underline{R}^{-1} C = 0, \tag{B.10}$$

where $\underline{A}$ is defined by (B.3).

4. *Q satisfies* (B.9) *if and only if it satisifies the control algebraic Riccati equation.*

Proof. We shall prove the equivalence of the filter equations; the equivalence of the control equations is similar.

1. The equations (B.7) and (B.8) are equivalent if and only if the following holds:

$$\underline{A}P(I + C^* \underline{R}^{-1} CP)^{-1} \underline{A}^* = A_P P(I + C^* \underline{R}^{-1} CP) A_P^*. \tag{B.11}$$

We use Lemma B.1 (which tells us that $\underline{A} = A_P(I + PC^* \underline{R}^{-1} C)$) to write the left-hand side of (B.11) as

$$A_P(I + PC^* \underline{R}^{-1} C)P(I + C^* \underline{R}^{-1} CP)^{-1}(I + C^* \underline{R}^{-1} CP)A_P^*,$$

which is indeed equal to the right-hand side of (B.11).

2. To prove the equivalence of (B.7) and the filter algebraic Riccati equation we substitute in (B.7) for A_P from (B.1) and for $(I+C^*\underline{R}^{-1}CP)A_P^*$, we substitute $\underline{A}^*$ (using (B.4)) and then substitute (B.3) for $\underline{A}$. We then get

$$(A-(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}C)P(A^*-C^*D\underline{S}^{-1}B^*)-P+B\underline{S}^{-1}B^*=0.$$

Rewriting this gives

$$\begin{aligned} APA^*-P+BB^* \;=\;& (BD^*+APC^*)(\underline{R}+CPC^*)^{-1}CPA^* \\ & -(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}CPC^*D\underline{S}^{-1}B^* \\ & +APC^*D\underline{S}^{-1}B^*-B\underline{S}^{-1}B^*+BB^*. \end{aligned}$$

We now focus on the last two lines of this last equation. We note that $I-\underline{S}^{-1}=D^*D\underline{S}^{-1}$ and we can thus rewrite these last two lines as

$$-(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}CPC^*D\underline{S}^{-1}B^*+APC^*D\underline{S}^{-1}B^*+BD^*D\underline{S}^{-1}B^*,$$

and this can be rewritten as

$$(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}[-CPC^*+\underline{R}+CPC^*]D\underline{S}^{-1}B^*.$$

Noting that $\underline{R}D\underline{S}^{-1}=D$, we see that this is equal to

$$(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}DB^*.$$

This completes the proof of the equivalence of (B.7) and the filter algebraic Riccati equation. □

Lemma B.4. *Assume the discrete-time system Σ has a solution Q of its control algebraic Riccati equation and P be of its filter algebraic Riccati equation. Define A_P and A_Q by* (B.1) *and* (B.2)*, respectively. Then*

$$(I+PQ)A_Q=A_P(I+PQ). \tag{B.12}$$

Proof. We use the equivalent version of the filter algebraic Riccati equation (B.7) to write

$$P=A_PP(I+C^*\underline{R}^{-1}CP)A_P^*+B\underline{S}^{-1}B^*,$$

which leads to

$$I+PQ=I+A_PP(I+C^*\underline{R}^{-1}CP)A_P^*Q+B\underline{S}^{-1}B^*Q$$

and so

$$(I+PQ)A_Q=(I+B\underline{S}^{-1}B^*Q)A_Q+A_PP(I+C^*\underline{R}^{-1}CP)A_P^*QA_Q.$$

We use (B.5) to write the right-hand side as

$$A_P(I + PC^*\underline{R}^{-1}C) + A_P P(I + C^*\underline{R}^{-1}CP)A_P^* Q A_Q.$$

Rearranging gives

$$A_P + A_P P[C^*\underline{R}^{-1}C + (I + C^*\underline{R}^{-1}CP)A_P^* Q A_Q],$$

and using (B.5) again we obtain

$$A_P + A_P P[C^*\underline{R}^{-1}C + A_Q^*(I + QB\underline{S}^{-1}B^*)QA_Q].$$

According to the version (B.9) of the control algebraic Riccati equation, the term in square brackets equals Q. So the above is equal to $A_P(I + PQ)$. $\square$

We now prove a relation concerning the difference of two solutions of a Riccati equation.

