The state space isomorphism theorem for discrete-time infinite-dimensional systems

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The final publication is available at Springer via
http://dx.doi.org/10.1007/s00020-015-2251-4

Abstract

It is well-known that the state space isomorphism theorem fails in infinite-dimensional Hilbert spaces: there exist minimal discrete-time systems (with Hilbert space state spaces) which have the same impulse response, but which are not isomorphic. We consider discrete-time systems on locally convex topological vector spaces which are Hausdorff and barrelled and show that in this setting the state space isomorphism theorem does hold. In contrast to earlier work on topological vector spaces, we consider a definition of minimality based on dilations and show how this necessitates certain definitions of controllability and observability.

1 Introduction

Consider the linear discrete-time system

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_n, \\ y_n &= Cx_n + Du_n. \end{aligned} \tag{1}$$

Here $A : \mathscr{X} \to \mathscr{X}$, $B : \mathbb{C} \to \mathscr{X}$, $C : \mathscr{X} \to \mathbb{C}$, $D : \mathbb{C} \to \mathbb{C}$ are linear operators, $\mathscr{X}$ is a vector space (the *state space*) and $n \in \mathbb{N}_0$. The above equations can be solved to obtain

$$y_n = CA^n x_0 + \sum_{k=0}^{n-1} CA^k B u_{n-k-1} + Du_n.$$

Assuming that $x_0 = 0$ we obtain

$$y_n = \sum_{k=1}^{n} CA^{k-1} B u_{n-k} + Du_n.$$

The output y (for initial condition zero) is thus characterized by the input u and the *impulse response sequence*

$$\theta_k := \begin{cases} D & k = 0, \\ CA^{k-1}B & k > 0. \end{cases}$$

Different quintuples $(A, B, C, D; \mathscr{X})$ can give the same impulse response sequence. Each of these quintuples is called a *realization* of the impulse response sequence. For example, if $S : \mathscr{X}_1 \to \mathscr{X}_2$ is an invertible operator, then the quintuple $(SAS^{-1}, SB, CS^{-1}, D; \mathscr{X}_2)$ produces the same impulse response sequence as the quintuple $(A, B, C, D; \mathscr{X}_1)$. We call the quintuples $(SAS^{-1}, SB, CS^{-1}, D; \mathscr{X}_2)$ and $(A, B, C, D; \mathscr{X}_1)$ related by an invertible operator S *isomorphic*. A fundamental question is whether isomorphism captures all the possible non-uniqueness. The answer to this is clearly *no* since for a given realization $(A, B, C, D; \mathscr{X})$ a non-isomorphic realization on the space $\mathscr{X} \times \mathbb{C}$ is

$$\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D; \begin{bmatrix} \mathscr{X} \\ \mathbb{C} \end{bmatrix} \right). \tag{2}$$

Let us for the moment restrict ourselves to the situation where the impulse response sequence has a realization with a finite-dimensional state space. Then we can define a *minimal realization* as a realization whose state space has minimal dimension amongst all realizations of the same impulse response sequence. It can be shown that all minimal realizations are isomorphic. This is a fundamental result in finite-dimensional linear systems theory sometimes referred to as the *state space isomorphism theorem*.

The infinite-dimensional case introduces several complications. Typically $\mathscr{X}$ is assumed to be a Hilbert space and A, B and C are assumed to be continuous. An isomorphism is then a continuous operator with a continuous inverse. The notion of minimal realization clearly becomes problematic. What has emerged as the most satisfactory notion of minimal realization is a definition based on dilations of operators (see Definition 2). With this definition, (2) is non-minimal since its restriction to the proper subspace $\mathscr{X}$ is also a realization. However, with this notion of minimality it is no longer true that all minimal realizations are isomorphic (also known as *similar*). What can be deduced is that minimal realizations are *pseudo-similar* (allowing for both S and S^{-1} to be unbounded) [2, Section 3]. In Section 2 we re-consider an example from [2, Section 2.7] which illustrates this.

The standard dilation definition of minimality only considers Hilbert spaces as allowable state spaces. In this article we consider more generally topological vector spaces which are locally convex, Hausdorff and barrelled. We show that in this context any sequence $(\theta_k)_{k=0}^{\infty}$ has a minimal realization and that all minimal realizations of the same sequence are isomorphic. This in essence shows that the "correct" setting for studying minimal realizations is that of topological vector spaces which are locally convex, Hausdorff and barrelled rather than that of Hilbert spaces.

The assumption that a topological vector space is locally convex and Hausdorff is standard. Our assumption that state spaces have to be barrelled might at first sight seem surprising. However we show in Remark 26 that the state space isomorphism theorem isn't true if this assumption is dropped.

Intuitively, infinite-dimensional Hilbert spaces are "large" in the sense that any Hamel basis for it must be uncountable. We show in Theorem 22 that any impulse response sequence can be realized on a topological vector space which is locally convex, Hausdorff and barrelled and additionally has a *countable* Hamel basis. Therefore, intuitively, any infinite-dimensional Hilbert space realization is “too large” (it has a Hamel basis of too large cardinality) for it to be minimal.

Our results explain why minimal Hilbert space realizations may not be isomorphic: when considered in the appropriate context (of more general topological vector spaces) these realizations are no longer minimal and from this more general point of view it should therefore not be surprising that the state space isomorphism theorem doesn't hold for Hilbert space realizations.

To illustrate our results, in Section 2 we first consider an example from [2, Section 2.7]. In [2] that example was used to illustrate the failure of the state space isomorphism theorem in Hilbert spaces. We use it to indicate how considering more general topological vector spaces resolves this issue. In Section 3 we precisely define the notion of minimal realization and give an equivalent characterization in terms of controllability and observability. Section 4 deals with existence of minimal realizations and Section 5 contains the state space isomorphism theorem. Finally in Section 6 we compare our results to other results regarding realizations on topological vector spaces available in the literature. In Appendix A some results from the theory of topological vector spaces that are needed in this article are collected for easy reference.

