Bounded real and positive real balanced truncation for infinite-dimensional systems

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Abstract

Bounded real balanced truncation for infinite-dimensional systems is considered. This provides reduced order finite-dimensional systems that retain bounded realness. We obtain an error bound analogous to the finite-dimensional case in terms of the bounded real singular values. By using the Cayley transform a gap metric error bound for positive real balanced truncation is subsequently obtained. For a class of systems with an analytic semigroup, we show rapid decay of the bounded real and positive real singular values. Together with the established error bounds, this proves rapid convergence of the bounded real and positive real balanced truncations.

1 Introduction

In model reduction the aim is to approximate a system with many degrees of freedom by a system with few degrees of freedom. In this article we are interested in the case where the original system has infinitely many degrees of freedom. Examples of such systems are systems described by partial differential equations or delay differential equations.

Approximation of controlled partial differential equations by standard numerical methods such as finite elements often gives results that are far from optimal [20]. A rigorous verification of this observation depends on two things: 1) an error analysis of these standard numerical methods and 2) determining what the optimal approximation results (approximately) are. Lyapunov balanced truncations are close to optimal approximations and are therefore important in rigorously verifying the above fundamental observation.

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Lyapunov balanced truncation was introduced for finite-dimensional systems by Moore [18] and a crucial aspect is the error bound

$$\|G - G_n\|_\infty \leq 2 \sum_{k=n+1}^{N} \sigma_k, \tag{1}$$

which was independently derived by Enns [7] and Glover [8]. In (1), σ_k are the singular values of the Hankel operator of the system and N and n are the orders of the original and truncated systems respectively.

The theory of Lyapunov balanced truncation has been extended to the infinite-dimensional case by Glover, Curtain and Partington [9]. Some assumptions made there were proven to be redundant in [12]. The upshot is that the balanced truncation error-bound (1) (now with $N = \infty$) continues to hold in the infinite-dimensional case. There is also the trivial lower bound

$$\sigma_{n+1} \leq \|G - G_n\|_\infty,$$

which holds for any reduced order system of dimension n. Combined these bounds show that Lyapunov balanced truncation is indeed close to optimal. An analysis of the singular values of Hankel operators shows that in many applications these singular values converge to zero at a rate faster than any polynomial rate (whether the rate is in fact exponential is –for partial differential equation examples– an open problem) [20, 21]. This implies that Lyapunov balanced truncations in these applications converge at a rate faster than any polynomial rate. Standard numerical methods such as finite elements do not converge as fast in these applications (so-called higher order methods are in fact not higher order for these applications because of lack of smoothness). See [20] and also the example in Section 8.

A downside of Lyapunov balanced truncation is that in general any energy relation in the original system is not necessarily retained in the reduced order system. In the finite-dimensional case the alternative methods called bounded real [19] and positive real balanced truncation [6] do retain such an energy relation. In this article we generalise these methods to the infinite-dimensional case. In particular, we prove the corresponding error bounds. For a special class of systems we also provide an analysis of the singular values involved. We conclude that for a large class of systems there exist approximations that preserve the relevant energy relation and that converge much faster than those provided by the standard numerical methods. This is illustrated by the numerical example of a boundary controlled heat equation in Section 8.

1.1 Statements of main results

There are two classical notions of dissipativity in control theory: on the one hand there are the systems called impedance passive, passive or positive real and on the other hand there are the systems called scattering passive, contractive or bounded real.

Our first main result considers the bounded real case. Precise definitions of the notions involved are given later in this article.

Theorem 1.1. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ be a strictly bounded real function with summable bounded real singular values and with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Then for each nonnegative integer n there exists a bounded real rational function of McMillan degree $\leq n$, denoted G_n and called the reduced order transfer function obtained by bounded real balanced truncation, such that*

$$\|G - G_n\|_{H^\infty} \leq 2 \sum_{k \geq n+1} \sigma_k, \tag{2}$$

where σ_k are the bounded real singular values.

Our second main result considers the positive real case. Again, precise definitions of the notions involved are given later in this article.