Lemma B.5. *Assume the discrete-time system* Σ *has solutions* Q_1 *and* Q_2 *of its control algebraic Riccati equation. Define* A_{Q_1} *and* A_{Q_2} *similarly to* (B.2)*. Then*

$$Q_1 - Q_2 = A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}.$$

Proof. Subtract the form (B.9) of the control algebraic Riccati equation for Q_1 and Q_2 to obtain

$$Q_1 - Q_2 = A_{Q_1}^*(I + Q_1 B\underline{S}^{-1}B^*)Q_1 A_{Q_1} - A_{Q_2}^*(I + Q_2 B\underline{S}^{-1}B^*)Q_2 A_{Q_2}. \tag{B.13}$$

According to Lemma B.1 (say with $P = I$) we have

$$\begin{aligned} A_{Q_2} &= (I + B\underline{S}^{-1}B^*Q_2)^{-1}A_P(I + PC^*\underline{R}^{-1}C) \\ &= (I + B\underline{S}^{-1}B^*Q_2)^{-1}(I + B\underline{S}^{-1}B^*Q_1)A_{Q_1}. \end{aligned} \tag{B.14}$$

Combining (B.13) and (B.14) we obtain

$$\begin{aligned} Q_1 - Q_2 &= A_{Q_2}^*(I + Q_2 B\underline{S}^{-1}B^*)Q_1 A_{Q_1} - A_{Q_2}^* Q_2(I + B\underline{S}^{-1}B^*Q_1)A_{Q_1} \\ &= A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}. \end{aligned}$$

$\square$

Proof of Proposition 6.39. Denote the Riccati closed-loop system associated with the solution Q by Σ_Q. We need to show that

$$A_Q P(I + PQ)^{-1} - P(I + PQ)^{-1} + B_Q B_Q^* = 0.$$

From (B.5) and (B.12) we see that this is equivalent to

$$(I+B\underline{S}^{-1}B^*Q)^{-1}A_P(I+PC^*\underline{R}^{-1}C)PA_P^*(I+QP)^{-1}-P(I+QP)^{-1}+B_QB_Q^*=0. \tag{B.15}$$

It is easily proven that

$$(I+B\underline{S}^{-1}B^*Q)B_QB_Q^* = B\underline{S}^{-1}B^*.$$

Using this we see that (B.15) is equivalent to

$$\begin{array}{rl} (I+B\underline{S}^{-1}B^*Q)^{-1} & \times \\ \left[A_P(I+PC^*\underline{R}^{-1}C)PA_P^* - (I+B\underline{S}^{-1}B^*Q)P + B\underline{S}^{-1}B^*(I+QP)\right] & \times \\ (I+QP)^{-1} & =0. \end{array}$$

The term in square brackets is zero since it is the equivalent version (B.7) of the filter Riccati equation. □

Lemma B.6. *Suppose that the discrete-time system with system operator* $[\check{A},\check{B};[\check{C}_1;\check{C}_2],[\check{D}_1;\check{D}_2]]$ *is such that* $\check{D}_1$ *has a bounded inverse and that there exists a nonnegative self-adjoint operator* V *such that*

$$\check{B}^*V\check{A}+\check{D}^*\check{C}=0. \tag{B.16}$$

Define A,B,C,D *as in Proposition 2.23,* $\underline{S}:=I+D^*D$ *and* $\underline{R}:=I+DD^*$*. Then*

1. $\check{A}=A-B(\underline{S}+B^*VB)^{-1}(D^*C+B^*VA)$ *and*
2. $\check{A}^*V\check{A}-V+\check{C}^*\check{C}=\check{A}^*(I+VB\underline{S}^{-1}B^*)V\check{A}-V+C^*\underline{R}^{-1}C.$