2 An example

We consider the example from [2, Section 2.7] (but with a sequence space rather than the equivalent space of holomorphic functions as state space). The entire function

$$\hat{\theta}(z) = \mathrm{e}^{z-1},$$

has power series coefficients

$$\theta_k = \mathrm{e}^{-1}\frac{1}{k!};$$

this is the impulse response sequence that we wish to realize. Let $\mathscr{X} := \ell^2(\mathbb{N}_0)$ and for $\rho > 0$ define the operators

$$(A_\rho x)_k = \rho x_{k+1}, \quad (B_\rho u)_k = \rho^{-k}\theta_{k+1}u, \quad Cx = x_0, \quad Du = \theta_0 u.$$

It is then easily checked that for $n \in \mathbb{N}_0$

$$(A_\rho^n x)_k = \rho^n x_{k+n},$$

so that

$$(A_\rho^n B_\rho u)_k = \rho^{-k}\theta_{k+n+1}u,$$

which implies that

$$CA_\rho^n B_\rho u = \theta_{n+1}u.$$

This shows that for any $\rho > 0$ the quintuple $(A_\rho, B_\rho, C, D; \mathscr{X})$ is a realization of θ. It is shown in [2, Section 2.7] that all these realizations are minimal in the sense used in [2, Section 2.7] (in contrast to the above calculations, this does use the specific form of θ). The spectrum of A_ρ equals $\{z \in \mathbb{C} : |z| \le \rho\}$. Therefore, A_{ρ_1} and A_{ρ_2} cannot be isomorphic for $\rho_1 \neq \rho_2$.

Define the sequence b_ρ by

$$(b_\rho)_k = \rho^{-k}\theta_{k+1},$$

and consider the space $\mathscr{X}_\rho$ defined by the Hamel basis

$$(A_\rho^k b_\rho)_{k=0}^{\infty},$$

i.e.

$$\mathscr{X}_\rho := \{x : \mathbb{N}_0 \to \mathbb{C} \text{ such that } x = \sum_{k=0}^{\infty} c_k A_\rho^k b_\rho \text{ finitely many } c_k \text{ nonzero}\}.$$

Then clearly B_ρ maps into $\mathscr{X}_\rho$ and A_ρ maps $\mathscr{X}_\rho$ into itself. Therefore, by the earlier computations, $CA_\rho^n B_\rho = \theta_{n+1}$ and the quintuple $(A_\rho, B_\rho, C, D; \mathscr{X}_\rho)$ defines a realization of θ. The space $\mathscr{X}_\rho$ can be given the structure of a locally convex topological vector space which is Hausdorff and barrelled and with respect to this topology A_ρ, B_ρ and C are continuous (this follows as in the proof of Theorem 22 below). The operator $S_\rho : \mathscr{X}_\rho \to \mathscr{X}_1$

$$(S_\rho x)_k := \rho^k x_k,$$

is continuous and has a continuous inverse (this also follows as in the proof of Theorem 22 below) and satisfies

$$S_\rho A_\rho = A_1 S_\rho, \quad S_\rho B_\rho = B_1, \quad C = CS_\rho,$$

so that all the quintuples $(A_\rho, B_\rho, C, D; \mathscr{X}_\rho)$ are isomorphic. Because of the properties of θ, $\mathscr{X}_\rho$ is dense in $\ell^2(\mathbb{N}_0)$ for all $\rho > 0$ (this is proven in [2, Section 2.7] as part of the Hilbert space minimality proof). Note that if $\rho > 1$, then S_ρ doesn't extend to a continuous operator on $\mathscr{X} = \ell^2(\mathbb{N}_0)$ and if $\rho < 1$, then S_ρ^{-1} doesn't extend to a continuous operator on $\mathscr{X} = \ell^2(\mathbb{N}_0)$. This explains why the quintuples $(A_\rho, B_\rho, C, D; \mathscr{X})$ with state space $\mathscr{X} = \ell^2(\mathbb{N}_0)$ for different ρ's are not isomorphic.

Note that the minimal Hilbert space state space $\ell^2(\mathbb{N}_0)$ for this example is "larger" than any of the spaces $\mathscr{X}_\rho$. Therefore it is intuitively clear that if we allow the spaces $\mathscr{X}_\rho$ as state spaces, then $\ell^2(\mathbb{N}_0)$ is no longer a "minimal state space" for the impulse response sequence θ. Since the quintuples $(A_\rho, B_\rho, C, D; \ell^2(\mathbb{N}_0))$ are not minimal from this broader perspective, there is no intrinsic reason for them to be isomorphic.

3 Minimal systems

In this section we define the notion of minimal realization and relate it to specific notions of controllability and observability.

Definition 1. A *system* is a quintuple $(A, B, C, D; \mathscr{X})$ consisting of a vector space $\mathscr{X}$ and linear operators $A : \mathscr{X} \to \mathscr{X}$, $B : \mathbb{C} \to \mathscr{X}$, $C : \mathscr{X} \to \mathbb{C}$, $D : \mathbb{C} \to \mathbb{C}$. The vector space $\mathscr{X}$ is called the *state space* of the system.

Note that at this point we make no topological assumptions. The following definition is that of a dilation as in e.g. [2, Section 2.2]. However, note that there an orthogonality assumption was made which we don't make (since orthogonality doesn't make sense in a general vector space).