Theorem 1.2. *Let $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ be a strictly positive real function with summable positive real singular values and with $\mathscr{U}$ finite-dimensional. Then for each nonnegative integer n there exists a positive real rational function of McMillan degree $\leq n$, denoted J_n and called the reduced order transfer function obtained by positive real balanced truncation, such that*

$$\hat{\delta}(J, J_n) \leq 2 \sum_{k \geq n+1} \sigma_k, \tag{3}$$

where $\hat{\delta}$ is the gap metric [16, p.197, p.201] and σ_k are the positive real singular values.

1.2 Organisation of the article

In Section 2 we review bounded real and positive real balanced truncation in the finite-dimensional case. We do this so that we can highlight some features that are essential in the infinite-dimensional case, but typically are not given much prominence in the finite-dimensional case. Section 3 summarises the aspects of well-posed linear systems, optimal control and spectral factor systems that are needed to prove the main results of this article. Section 4 is the technical heart: here we construct the bounded real balanced truncation and prove Theorem 1.1. These results are converted via the Cayley transform, discussed in Section 5, to the positive real case in Section 6, which contains the proof of Theorem 1.2. In Section 7 we discuss the asymptotic behavior of the bounded real and positive singular values for a class of systems. Finally, Section 8 contains the already mentioned specific example of a boundary controlled heat equation.

2 Review of the finite-dimensional case

Model reduction by bounded real balanced truncation and positive real balanced truncation for rational functions, introduced in Opdenacker & Jonckheere [19]

and Desai & Pal [6] respectively, is reviewed. The survey article by Gugercin & Antoulas [10], as well as Antoulas [2] include summaries of some of the material in this section. In our review we emphasise the aspects that are important in the generalisation to non-rational functions.

Let $\mathscr{U}$ and $\mathscr{Y}$ denote finite-dimensional Hilbert spaces, which are the input and output spaces respectively. We recall that a rational function $G : \mathbb{C}_0^+ \to B(\mathscr{U}, \mathscr{Y})$ belongs to H^∞ if and only if G is proper and every pole of G is in the open left-half complex plane. Given such a G it is possible to write

$$G(s) = D + C(sI - A)^{-1}B, \quad s \in \mathbb{C}_0^+,$$

for some finite-dimensional space $\mathscr{X}$ and operators

$$A : \mathscr{X} \to \mathscr{X}, \quad B : \mathscr{U} \to \mathscr{X}, \quad C : \mathscr{X} \to \mathscr{Y}, \quad D : \mathscr{U} \to \mathscr{Y}, \tag{4}$$

with A Hurwitz. The quadruple of operators (4) (and implicitly the space $\mathscr{X}$) is called a realisation of G and is denoted by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Moreover, we can always choose $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that the associated input-state-output system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0, \end{aligned} \tag{5}$$

is minimal (i.e. controllable and observable).

2.1 Bounded real balanced truncation

Definition 2.1. A function $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$, where $\mathscr{U}$ and $\mathscr{Y}$ are Banach spaces, is bounded real if

$$\|G\|_{H^\infty} \leq 1, \tag{6}$$

and such a $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ is strictly bounded real if the above inequality is strict.

Remark 2.2.

1. Synonymously with the term 'bounded real' the terms Schur, contractive and scattering passive are used. In the model reduction literature [2] the term 'bounded real balanced truncation' seems to have become standard and therefore we use this terminology.

2. Note that, in spite of the terminology, there is no realness assumption in Definition 2.1. However, if such an assumption is made about the original system, then realness of the reduced order system can be concluded.

Bounded real balanced truncation makes use of the well-known Bounded Real Lemma, see Anderson & Vongpanitlerd [1], which gives a state space characterisation of bounded real functions.

Lemma 2.3 (Bounded Real Lemma)**.** *Given rational $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ with a minimal realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, the following are equivalent.*

(i) G is bounded real.