Proof. We first prove the equality

$$\underline{S}\check{C}_1=-(B^*V\check{A}+D^*C). \tag{B.17}$$

From (B.16) and (2.5) we obtain

$$\check{D}_1^*B^*V\check{A}+\check{D}_1^*\check{C}_1+\check{D}_2^*\check{C}_2=0.$$

Thus

$$\check{C}_1=-B^*V\check{A}-D^*\check{C}_2=-B^*V\check{A}-D^*(C+D\check{C}_1)$$

and this yields (B.17):

$$\underline{S}\check{C}_1=(I+D^*D)\check{C}_1=-B^*V\check{A}-D^*C.$$

We now prove the first equality stated in the lemma. We take the equality just proved (B.17) and substitute $\check{A} = A + B\check{C}_1$ to obtain

$$\underline{S}\check{C}_1 = -(B^*V(A + B\check{C}_1) + D^*C).$$

Thus

$$(\underline{S} + B^*VB)\check{C}_1 = -(B^*VA + D^*C).$$

We now solve for $\check{C}_1$ and substitute to obtain

$$\check{A} = A + B\check{C}_1 = A - B(\underline{S} + B^*VB)^{-1}(B^*VA + D^*C).$$

We now prove the equality

$$\check{C}^*\check{C} = \check{A}^*VB\underline{S}^{-1}B^*V\check{A} + C^*\underline{R}^{-1}C. \tag{B.18}$$

We have

$$\check{C}^*\check{C} = \check{C}_1^*\check{C}_1 + \check{C}_2^*\check{C}_2$$

and substituting for $\check{C}_2$ from (2.5) gives

$$\check{C}^*\check{C} = \check{C}_1^*\check{C}_1 + (C + D\check{C}_1)^*(C + D\check{C}_1).$$

Finally, substituting for $\check{C}_1$ from (B.17) and simplifying gives the result. The second equality stated in the lemma follows easily from (B.18). □

Lemma B.7. *Suppose that the discrete-time system with system operator $[\check{A}, \check{B}; [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2]]$ is such that $\check{D}_1$ has a bounded inverse and assume that a nonnegative self-adjoint operator V exists such that*

$$\check{B}^*V\check{B} + \check{D}^*\check{D} = I.$$

Define A, B, C, D as in Proposition 2.23. Then we have

1. *$B^*VB + \underline{S} = \check{D}_1^{-*}\check{D}_1^{-1}$ and*
2. *$\check{B}\check{B}^*(I + VB\underline{S}^{-1}B^*) = B\underline{S}^{-1}B^*$.*

Proof. 1. The given equation for V translates to

$$\check{D}_1^*B^*VB\check{D}_1 + \check{D}_1^*\check{D}_1 + \check{D}_1^*D^*D\check{D}_1 = I,$$

and multiplying from the left with $\check{D}_1^{-*}$ and from the right with $\check{D}_1^{-1}$ gives the result.

2. The first equality implies that $(\underline{S}+B^*VB)^{-1} = \check{D}_1\check{D}_1^*$ and so $B(\underline{S}+B^*VB)^{-1}B^*$
$= \check{B}\check{B}^*$. Hence

$$\begin{aligned}\check{B}\check{B}^*(I+VB\underline{S}^{-1}B^*) &= B(\underline{S}+B^*VB)^{-1}B^*(I+VB\underline{S}^{-1}B^*)\\ &= B(\underline{S}+B^*VB)^{-1}(\underline{S}+B^*VB)\underline{S}^{-1}B^* = B\underline{S}^{-1}B^*,\end{aligned}$$

which proves the second equality. □

Proof of Lemma 6.46. It easily follows from Proposition 6.45 that $\check{\Sigma}$ is the Riccati closed-loop system of Σ corresponding to the solution of the control Riccati equation $Q := L_c$. This in particular implies that $\check{A} = A_Q$, where A_Q is defined by (B.2). Define $P := L(I-QL)^{-1}$ and define A_P by (B.1). We establish the identity

$$(I-LQ)A_P = \check{A}(I-LQ) \tag{B.19}$$

or by (B.6) the equivalent identity

$$(I-LQ)(I+B\underline{S}^{-1}B^*Q)\check{A} = \check{A}(I-LQ)(I+PC^*\underline{R}^{-1}C). \tag{B.20}$$