Definition 2 (Dilation)**.** Let $\Sigma := (A, B, C, D; \mathscr{X})$ and $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ be two given systems. Then $\widetilde{\Sigma}$ is called a *dilation* of Σ and Σ is called a *restriction* of $\widetilde{\Sigma}$ if $\widetilde{D} = D$ and $\widetilde{\mathscr{X}}$ has a direct sum decomposition $\widetilde{\mathscr{X}} = \mathscr{E} + \mathscr{X} + \mathscr{E}_*$ such that relative to this decomposition

$$\widetilde{A} = \begin{bmatrix} A_1 & A_3 & A_4 \\ 0 & A & A_5 \\ 0 & 0 & A_2 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_1 \\ B \\ 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 0 & C & C_1 \end{bmatrix},$$

for linear operators $A_1 : \mathscr{E} \to \mathscr{E}$, $A_2 : \mathscr{E}_* \to \mathscr{E}_*$, $A_3 : \mathscr{X} \to \mathscr{E}$, $A_4 : \mathscr{E}_* \to \mathscr{E}$, $A_5 : \mathscr{E}_* \to \mathscr{X}$, $B_1 : \mathbb{C} \to \mathscr{E}$ and $C_1 : \mathscr{E}_* \to \mathbb{C}$.

Remark 3. The Kalman decomposition of the system $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ (see e.g. [21, Theorem 3.10]) is strongly related to the above notion of dilation. In the Kalman decomposition the state space $\widetilde{\mathscr{X}}$ is written as the direct sum of four spaces: $\mathscr{X}_{co}$ (the controllable and observable subspace), $\mathscr{X}_{\bar{c}o}$ (the uncontrollable but observable subspace), $\mathscr{X}_{c\bar{o}}$ (the controllable but unobservable subspace) and $\mathscr{X}_{\bar{c}\bar{o}}$ (the uncontrollable and unobservable subspace). From this decomposition one can obtain several restrictions of the original system. One may for example choose $\mathscr{X} := \mathscr{X}_{co}$, $\mathscr{E} := \mathscr{X}_{c\bar{o}}$, $\mathscr{E}_* := \mathscr{X}_{\bar{c}o} + \mathscr{X}_{\bar{c}\bar{o}}$, but alternatively one may also choose $\mathscr{X} := \mathscr{X}_{co}$, $\mathscr{E} := \mathscr{X}_{c\bar{o}} + \mathscr{X}_{\bar{c}\bar{o}}$, $\mathscr{E}_* := \mathscr{X}_{\bar{c}o}$. The "finer" decomposition of the state space given by the Kalman decomposition turns out not to be needed for our purposes, the concept of dilation from Definition 2 is sufficient.

Two concepts related to the above definition of dilation are now defined. Also these concepts can be seen from the point of view of the Kalman decomposition (see e.g. [21, Theorem 3.8 and Corollary 3.9]).

Definition 4 (C-dilation)**.** Let $\Sigma := (A, B, C, D; \mathscr{X})$ and $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ be two given systems. Then $\widetilde{\Sigma}$ is called a *C-dilation* of Σ and Σ is called a *C-restriction* of $\widetilde{\Sigma}$ if $\widetilde{D} = D$ and $\widetilde{\mathscr{X}}$ has a direct sum decomposition $\widetilde{\mathscr{X}} = \mathscr{X} + \mathscr{E}_*$ such that relative to this decomposition

$$\widetilde{A} = \begin{bmatrix} A & A_5 \\ 0 & A_2 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & C_1 \end{bmatrix},$$

for linear operators $A_2 : \mathscr{E}_* \to \mathscr{E}_*$, $A_5 : \mathscr{E}_* \to \mathscr{X}$ and $C_1 : \mathscr{E}_* \to \mathbb{C}$.

Definition 5 (O-dilation)**.** Let $\Sigma := (A, B, C, D; \mathscr{X})$ and $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ be two given systems. Then $\widetilde{\Sigma}$ is called an *O-dilation* of Σ and Σ is called an *O-restriction* of $\widetilde{\Sigma}$ if $\widetilde{D} = D$ and $\widetilde{\mathscr{X}}$ has a direct sum decomposition $\widetilde{\mathscr{X}} = \mathscr{E} \dot{+} \mathscr{X}$ such that relative to this decomposition

$$\widetilde{A} = \begin{bmatrix} A_1 & A_3 \\ 0 & A \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_1 \\ B \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 0 & C \end{bmatrix},$$

for linear operators $A_1 : \mathscr{E} \to \mathscr{E}$, $A_3 : \mathscr{X} \to \mathscr{E}$, and $B_1 : \mathbb{C} \to \mathscr{E}$.

Definition 6. A restriction (in the sense of Definition 2) is called *proper* if either $\mathscr{E} \neq \{0\}$ or $\mathscr{E}_* \neq \{0\}$ (or both). A C-restriction is called *proper* if $\mathscr{E}_* \neq \{0\}$ and an O-restriction is called *proper* if $\mathscr{E} \neq \{0\}$.

Definition 7. A system is called *minimal* if it does not have a proper restriction.

Definition 8. The *impulse response sequence* of a system $(A, B, C, D; \mathscr{X})$ is the sequence

$$\theta : \mathbb{N}_0 \to \mathbb{C}, \quad \theta_k := \begin{cases} D1 & k = 0, \\ CA^{k-1}B1 & k > 0. \end{cases}$$

The following fact is easily verified:

- A system and any of its dilations, C-dilations and O-dilations have the same impulse response sequence.

Definition 9. The *controllable subspace* of the system $(A, B, C, D; \mathscr{X})$ is (here $b := B1$)

$$\left\{ z \in \mathscr{X} : z = \sum_{k=0}^{\infty} c_k A^k b \text{ finitely many } c_k \text{ nonzero} \right\}.$$

The system is called *controllable* if the controllable subspace equals $\mathscr{X}$.

The *unobservable subspace* is

$$\bigcap_{n=0}^{\infty} \ker(CA^n).$$

The system is called *observable* if the unobservable subspace equals $\{0\}$.

Note that in terms of the dynamical system (1), the controllable subspace is the space of states reachable from $x_0 = 0$ in a finite time by applying a control u. The unobservable subspace is the set of initial states x_0 which, for zero input u, give as output y the zero sequence.

Note further that our definition of controllable subspace and controllable system is different from that in [2, Section 2.3]: in [2] the controllable subspace is defined to be the closure of what we define as the controllable subspace and they use the term *reachable manifold* for what we call the controllable subspace.