(ii) There exists a triple of operators (P, K, W) with

$$P : \mathscr{X} \to \mathscr{X}, \quad K : \mathscr{X} \to \mathscr{U}, \quad W : \mathscr{U} \to \mathscr{U},$$

and P positive and self-adjoint satisfying the bounded real Lur'e equations

$$A^*P + PA + C^*C = -K^*K, \tag{7a}$$
$$PB + C^*D = -K^*W, \tag{7b}$$
$$I - D^*D = W^*W. \tag{7c}$$

Moreover, if either of the above hold then there are positive self-adjoint solutions P_m, P_M to (7) such that for any self-adjoint solution P of (7) we have

$$0 < P_m \leq P \leq P_M. \tag{8}$$

The extremal operators P_m, P_M are the optimal cost operators of the bounded real optimal control problems, namely:

$$\langle P_M x_0, x_0 \rangle_{\mathscr{X}} = \inf_{u \in L^2(\mathbb{R}^-; \mathscr{U})} \int_{\mathbb{R}^-} \|u(s)\|_{\mathscr{U}}^2 - \|y(s)\|_{\mathscr{Y}}^2 \, ds, \tag{9a}$$
$$-\langle P_m x_0, x_0 \rangle_{\mathscr{X}} = \inf_{u \in L^2(\mathbb{R}^+; \mathscr{U})} \int_{\mathbb{R}^+} \|u(s)\|_{\mathscr{U}}^2 - \|y(s)\|_{\mathscr{Y}}^2 \, ds. \tag{9b}$$

The minimisation problems (9) are subject to the minimal input-state-output realisation (5).

Proof. See [1]. There it is assumed that $\dim \mathscr{U} = \dim \mathscr{Y}$, but the result is true in general. □

If $P = P^* > 0$ is a solution of (7), for some K, W then an elementary calculation shows that $P^{-1} > 0$ solves the dual bounded real Lur'e equations

$$AQ + QA^* + BB^* = -LL^*, \tag{10a}$$
$$QC^* + BD^* = -LX^*, \tag{10b}$$
$$I - DD^* = XX^*, \tag{10c}$$

for some operators $L : \mathscr{Y} \to \mathscr{X}$, $X : \mathscr{Y} \to \mathscr{Y}$. By the Bounded Real Lemma, there are extremal self-adjoint solutions Q_m, Q_M to (10) such that for any self-adjoint solution Q to (10); $0 < Q_m \leq Q \leq Q_M$. In particular, it is not difficult to see that

$$P_m = Q_M^{-1}, \quad \text{and} \quad P_M = Q_m^{-1}. \tag{11}$$

Remark 2.4. Solutions of the bounded real Lur'e equations are generally not unique. We expand on this further, as it will be important later in this article. Given solutions (P, K, W) and (Q, L, X) of the bounded real Lur'e equations (7) and dual bounded real Lur'e equations (10) respectively, the first components P and Q do not in general uniquely determine the other two respective components. If we fix (P, K, W) and (Q, L, X) as above then the operators K', W' and L', X' defined by

$$\begin{aligned} K' &= UK, \quad W' = UW \\ L' &= LV, \quad\ \ X' = XV, \end{aligned}$$

for $U : \mathscr{U} \to \mathscr{U}$, $V : \mathscr{Y} \to \mathscr{Y}$ unitary, are such that (P, K', W') and (Q, L', X') are also solutions of (7) and (10) respectively.

Definition 2.5. The minimal realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of a bounded real rational transfer function is bounded real balanced, or in bounded real balanced co-ordinates, if

$$P_m = P_M^{-1} =: \Pi. \tag{12}$$

The nonnegative square roots of the eigenvalues of the product $P_m P_M^{-1}$ are called the bounded real singular values, which we denote by $(\sigma_k)_{k=1}^m$, each with (geometric) multiplicity r_k (so that $\sum_{k=1}^m r_k = \dim \mathscr{X}$). The bounded real singular values are ordered such that $\sigma_k > \sigma_{k+1} > 0$ for each k.

To define the bounded real balanced truncation, for $n < m$ let $\mathscr{X}_n$ and $\mathscr{Z}_n$ denote the sum of the first n and last $m-n$ eigenspaces of Π respectively. Then with respect to the orthogonal decomposition $\mathscr{X} = \mathscr{X}_n \oplus \mathscr{Z}_n$, the operators A, B, C and Π split as

$$\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

The truncated system with realisation $\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D \end{smallmatrix}\right]$ is called the bounded real balanced truncation and its transfer function is called the *reduced order transfer function obtained by bounded real balanced truncation.* We note that this reduced order transfer function is uniquely determined by the original transfer function, i.e., it does not depend on the particular bounded real balanced realisation that is chosen for truncation.