Since $P = L(I-QL)^{-1} = (I-LQ)^{-1}L$ the right-hand side of (B.20) is equal to

$$\check{A} - \check{A}LQ + \check{A}LC^*\underline{R}^{-1}C.$$

We substitute $Q - C^*\underline{R}^{-1}C = \check{A}^*(I+QB\underline{S}^{-1}B^*)Q\check{A}$ (this identity holds because Q is a solution of the filter Riccati equation; see (B.9)) to obtain for the right-hand side of (B.20)

$$\check{A} - \check{A}L\check{A}^*(I+QB\underline{S}^{-1}B^*)Q\check{A}.$$

The control Lyapunov equation tells us that $\check{A}L\check{A}^* = L - \check{B}\check{B}^*$ and so the right-hand side of (B.20) is equal to

$$\check{A} - L(I+QB\underline{S}^{-1}B^*)Q\check{A} + \check{B}\check{B}^*(I+QB\underline{S}^{-1}B^*)Q\check{A}.$$

Substituting $\check{B}\check{B}^*(I+QB\underline{S}^{-1}B^*) = B\underline{S}^{-1}B^*$ from Lemma B.7 with $V = Q$ we obtain for the right-hand side of (B.20)

$$\check{A} - L(I+QB\underline{S}^{-1}B^*)Q\check{A} + B\underline{S}^{-1}B^*Q\check{A},$$

which is equal to the left-hand side of (B.20). This proves (B.19). We show that P is a solution of the filter algebraic Riccati equation. We start with the control Lyapunov equation

$$\check{A}L\check{A}^* - L + \check{B}\check{B}^* = 0$$

and substitute $\check{A} = (I - LQ)A_P(I - LQ)^{-1}$ from (B.19) and $\check{A}^* = (I + C^*\underline{R}^{-1}CP)A_P^*(I + QB\underline{S}^{-1}B^*)^{-1}$ from (B.6) to obtain

$$(I-LQ)A_P(I-LQ)^{-1}L(I+C^*\underline{R}^{-1}CP)A_P^*(I+QB\underline{S}^{-1}B^*)^{-1}-L+\check{B}\check{B}^* = 0.$$

We multiply by $(I - LQ)^{-1}$ from the left and by $(I + QB\underline{S}^{-1}B^*)$ from the right to obtain

$$A_P P(I+C^*\underline{R}^{-1}CP)A_P^*-P(I+QB\underline{S}^{-1}B^*)+(I-LQ)^{-1}\check{B}\check{B}^*(I+QB\underline{S}^{-1}B^*)=0.$$

We again use the fact that $\check{B}\check{B}^*(I + QB\underline{S}^{-1}B^*) = B\underline{S}^{-1}B^*$ to obtain

$$A_P P(I+C^*\underline{R}^{-1}CP)A_P^*-P-PQB\underline{S}^{-1}B^*+(I-LQ)^{-1}B\underline{S}^{-1}B^* = 0. \tag{B.21}$$

Using that $P = (I - LQ)^{-1}L$ we see that the sum of the two last terms of the left-hand side of (B.21) equals $B\underline{S}^{-1}B^*$. This proves that P is a solution of the equivalent version (B.7) of the filter algebraic Riccati equation. $\square$

Lemma B.8. *Let Q be a solution of the control algebraic Riccati equation of Σ. Define A_Q by* (B.2). *Denote by $\overline{\mathbb{D}}$ the closed unit disc. Then the component of $1/\rho(A) \cap \overline{\mathbb{D}}$ that contains zero is contained in the component of $1/\rho(A_Q) \cap \overline{\mathbb{D}}$ that contains zero.*

Proof. We first show that if $\lambda \in \overline{\mathbb{D}}$ is in the approximate point spectrum of A_Q, then it is in the appoximate point spectrum of A.