Lemma 10. *A system is controllable if and only if it has no proper C-restriction.*

Proof. We prove the contrapositive: the system $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ has a proper C-restriction if and only if it is not controllable.

First assume that $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ has a proper C-restriction. We use the notation from Definition 4. We have

$$\widetilde{A}^k \widetilde{B} = \begin{bmatrix} A^k B \\ 0 \end{bmatrix},$$

so that no nonzero element of $\mathscr{E}_*$ is in the controllable subspace. Since $\mathscr{E}_* \neq \{0\}$, it follows that the controllable subspace doesn't equal $\widetilde{\mathscr{X}}$. Therefore, $\widetilde{\Sigma}$ is not controllable.

Now assume that $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ is not controllable. Denote its controllable subspace by $\mathscr{X}$. Then there exists a vector space $\mathscr{E}_*$ such that $\widetilde{\mathscr{X}}$ is the direct sum of $\mathscr{X}$ and $\mathscr{E}_*$ (this follows from Zorn's Lemma). Since $\widetilde{\Sigma}$ is not controllable, $\mathscr{E}_*$ is nonzero. The operators decompose as follows with respect to the above direct sum:

$$\widetilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

for linear operators $A_{11} : \mathscr{X} \to \mathscr{X}$, $A_{12} : \mathscr{E}_* \to \mathscr{X}$, $A_{21} : \mathscr{X} \to \mathscr{E}_*$, $A_{22} : \mathscr{E}_* \to \mathscr{E}_*$, $B_1 : \mathbb{C} \to \mathscr{X}$, $B_2 : \mathbb{C} \to \mathscr{E}_*$, $C_1 : \mathscr{X} \to \mathbb{C}$, $C_2 : \mathscr{E}_* \to \mathbb{C}$. Since $\widetilde{B}$ maps into the controllable subspace, we have $B_2 = 0$. Similarly, since $\widetilde{A}$ maps the controllable subspace into itself we have $A_{21} = 0$. It follows that $(A_{11}, B_1, C_1, \widetilde{D}; \mathscr{X})$ is a proper C-restriction of $\widetilde{\Sigma}$. □

Lemma 11. *A system is observable if and only if it has no proper O-restriction.*

Proof. We prove the contrapositive: the system $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ has a proper O-restriction if and only if it is not observable.

First assume that $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ has a proper O-restriction. We use the notation from Definition 5. We have

$$\widetilde{C}\widetilde{A}^k = \begin{bmatrix} 0 & CA^k \end{bmatrix},$$

from which it follows that $\mathscr{E}$ is contained in the unobservable subspace. Since $\mathscr{E}$ is nonzero it follows that the unobservable subspace is nonzero. Therefore, $\widetilde{\Sigma}$ is not observable.

Now assume that $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ is not observable. Denote the unobservable subspace by $\mathscr{E}$. Then there exists a vector space $\mathscr{X}$ such that $\widetilde{\mathscr{X}}$ is the direct sum of $\mathscr{E}$ and $\mathscr{X}$ (this follows from Zorn's Lemma). The operators decompose as follows with respect to the above direct sum:

$$\widetilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

for linear operators $A_{11} : \mathscr{E} \to \mathscr{E}$, $A_{12} : \mathscr{X} \to \mathscr{E}$, $A_{21} : \mathscr{E} \to \mathscr{X}$, $A_{22} : \mathscr{X} \to \mathscr{X}$, $B_1 : \mathbb{C} \to \mathscr{E}$, $B_2 : \mathbb{C} \to \mathscr{X}$, $C_1 : \mathscr{E} \to \mathbb{C}$, $C_2 : \mathscr{X} \to \mathbb{C}$. Since $\mathscr{E}$ is contained in the kernel of $\widetilde{C}$ we have $C_1 = 0$. Similarly, since $\widetilde{A}$ maps the unobservable subspace into itself we have $A_{12} = 0$. It follows that $(A_{22}, B_2, C_2, \widetilde{D}; \mathscr{X})$ is a proper O-restriction of $\widetilde{\Sigma}$. □

Theorem 12. *A system is minimal if and only if it is both controllable and observable.*

Proof. A system has a proper restriction if and only if it has either a proper C-restriction or a proper O-restriction (or both). The result then follows from Lemmas 10 and 11. □

Remark 13. Note that Theorem 12 at first sight seems to be the same as [2, Theorem 2.1]. However our definition of controllable is different from that in [2] and so is our definition of minimal (since our definition of dilation is different).

Lemma 14. *Let* $\Sigma = (A, B, C, D; \mathscr{X})$ *be a controllable system. Denote* $b := B1$. *Then either* $\dim \mathscr{X} < \infty$ *and* $(A^k b)_{k=0}^{\dim \mathscr{X} -1}$ *is a Hamel basis for* $\mathscr{X}$ *or* $\dim \mathscr{X} = \infty$ *and* $(A^k b)_{k=0}^{\infty}$ *is a Hamel basis for* $\mathscr{X}$.

Proof. We only consider the case $\dim \mathscr{X} = \infty$ (the finite-dimensional case can be proven similarly). Since the system is controllable, $\mathscr{X}$ equals the controllable subspace and clearly the Hamel-span of $(A^k b)_{k=0}^{\infty}$ equals this controllable subspace. It remains to show that the vectors in the set $\{A^k b : k = 0, \ldots, \infty\}$ are (Hamel-) linearly independent. Assume that they are not; then there exist a $m \in \mathbb{N}$ and $(c_k)_{k=0}^{m-1} \subset \mathbb{C}$ such that

$$A^m b = \sum_{k=0}^{m-1} c_k A^k b.$$

It follows that any vector $A^n b$ for $n \in \mathbb{N}$ with $n \geq m$ can be written as a linear combination of $\{A^k b : k = 0, \ldots, m-1\}$. This contradicts infinite-dimensionality of $\mathscr{X}$. □

3.1 The Hankel operator

The following definition uses the space

$$c_{00} := \{v : \mathbb{N}_0 \to \mathbb{C} : \text{ finitely many } v_k \text{ nonzero}\}.$$

Definition 15. The *controllability map* $\mathcal{C} : c_{00} \to \mathscr{X}$ of the system $(A, B, C, D; \mathscr{X})$ is defined by

$$\mathcal{C} v := \sum_{k=0}^{\infty} A^k B v_k.$$

Note that a system is controllable if and only if its controllability map is onto.