Given a bounded real balanced realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, observe that from (7a), the optimal cost operator $\Pi = P_m$ satisfies the Lyapunov equation

$$A^*\Pi + \Pi A + \begin{bmatrix} C^* & K^* \end{bmatrix} \begin{bmatrix} C \\ K \end{bmatrix} = 0. \tag{13}$$

Similarly from (10a), $\Pi = Q_m$ satisfies the Lyapunov equation

$$A\Pi + \Pi A^* + \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} B^* \\ L^* \end{bmatrix} = 0. \tag{14}$$

Since A is stable it follows that Π is both the controllability and observability Gramian of the extended system

$$\left[\begin{array}{c|cc} A & B & L \\ \hline C & D & X \\ K & W & 0 \end{array}\right], \tag{15}$$

which we denote by Σ_E. Note that Σ_E itself depends on Π through K and L and also by Remark 2.4, Σ_E is not uniquely determined by A, B, C, D and Π. For every choice of K, L, X and W, however, Σ_E has input and output spaces $\left[\begin{smallmatrix} \mathscr{U} \\ \mathscr{Y} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]$ respectively, and the same state-space as $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

From the Lyapunov equations (13) and (14) we see that Σ_E is Lyapunov balanced [18],[22] and that the bounded real singular values of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ are the Lyapunov singular values of Σ_E (i.e. the singular values of the Hankel operator of Σ_E). The Lyapunov balanced truncation of (15) is

$$\left[\begin{array}{c|cc} A_{11} & B_1 & L_1 \\ \hline C_1 & D & X \\ K_1 & W & 0 \end{array}\right],$$

from which the bounded real balanced truncation $\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is recovered by omitting the blocks L_1, K_1, X, W and zero. This corresponds to restricting to and projecting onto the original input and output spaces $\mathscr{U}$ and $\mathscr{Y}$ respectively. Therefore bounded real balanced truncation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ can be seen as *Lyapunov* balanced truncation of Σ_E. This relation is used in [19] in proving the following theorem, which is the main result for bounded real balanced truncation in the finite-dimensional case.

Theorem 2.6. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a rational bounded real function and let $(\sigma_j)_{j=1}^m$ denote its bounded real singular values, each with multiplicity r_j. For $r < m$ let G_r denote the reduced order transfer function obtained by bounded real balanced truncation. Then G_r is bounded real and the following error bound holds*

$$\|G - G_r\|_{H^\infty} \leq 2 \sum_{j=r+1}^{m} \sigma_j. \tag{16}$$

If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a minimal, bounded real balanced realisation of G, then the bounded real balanced truncation $\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D \end{smallmatrix}\right]$ is stable. If additionally G is strictly bounded real, then G_r has MacMillan degree $\boldsymbol{r} = \sum_{j=1}^r r_j$ and $\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D \end{smallmatrix}\right]$ is minimal and bounded real balanced.

Proof. See Theorem 2 and Section IV of [19]. The assumption there that G is strictly bounded real is not needed to prove that G_r is bounded real and that A_{11} is stable. The authors also assume throughout that $\mathscr{U} = \mathscr{Y}$, but this isn't needed and the proof for the general case is essentially the same. □

Our approach to the infinite-dimensional case makes extensive use of the above connection with Lyapunov balanced truncation (more so than the finite-dimensional case does). This approach necessitates careful consideration of the non-uniqueness of the extended system.

2.2 Positive real balanced truncation

Definition 2.7. An operator valued analytic function $J : \mathbb{C}_0^+ \to B(\mathscr{U})$, where $\mathscr{U}$ is a Hilbert space, is positive real if

$$J(s) + [J(s)]^* \geq 0, \quad \forall\, s \in \mathbb{C}_0^+. \tag{17}$$

We say that the analytic function $J : \mathbb{C}_0^+ \to B(\mathscr{U})$ is strictly positive real if there exists $\eta > 0$ such that

$$J(s) + [J(s)]^* \geq \eta I, \quad \forall\, s \in \mathbb{C}_0^+. \tag{18}$$

Remark 2.8.

1. The term strictly positive real is used for various slightly different concepts in the literature, as described in, for example, Wen [38]. The condition (18) is equivalent to the concept sometimes called extended strictly positive real, as in Sun *et al.* [32, Definition 2.1].