We first note that for all $x \in \mathscr{X}$ and $\lambda \in \mathbb{C}$

$$\begin{aligned} ||(\lambda I - A)x|| &\leq ||(A_Q - A)x|| + ||(\lambda I - A_Q)x|| \leq \\ &||B(\underline{S} + B^*QB)^{-1}||\ (||D^*Cx|| + ||B^*QAx||) + ||(\lambda I - A_Q)x||. \end{aligned} \tag{B.22}$$

Second we note that the control algebraic Riccati equation (it follows most easily from the equivalent version (B.9)) implies that for every $x \in \mathscr{X}$

$$\|\underline{S}^{-1/2}B^*QA_Qx\|^2 + \|\underline{R}^{-1/2}Cx\|^2 = \|Q^{1/2}x\|^2 - \|Q^{1/2}A_Qx\|^2. \tag{B.23}$$

For every λ in the exterior of the open unit disc we have that the right-hand side of (B.23) is smaller than or equal to $|\lambda|^2\ \|Q^{1/2}x\|^2 - \|Q^{1/2}A_Qx\|^2$, which equals

$$-\langle Q(\lambda I - A_Q)x, (\lambda I - A_Q)x\rangle + \lambda\langle Qx, (\lambda I - A_Q)x\rangle + \bar{\lambda}\langle(\lambda I - A_Q)x, Qx\rangle.$$

It is easily computed that $B^*QA_Q = \underline{S}(\underline{S} + B^*QB)^{-1}B^*QA$. It follows that the left-hand side of (B.23) is equal to

$$||\underline{S}^{1/2}(\underline{S} + B^*QB)^{-1}B^*QAx||^2 + ||R^{1/2}Cx||^2.$$

Hence we obtain

$$\begin{aligned}&||\underline{S}^{1/2}(\underline{S}+B^*QB)^{-1}B^*QAx\|^2+\|\underline{R}^{-1/2}Cx\|^2 \\ &\le -\|Q^{1/2}(\lambda I-A_Q)x\|^2+\lambda\langle Qx,(\lambda I-A_Q)x\rangle+\bar{\lambda}\langle(\lambda I-A_Q)x,Qx\rangle.\end{aligned} \tag{B.24}$$

Assume that $\lambda \in \overline{\mathbb{D}}$ is in the approximate point spectrum of A_Q. Then there exists an sequence $x_n \in \mathscr{X}$ with $\|x_n\| = 1$ such that $\|(\lambda I - A_Q)x_n\| \to 0$. It follows from (B.24) that $\|B^*QAx_n\| \to 0$ and $\|Cx_n\| \to 0$. It then follows using (B.22) that $\|(\lambda I-A)x_n\| \to 0$. This means that λ is in the approximate point spectrum of A. So we have $\sigma_a(A_Q) \cap \overline{\mathbb{D}^+} \subset \sigma_a(A) \cap \overline{\mathbb{D}^+}$.

Let μ be an element of the component of $1/\rho(A) \cap \overline{\mathbb{D}}$ that contains zero. Then there exists a path l in $1/\rho(A)\cap\overline{\mathbb{D}}$ that has zero and μ as its endpoints. Assume that μ is not an element of the component of $1/\rho(A_Q) \cap \overline{\mathbb{D}}$ that contains zero. Consider the sets

$$V_\sigma := \{z \in l_\mu : 1/z \in \sigma(A_Q)\cap\overline{\mathbb{D}^+}\}, \quad V_\rho := \{z \in l_\mu : 1/z \in \rho(A_Q)\cap\overline{\mathbb{D}^+}\}.$$