In the following definition we use the notation $\mathbb{C}^{\mathbb{N}_0}$ for the space of all sequences from $\mathbb{N}_0$ to $\mathbb{C}$.

Definition 16. The *observability map* $\mathcal{O} : \mathscr{X} \to \mathbb{C}^{\mathbb{N}_0}$ of the system $(A, B, C, D; \mathscr{X})$ is defined by

$$(\mathcal{O}x)_k := CA^k x.$$

Note that a system is observable if and only if its observability map is injective.

Definition 17. The *Hankel operator* $\mathcal{H} : c_{00} \to \mathbb{C}^{\mathbb{N}_0}$ of the system $(A, B, C, D; \mathscr{X})$ is defined by $\mathcal{H} = \mathcal{O}\mathcal{C}$, where $\mathcal{C}$ is the controllability map and $\mathcal{O}$ is the observability map of the system.

Remark 18. Note that with $(\theta_k)_{k=0}^{\infty}$ the impulse response sequence of the system we have

$$(\mathcal{H}v)_k = \sum_{j=0}^{\infty} \theta_{k+j+1} v_j.$$

Therefore the impulse response sequence uniquely determines the Hankel operator and the Hankel operator together with θ_0 uniquely determines the impulse response sequence.

We note that it follows from Lemma 14 that the state space of a controllable system is either finite-dimensional or its dimension is countably infinite. Therefore, the dimension of a controllable system is a well-defined element of $\mathbb{N}_0 \cup \{\infty\}$ where ∞ is used to denote that the dimension is countably infinite. Also, the dimension of the range of the Hankel operator is either finite-dimensional or its dimension is countably infinite. Therefore, we can use the same convention for its dimension.

Lemma 19. *Let* $\Sigma = (A, B, C, D; \mathscr{X})$ *be a minimal system with Hankel operator* $\mathcal{H}$*. Then* $\dim \mathscr{X} = \dim \operatorname{ran}(\mathcal{H})$*.*

Proof. We use that by Theorem 12 minimality is equivalent to controllability plus observability. Since $\mathcal{H} = \mathcal{O}\mathcal{C}$ and the observability map $\mathcal{O}$ is injective by observability, $\dim \operatorname{ran}(\mathcal{H}) = \dim \operatorname{ran}(\mathcal{C})$. From Lemma 14 we see that $\dim \operatorname{ran}(\mathcal{C})$ equals $\dim \mathscr{X}$ by controllability. □

Corollary 20. *Let* $\Sigma = (A, B, C, D; \mathscr{X})$ *and* $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ *be minimal systems with the same impulse response sequence. Then* $\dim \mathscr{X} = \dim \widetilde{\mathscr{X}}$*.*

Proof. By Remark 18 the two systems have the same Hankel operator. Therefore by Lemma 19, $\dim \mathscr{X} = \dim \operatorname{ran}(\mathcal{H}) = \dim \widetilde{\mathscr{X}}$. □

4 Existence of minimal realizations

Definition 21. A *topological system* is a quintuple $(A, B, C, D; \mathscr{X})$ consisting of a topological vector space $\mathscr{X}$ and continuous linear operators $A : \mathscr{X} \to \mathscr{X}$, $B : \mathbb{C} \to \mathscr{X}$, $C : \mathscr{X} \to \mathbb{C}$, $D : \mathbb{C} \to \mathbb{C}$ where $\mathbb{C}$ is equipped with its usual topology.

Note that a topological system is a system according to Definition 1, but that the converse is not true. When we refer to a *minimal topological system* we mean a topological system which after applying the "forgetful functor" (i.e. we "forget" that there is a topology) is minimal in the sense of Definition 7 (in particular: the considered restrictions do not have to be topological systems).

The following is one of our two main theorems. We note that the definition of strict LF-space is recalled in the appendix (Definition 28).

Theorem 22. *Every sequence $(\theta_k)_{k=0}^\infty$ is the impulse response sequence of some minimal topological system. Moreover, the state space can be taken to be a strict LF-space with a countable Hamel basis.*

Proof. Define $D : \mathbb{C} \to \mathbb{C}$ by $Du = \theta_0 u$. Define the space $\widetilde{\mathscr{X}}$ as the space of all sequences $\mathbb{N}_0 \to \mathbb{C}$ and define the operators $\widetilde{A} : \widetilde{\mathscr{X}} \to \widetilde{\mathscr{X}}$, $\widetilde{B} : \mathbb{C} \to \widetilde{\mathscr{X}}$ and $\widetilde{C} : \widetilde{\mathscr{X}} \to \mathbb{C}$ by

$$(\widetilde{A}x)_k := x_{k+1}, \quad (\widetilde{B}u)_k := \theta_{k+1}u, \quad \widetilde{C}x := x_0.$$

Then as in Section 2, we see that the system $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, D; \widetilde{\mathscr{X}})$ has $(\theta_k)_{k=0}^\infty$ as its impulse response sequence. We note that $\widetilde{\Sigma}$ is observable since $x \in \ker(CA^n)$ implies $x_n = 0$ and therefore $x \in \cap_{n=0}^\infty \ker(CA^n)$ implies $x = 0$. Let $\mathscr{X}$ be the controllable subspace of $\widetilde{\Sigma}$. Since $\widetilde{A}$ maps the controllable subspace to itself, we can define $A : \mathscr{X} \to \mathscr{X}$ as the restriction of $\widetilde{A}$ to $\mathscr{X}$. Since $\widetilde{B}$ maps into the controllable subspace, we can define $B : \mathbb{C} \to \mathscr{X}$ as $\widetilde{B}$ seen as an operator to $\mathscr{X}$. We further define C as the restriction of $\widetilde{C}$ to $\mathscr{X}$. It is trivial that $\Sigma := (A, B, C, D; \mathscr{X})$ also has $(\theta_k)_{k=0}^\infty$ as its impulse response sequence. By construction, Σ is controllable. Since $\widetilde{\Sigma}$ is observable we see that Σ is observable. Therefore, by Theorem 12, the system Σ is minimal.

It remains to show that $\mathscr{X}$ can be made into a topological vector space in such a way as to make Σ a topological system. Note that $\mathscr{X}$ is the strict inductive limit of the finite-dimensional spaces (here as before $b := B1$)

$$\mathscr{X}_n := \left\{ \sum_{k=0}^{n-1} c_k A^k b : c_k \in \mathbb{C} \right\}.$$

Therefore, we can equip $\mathscr{X}$ with the inductive limit topology making it into a strict LF-space (Definition 28). The space $\mathscr{X}$ clearly has the countable Hamel basis $(A^k b)_{k=0}^\infty$ (or $(A^k b)_{k=0}^{\dim \mathscr{X} - 1}$ if $\dim \mathscr{X} < \infty$). By Theorem 32 (from the appendix) it follows that A and C are continuous. By Lemma 31 (from the appendix), B is continuous. $\square$

5 The state space isomorphism theorem

Definition 23. Two topological systems $(A, B, C, D; \mathscr{X})$ and $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ are called *topologically isomorphic* if there exists a continuous operator $S : \mathscr{X} \to$

$\widetilde{\mathscr{X}}$ with a continuous inverse such that

$$\widetilde{A}S = SA, \quad \widetilde{B} = SB \quad \widetilde{C}S = C, \quad \widetilde{D} = D. \tag{3}$$

The following is our second main theorem. We note that the definition of a barrelled space is recalled in the appendix (Definition 36).

Theorem 24. *Let $\Sigma = (A, B, C, D; \mathscr{X})$ and $\widetilde{\Sigma} := (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathscr{X}})$ be two minimal topological systems with $\mathscr{X}$ and $\widetilde{\mathscr{X}}$ locally convex topological vector spaces which are additionally Hausdorff and barrelled. If Σ and $\widetilde{\Sigma}$ have the same impulse response sequence, then Σ and $\widetilde{\Sigma}$ are topologically isomorphic.*

Proof. By Corollary 20 we have $\dim \mathscr{X} = \dim \widetilde{\mathscr{X}}$. We consider only the case where this dimension is infinite (the case where it is finite is similar). By Theorem 12, Σ and $\widetilde{\Sigma}$ are controllable. Therefore, by Lemma 14, $\mathscr{X}$ has Hamel basis $(A^k b)_{k=0}^{\infty}$ and $\widetilde{\mathscr{X}}$ has Hamel basis $(\widetilde{A}^k \widetilde{b})_{k=0}^{\infty}$ where $b := B1$ and $\widetilde{b} := \widetilde{B}1$. Define $S : \mathscr{X} \to \widetilde{\mathscr{X}}$ by

$$S\left(\sum_{k=0}^{\infty} c_k A^k b\right) = \sum_{k=0}^{\infty} c_k \widetilde{A}^k \widetilde{b}, \tag{4}$$

where the sequence of coefficients $(c_k)_{k=0}^{\infty}$ is finitely nonzero. By the defining properties of a Hamel basis, S is well-defined and bijective.

We show that the isomorphism equations (3) are satisfied. We denote the impulse response sequence by $(\theta_k)_{k=0}^{\infty}$. We have $D1 = \theta_0 = \widetilde{D}1$, so that $D = \widetilde{D}$. From (4) with $c_0 = 1$ and $c_k = 0$ for $k > 0$ we obtain $Sb = \widetilde{b}$, which implies $SB = \widetilde{B}$. Let x be an arbitrary element of $\mathscr{X}$. Then $x = \sum_{k=0}^{\infty} x_k A^k b$ for $(x_k)_{k=0}^{\infty}$ a finitely nonzero sequence. Therefore

$$\begin{aligned} SAx = S\left(\sum_{k=0}^{\infty} x_k A^{k+1} b\right) &= S\left(\sum_{j=1}^{\infty} x_{j-1} A^j b\right) \\ &= \sum_{j=1}^{\infty} x_{j-1} \widetilde{A}^j \widetilde{b} = \sum_{k=0}^{\infty} x_k \widetilde{A}^{k+1} \widetilde{b} = \widetilde{A} \sum_{k=0}^{\infty} x_k \widetilde{A}^k \widetilde{b} = \widetilde{A}Sx. \end{aligned}$$

We conclude that $SA = \widetilde{A}S$. Finally, we have for x as above

$$Cx = \sum_{k=0}^{\infty} x_k C A^k b = \sum_{k=0}^{\infty} x_k \theta_{k+1} = \sum_{k=0}^{\infty} x_k \widetilde{C} \widetilde{A}^k \widetilde{b} = \widetilde{C} \sum_{k=0}^{\infty} x_k \widetilde{A}^k \widetilde{b} = \widetilde{C}Sx.$$

Therefore $C = \widetilde{C}S$. Hence the isomorphism equations (3) are satisfied.