Since l is contained in the closed unit disc it follows that $l = V_\sigma \cup V_\rho$. It is easily seen that V_σ is closed. Let $p : [0, 1] \to l$ be a parametrization of l. We have $p(0) = 0 \in V_\rho$ and $p(1) = \mu \in V_\sigma$. We have $[0, 1] = p^{-1}(V_\sigma) \cup p^{-1}(V_\rho)$. Since $p^{-1}(V_\sigma)$ is closed it has a smallest element. Denote this smallest element by $t_{\min}$. Since $0 \in p^{-1}(V_\rho)$ we have $t_{\min} > 0$. Denote $\lambda_{\min} = p(t_{\min})$. It follows by construction that $\lambda_{\min}$ is an element of the boundary of $\sigma(A_Q)$. Since the boundary of the spectrum consists of approximate eigenvalues we have $\lambda_{\min} \in \sigma_a(A_Q)\cap\overline{\mathbb{D}^+}$. It follows, using the above established relation between the approximate eigenvalues, that $\lambda_{\min} \in \sigma_a(A) \cap \overline{\mathbb{D}^+}$. But this contradicts the fact that l is contained in $1/\rho(A) \cap \overline{\mathbb{D}}$. The desired result follows. $\square$

Summary

The main aim of this thesis is, as the title suggests, the presentation of results on model reduction for controller design for infinite-dimensional systems. These results are presented for discrete-time systems in Chapter 10 and for continuous-time systems in Section 14.7. They are perfect generalizations of the finite-dimensional results: we obtained existence and uniqueness of minimal LQG-balanced realizations under conditions that are obviously necessary (but it is far from obvious that they are sufficient!) and an error-bound for truncated LQG-balanced realizations. The results are illustrated by a controller design for a beam in Chapter 15.

Along the way we generalized several important theorems and introduced a few promising new concepts. Arguably the most important theorem that we generalize is that on the existence of (strongly) coprime factorizations. The results in Chapter 7 solve this long outstanding problem for which many partial results exist in the literature. The most important new concept resulting from this Ph.D. work is probably that of a (distributional) resolvent linear system. As shown in part II many systems described by partial differential equations fall into this class of systems *and* one can reasonably easily prove theorems for this class of systems. That this new concept brings together well-established concepts such as distribution semigroups, the Cayley transform and nonhomogeneous elliptic boundary value problems strengthens our belief that we have discovered an important new class of systems.

Samenvatting

Het hoofddoel van dit proefschrift is, zoals de titel aangeeft, de presentatie van resultaten over modelreductie voor regelaarontwerp voor oneindig-dimensionale systemen. Deze resultaten worden voor discrete-tijd systemen gegeven in hoofdstuk 10 en voor continue-tijd systemen in sectie 14.7. Deze resultaten zijn perfecte generalizaties van de overeenkomstige eindig-dimensionale resultaten: we tonen existentie en eenduidigheid van minimale LQG-gebalanceerde realizaties aan onder condities die overduidelijk noodzakelijk zijn (maar het is verre van overduidelijk dat deze condities ook voldoende zijn!) en we geven een foutafschatting voor afgekapte LQG-gebalanceerde realizaties. De theoretische resultaten worden in hoofdstuk 15 geïllustreerd middels een regelaarontwerp voor een balk.

Als tussenresultaten presenteren we generalizaties van een aantal belangrijke stellingen. Waarschijnlijk de belangrijkste stelling die we generalizeren is die over het bestaan van (sterke) copriem factorizaties. De resultaten in hoofdstuk 7 geven een volledige oplossing van dit al lang openstaande probleem waarvoor vele deeloplossingen bestaan in de literatuur.

Ook introduceren we in dit proefschrift een aantal nieuwe concepten. Het belangrijkste nieuwe concept in dit proefschrift is waarschijnlijk dat van een (distributional) resolvent linear system. Zoals beschreven in deel II van dit proefschrift vallen vele systemen beschreven door partiële differentiaalvergelijkingen binnen deze klasse van systemen *en* kan men relatief eenvoudig stellingen bewijzen voor deze klasse van systemen. Dat dit nieuwe concept enkele bestaande concepten zoals distributie halfgroepen, de Cayley transformatie en niet-homogene elliptische randwaardeproblemen bij elkaar brengt sterkt ons in de overtuiging dat we een belangrijke nieuwe klasse van systemen ontdekt hebben.