It remains to verify that S and its inverse are continuous. By assumption $\mathscr{X}$ is a locally convex topological vector space which is additionally Hausdorff and barrelled. We saw above that controllability implies that $\mathscr{X}$ has a countable Hamel basis. By Theorem 37 (from the appendix) we see that $\mathscr{X}$ is a strict LF-space with a countable Hamel basis. It follows from Theorem 32 (from the appendix)that S is continuous. We similarly have that $\widetilde{\mathscr{X}}$ is a strict LF-space with a countable Hamel basis. Therefore it follows from Theorem 32 (from the appendix) that S^{-1} is continuous. □

Remark 25. We note that the first part of the proof of Theorem 24 shows that two minimal systems are *algebraically isomorphic.*

Remark 26. It is not true that two minimal topological systems which have the same impulse response are topologically isomorphic (i.e. we cannot omit the additional assumptions on the state spaces made in Theorem 24). Consider the minimal realization constructed in the proof of Theorem 22. However, now equip $\mathscr{X}$ with the topology induced by $\mathbb{C}^{\mathbb{N}_0}$ (which carries its natural Fréchet space topology). If the impulse response sequence is such that $\dim \mathscr{X} = \infty$ (for example: the impulse response sequence from Section 2), then $\mathscr{X}$ is not complete (since otherwise it would be an infinite dimensional Fréchet space with a countable Hamel basis, which does not exist). Therefore $\mathscr{X}$ with this induced topology is not topologically isomorphic to the state space constructed in the proof of Theorem 22 (which as a strict LF-space is complete). Hence we have constructed two minimal topological systems which have the same impulse response and which are not topologically isomorphic.

Remark 27. The natural *dual* of a quintuple $(A, B, C, D; \mathscr{X})$ is the quintuple $(A', C', B', D'; \mathscr{X}')$. We note that (in our context of locally convex topological vector spaces which are additionally Hausdorff and barrelled) for a minimal quintuple its dual is minimal if and only if $\mathscr{X}$ is finite-dimensional. One direction of this is clear: if $\mathscr{X}$ is finite-dimensional and the quintuple is minimal, then so is its dual. By our results, if $(A, B, C, D; \mathscr{X})$ is minimal and $\mathscr{X}$ is a locally convex topological vector space which is additionally Hausdorff and barrelled, then $\mathscr{X}$ is in fact a strict LF-space with a countable Hamel basis. If $\mathscr{X}$ is infinite-dimensional, it is therefore isomorphic to c_{00}. It follows that in this case $\mathscr{X}'$ is isomorphic to $\mathbb{C}^{\mathbb{N}_0}$. This implies that $\mathscr{X}'$ does not have a countable Hamel basis and therefore that $(A', C', B', D'; \mathscr{X}')$ is not minimal.

6 Conclusion and comparison with the literature

We note that if the impulse response sequence has values which are linear operators between finite-dimensional spaces, i.e. $\theta_k : \mathbb{C}^m \to \mathbb{C}^p$ for $m, p \in \mathbb{N}$ independent of k, then the situation is entirely similar to the scalar case described in this article. Since the notation becomes slightly more cumbersome in this operator-valued situation, we have however chosen to only present the scalar case. It is also possible to consider impulse response sequences with $\theta_k : \mathscr{U} \to \mathscr{Y}$ where $\mathscr{U}$ and $\mathscr{Y}$ are vector spaces independent of k. To obtain a topological isomorphism result, some assumptions have to be made about the topology of $\mathscr{U}$ and $\mathscr{Y}$. In contrast to the finite-dimensional case, this situation is more complicated than the scalar case discussed in this article and may be the subject of future work.

Most of the literature on realization theory for infinite-dimensional systems deals with the notion of a *canonical* system (i.e. a system which is both controllable and observable) rather than that of a minimal system (in the sense of our Definition 7). There are however different natural notions of controllability and observability. We considered the controllability map (Definition 15) to have domain c_{00}, but other choices are possible, for example $\ell^2(\mathbb{N}_0)$ (provided that the system is "stable" so that the infinite sum defining the controllability map converges for all $v \in \ell^2$). If we define a system to be controllable if its controllability map is onto, then these different choices of the domain of the controllability map lead to different notions of controllability. Another natural notion of controllability is for the controllability map to have dense range rather than to be onto. Again, this leads to a different notion of controllability. Similar remarks can be made regarding observability. With certain particular pairs of choices of controllability and observability, it can be shown that canonical realizations are unique up to topological isomorphism. However, systems which are canonical in different senses are generally not topologically isomorphic. In the context of continuous-time systems, these issues were investigated in [8]. Also in the context of continuous-time systems, [20] argued for a particular choice (on engineering grounds).

We took the notion of minimal (in the sense of our Definition 7) as the central notion. As mentioned, this notion originates in Hilbert space operator theory and is also closely connected to the Kalman decomposition. Theorem 12 then shows that the choices of controllability and observability that we made are the correct ones (in our context). We note that these notions of controllability and observability are the same as those of for example [10, Chapter 10].

The key role played by barrelled spaces in canonical realization theory was partially recognized in [8, Lemma 4.3 and Theorem 4.6]. A result very closely related to our main results (Theorems 22 and 24) is mentioned in [1, page 206]. However, for details [1, page 206] refers to a book ([13] in our bibliography) in which we couldn't find them. We further note that the result stated in [1, page 206] concerns canonical realizations rather than minimal realizations. The importance of barrelled spaces is also recognized in [19] where, as in this article, the failure of the state space isomorphism theorem in Hilbert spaces is investigated by considering barrelled topological vector spaces. However, the viewpoint of [19] is very different from that given here.

We now briefly discuss some known conditions from the literature under which the Hilbert space state space isomorphism theorem does hold. The quintuple $(A, B, C, D; \mathscr{X})$ where $\mathscr{X}$ is a Hilbert space is called *scattering passive* if $\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq 1$. A sequence $(\theta_k)_{k=0}^{\infty}$ is called *Schur* if the function $s(z) := \sum_{k=0}^{\infty} \theta_k z^k$ is holomorphic on the open unit disc and satisfies $|s(z)| \leq 1$ there. It is well-known that the impulse response sequence of a scattering passive quintuple is Schur and that conversely, every Schur sequence has a minimal scattering passive realization. If some additional assumptions are made on the Schur sequence, then all its minimal scattering passive realizations are unitarily similar, see [3, 4]. We remark that scattering passivity is naturally a Banach space assumption and it is this additionally imposed structure which fundamentally distinguishes this case from the case where the quintuple is only assumed to be continuous (which is naturally a topological vector space assumption).

Finally we remark that although the example mentioned in Section 2 originates in [2], there are earlier examples (as acknowledged in [2]) where the state space isomorphism theorem fails in the Hilbert space context [6, Example 2.5].

A Some results from the theory of topological vector spaces

In this appendix we collect some results from the theory of topological vector spaces that are needed in this article. Standard references for this material include [5, 7, 11, 12, 14, 16, 17, 18].

We first recall the notion of a strict LF-space.

Definition 28. [17, page 126] Let $\mathscr{V}$ be a vector space and let $(\mathscr{V}_k)_{k=1}^{\infty}$ be a sequence of subspaces with the following properties:

- $\mathscr{V}_k \subset \mathscr{V}_{k+1}$ for all $k \in \mathbb{N}$,
- $\bigcup_{k=1}^{\infty} \mathscr{V}_k = \mathscr{V}$,
- Each $\mathscr{V}_k$ is a Fréchet space and the topology induced on $\mathscr{V}_k$ by $\mathscr{V}_{k+1}$ equals the original topology of $\mathscr{V}_k$.

Then we call $\mathscr{V}$ a *strict LF-space* and the sequence $(\mathscr{V}_k)_{k=1}^{\infty}$ a *defining sequence* for $\mathscr{V}$.

The topology of a strict LF-space is defined as follows: a convex subset U is a neighbourhood of zero if $U \cap \mathscr{V}_k$ is a neighbourhood of zero in $\mathscr{V}_k$ for all $k \in \mathbb{N}$.

We note that strict LF-spaces are complete [17, Theorem 13.1], Hausdorff and locally convex [17, p126].

Lemma 29. *Let $\mathscr{V}$ be a strict LF-space with defining sequence $(\mathscr{V}_k)_{k=1}^{\infty}$. If $\mathscr{V}$ has a countable Hamel basis, then $\mathscr{V}_k$ is finite-dimensional for all $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$. Since $\mathscr{V}$ has a countable Hamel basis and $\mathscr{V}_k \subset \mathscr{V}$, it follows that $\mathscr{V}_k$ has a countable Hamel basis. Since $\mathscr{V}_k$ is a Fréchet space, this implies that $\mathscr{V}_k$ is finite-dimensional [15, Chapter 2 Exercise 1]. □

Lemma 30. *[17, Proposition 13.1] Let $\mathscr{V}$ be a strict LF-space with defining sequence $(\mathscr{V}_k)_{k=1}^{\infty}$, let $\mathscr{W}$ be a locally convex topological vector space and let $T : \mathscr{V} \to \mathscr{W}$ be a linear operator. Then T is continuous if and only if for all $k \in \mathbb{N}$ the operator $T|_{\mathscr{V}_k} : \mathscr{V}_k \to \mathscr{W}$ is continuous.*

Lemma 31. *[15, Lemma 1.20] Every linear operator from a finite-dimensional Hausdorff topological vector space into a locally convex topological vector space is continuous.*

Theorem 32. *Let $\mathscr{V}$ be a strict LF-space with a countable Hamel basis, let $\mathscr{W}$ be a locally convex topological vector space and let $T : \mathscr{V} \to \mathscr{W}$ be a linear operator. Then T is continuous.*

Proof. Let $(\mathscr{V}_k)_{k=1}^{\infty}$ be a defining sequence for $\mathscr{V}$. By Lemma 29, $\mathscr{V}_k$ is finite-dimensional for all $k \in \mathbb{N}$. Hence by Lemma 31, $T|_{\mathscr{V}_k} : \mathscr{V}_k \to \mathscr{W}$ is continuous for all $k \in \mathbb{N}$. It then follows from Lemma 30 that $T : \mathscr{V} \to \mathscr{W}$ is continuous. □

We recall the definition of a barrelled space.

Definition 33. [17, Definition 3.2] A subset S of a vector space $\mathscr{X}$ is called *balanced* if for every $x \in S$ and every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ we have $\lambda x \in S$.

Definition 34. [17, Definition 3.1] A subset S of a vector space $\mathscr{X}$ is called *absorbing* if for every $x \in \mathscr{X}$ there exists a $c > 0$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq c$ we have $\lambda x \in S$.

Definition 35. [17, Definition 7.1] A subset S of a topological vector space $\mathscr{X}$ is called a *barrel* if it is absorbing, balanced, closed and convex.

Definition 36. [17, Definition 33.1] A topological vector space $\mathscr{X}$ is called *barrelled* if every barrel in $\mathscr{X}$ is a neighbourhood of zero in $\mathscr{X}$.

We note that Hilbert spaces, Banach spaces, Fréchet spaces and strict LF-spaces are barrelled [17, page 347].

Theorem 37. *A locally convex topological vector space which is Hausdorff and barrelled and has a countable Hamel basis is a strict LF-space.*

Proof. Let $\mathscr{X}$ be a strict LF-space with a countable Hamel basis and let $\mathscr{Y}$ be a locally convex topological vector space which is Hausdorff and barrelled and has a countable Hamel basis. As in the proof of Theorem 24, we obtain a bijective map $T : \mathscr{X} \to \mathscr{Y}$ by mapping one Hamel basis to the other. By Theorem 32, T is continuous. A strict LF-space with a countable Hamel basis is fully complete (a.k.a. B-complete or a Ptak space); this follows e.g. from [9, Corollary 9.1] combined with Lemma 29. Therefore the open mapping theorem implies that S^{-1} is continuous; see e.g. [14, Corollary 2 page 116]. Hence $\mathscr{Y}$ is topologically isomorphic to $\mathscr{X}$ and is therefore a strict LF-space. □

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