Tot slot

Dit proefschrift werd mede mogelijk gemaakt door...

Mijn promotor: Het onderwerp van dit proefschrift is door haar aangedragen en de meeste resultaten gepresenteerd in dit proefschrift zijn het product van ons gezamenlijk werk. Verder dank ik haar voor al de wijze lessen die ze geprobeerd heeft mij te leren.

De RuG-wiskundigen: Ik heb met veel plezier negen jaar op de wiskunde afdeling van de Rijks*universiteit* Groningen rondgelopen. Tijdens de colleges en daarbuiten heb ik veel geleerd van de hoogleraren en docenten van de afdeling. Een extra woord van dank gaat uit naar Erik Thomas die me ook nog tijdens mijn promotieonderzoek een aantal maal van essentiële kennis deelgenoot heeft gemaakt.

Mijn afstudeerder: De numerieke resultaten in sectie 15.3 van dit proefschrift zijn verkregen met behulp van een matlab-programma dat geschreven is door Simon van Mourik voor zijn afstuderen. Dank dat je dit numerieke werk uit mijn handen hebt genomen Simon!

De systeemtheorie groep: De systeemtheorie groep is in de vijf jaar dat ik er nu deel van uit maak erg veranderd, maar is altijd een bron van steun en inspiratie gebleven. Daar wil ik de officiële en onofficiële leden van de groep gedurende de afgelopen vijf jaar voor bedanken.

De wiskunde promovendi: Nu het mijn beurt is om te gaan heb ik heb een hele generatie van promovendi zijn komen of gaan. Thanks for everything guys!

De leescommissie: Ik wil de leden van de leescommissie danken voor het vrijmaken van een deel van hun schaarse tijd om mijn proefschrift te lezen en er commentaar op te geven. Verder wil ik Olof Staffans bedanken voor de twee maanden die ik bij hem in Finland mocht doorbrengen. Het schrijven van dit proefschrift en het uitwerken van onze toen behaalde resultaten kwamen elkaar danig in de weg te zitten, maar nu het proefschrift geschreven is zullen die artikelen met onze 'Finse' resultaten dan toch daadwerkelijk afgemaakt kunnen worden.

NWO: Dit onderzoek werd gefinancierd door een subsidie uit de open competitie van de Nederlandse organisatie voor Wetenschappelijk Onderzoek.

Mijn familie: In dit dankwoord last but certainly not least.

List of Notation

Symbol	Short description	Page
A	state operator	8
A	state function	10
$\mathfrak{A}$	resolvent of a discrete-time system	12
$\mathfrak{a}$	pseudoresolvent (of a continuous-time system)	124
B	input operator	8
$\mathscr{B}$	Banach space	
$\mathbb{B}$	behavior	7
$\mathcal{B}$	input map	8
B	input function	10
$\mathfrak{B}$	incoming wave function of a discrete-time system	12
$\mathfrak{b}$	incoming wave function of a continuous-time system	124
$\vec{B}(\cdot,\cdot)$	directed gap ball	95
$B(\cdot,\cdot)$	gap ball	95
C	output operator	8
$\mathbb{C}$	set of complex numbers	
$\mathbb{C}_\sigma^+$	open right half-plane $\{s \in \mathbb{C} : \mathrm{Re} s > \sigma\}$	
$\mathcal{C}$	output map	8
C	output function	10
$\mathfrak{C}$	outgoing wave function of a discrete-time system	12
$\mathfrak{c}$	outgoing wave function of a continuous-time system	124

Symbol	Short description	Page
d	Dirichlet form	133
D	feedthrough operator	8
$\mathbb{D}$	open unit disc $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$	
$\mathbb{D}_r$	open disc $\{z \in \mathbb{C} : \lvert z\rvert < r\}$	
$\overline{\mathbb{D}}$	closed unit disc $\{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$	
$\overline{\mathbb{D}}_r$	closed disc $\{z \in \mathbb{C} : \lvert z\rvert \leq r\}$	
$\mathcal{D}$	input-output map	9
D	transfer function	11
$\mathfrak{D}$	characteristic function of a discrete-time system	13
$\mathfrak{d}$	characteristic function of a continuous-time system	124
F	component of an admissible feedback pair	33
F	feedback operator associated with the Riccati equation	49
$\mathfrak{f}$	component of an admissible feedback pair	147
F^{min}	optimal cost feedback operator	47
G	component of an admissible feedback pair	33
$\mathfrak{g}$	component of an admissible feedback pair	147
G	holomorphic function	
$\mathscr{H}$	(separable) Hilbert space	
$\mathcal{H}$	Hankel map	9
H^2	Hardy space	169
H^∞	Hardy space	169
H^s	Sobolev space (only in Chapter 12)	
H_0	Space of functions holomorphic at zero	61
$\mathcal{I}^+$	minimizing operator	45
J	cost function	43
$\mathscr{K}$	subspace of a Hilbert space	
K	admissible feedback function	81

Symbol	Short description	Page
L	strongly elliptic operator	133
L_{F}	Laurent operator of the function F	172
$\mathcal{L}(\mathscr{B}_1, \mathscr{B}_2)$	set of bounded linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$	
$\mathcal{L}(\mathscr{B})$	set of bounded linear operators from $\mathscr{B}$ to itself	
$l^2(J, \mathscr{H})$	set of square summable sequences $J \subset \mathbb{Z} \to \mathscr{H}$	
$L^2(J, \mathscr{H})$	set of square summable functions $J \subset \mathbb{R} \to \mathscr{H}$	
M	Möbius operator	140
M	component of a right factor	62
$\tilde{\mathsf{M}}$	component of a left factor	62
N	component of a right factor	62
$\tilde{\mathsf{N}}$	component of a left factor	62
q	component of a control Riccati triple	49
$q^{\min}$	optimal cost sesquilinear form	46
Q	solution of the control algebraic Riccati equation	49
$Q^{\min}$	optimal cost operator	47
$\mathbb{R}$	set of real numbers	
$\mathbb{R}^+$	set of nonnegative real numbers $\{x \in \mathbb{R} : x \geq 0\}$	
$\mathbb{R}^-$	set of negative real numbers $\{x \in \mathbb{R} : x < 0\}$	
$r(T)$	spectral radius of the operator T	
r_{A}	radius of convergence of series for A	10
r_{B}	radius of convergence of series for B	10
r_{C}	radius of convergence of series for C	10
r_{D}	radius of convergence of series for D	11
s	component of a control Riccati triple	49
S	(often) system operator	8
S	sensitivity operator associated with the Riccati equation	49

Symbol	Short description	Page
$\mathbb{T}$	unit circle $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$	
T_{F}	Toeplitz operator of the function F	172
$\mathscr{U}$	input space (a separable Hilbert space)	7
$\mathscr{V}(x_0)$	set of stable input-output pairs	44
$\mathscr{X}$	state space (a separable Hilbert space)	7
X	component of a left Bezout factor	62
$\tilde{\mathsf{X}}$	component of a right Bezout factor	62
$\mathscr{Y}$	output space (a separable Hilbert space)	7
Y	component of a left Bezout factor	62
$\tilde{\mathsf{Y}}$	component of a right Bezout factor	62
$\mathbb{Z}$	set of integers	
$\mathbb{Z}^+$	set of non-negative integers $\{n \in \mathbb{Z} : n \geq 0\}$	
$\mathbb{Z}^-$	set of negative integers $\{n \in \mathbb{Z} : n < 0\}$	
δ	gap metric	89,92
$\vec{\delta}$	directed gap	89,92
Λ	set of definition of a resolvent linear system	124
Λ_E	exponential region	126
Σ	system	
$\dagger$	$f^\dagger(s) := f(\bar{s})^*$	16
$\wedge$	$\hat{h}$ is the Z-transform or Laplace transform of h	

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