2. We do not assume that a positive real function is real on the real axis as is sometimes done in the literature.

3. Synonymously with the term 'positive real function' the terms impedance passive function, Weyl function, Titchmarsh-Weyl function and Caratheodory-Nevanlinna function are used. In the model reduction literature the term 'positive real balanced truncation' seems to have become standard and therefore we use this terminology.

Positive real balanced truncation is identical in spirit to bounded real balanced truncation and was proposed in the finite-dimensional case in [6]. The key ingredient is the Positive Real Lemma, which analogously to the Bounded Real Lemma provides a state-space characterisation of positive real functions.

Lemma 2.9 (Positive Real Lemma)**.** *Given rational $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ with a minimal realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, the following are equivalent.*

(i) J is positive real.

(ii) There exists a triple of operators (P, K, W) with

$$P : \mathscr{X} \to \mathscr{X}, \quad K : \mathscr{X} \to \mathscr{U}, \quad W : \mathscr{U} \to \mathscr{U},$$

and P positive and self-adjoint satisfying the positive real Lur'e equations

$$A^*P + PA = -K^*K, \tag{19a}$$
$$PB - C^* = -K^*W, \tag{19b}$$
$$D + D^* = W^*W. \tag{19c}$$

If either of the above hold then there exist positive, self-adjoint solutions $\tilde{P}_m, \tilde{P}_M$ to (19) *such that any self-adjoint solution P to* (19) *satisfies*

$$0 < \tilde{P}_m \leq P \leq \tilde{P}_M. \tag{20}$$

The extremal operators $\tilde{P}_m$, $\tilde{P}_M$ are the optimal cost operators of the positive real optimal control problems, namely:

$$\langle \tilde{P}_M x_0, x_0 \rangle_{\mathscr{X}} = \inf_{u \in L^2(\mathbb{R}^-;\mathscr{U})} \int_{\mathbb{R}^-} 2 \operatorname{Re} \langle u(s), y(s) \rangle_{\mathscr{U}} \, ds, \tag{21a}$$

$$-\langle \tilde{P}_m x_0, x_0 \rangle_{\mathscr{X}} = \inf_{u \in L^2(\mathbb{R}^+;\mathscr{U})} \int_{\mathbb{R}^+} 2 \operatorname{Re} \langle u(s), y(s) \rangle_{\mathscr{U}} \, ds. \tag{21b}$$

The minimisation problems (21) *are subject to the minimal input-state-output realisation* (5) *of J.*

Proof. See [1]. □

The realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of J is positive real balanced if

$$\tilde{P}_m = \tilde{P}_M^{-1} =: \tilde{\Pi},$$

The nonnegative square roots of the eigenvalues of $\tilde{P}_m \tilde{P}_M^{-1}$ are called the positive real singular values, ordered according to magnitude in decreasing order. The positive real balanced truncation is defined in the same way as the bounded real balanced truncation. The main result in the finite-dimensional positive real case is stated below.

Theorem 2.10. *Let $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ denote a rational positive real function and let $(\sigma_j)_{j=1}^m$ denote its positive real singular values, each with multiplicity r_j. For $r < m$, let J_r denote the reduced order transfer function obtained by positive real balanced truncation. Then $J_r \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ and J_r is positive real.*

If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a minimal positive real balanced realisation of J, then the positive real balanced truncation $\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D \end{smallmatrix}\right]$ is stable. If additionally J is strictly positive real, then J_r has MacMillan degree $\boldsymbol{r} = \sum_{j=1}^r r_j$ and $\left[\begin{smallmatrix} A_{11} & B_1 \\ C_1 & D \end{smallmatrix}\right]$ is minimal and positive real balanced.

Proof. See Harshavardhana *et al.* [15], [14] and the references therein. □

Remark 2.11. The analogous H^∞ error bound does *not* hold for positive real balanced truncation; a counter-example can be found in Guiver & Opmeer [13]. This is because there is *not* the same connection to Lyapunov balanced truncation [18] as in the bounded real balanced truncation case. In fact in the positive real case an H^∞ error bound seems less natural as positive real functions need not belong to H^∞. Instead a gap metric error bound

$$\hat{\delta}(J, J_n) \leq 2 \sum_{k=n+1}^{m} \sigma_k, \tag{22}$$

holds, where $\hat{\delta}$ is the gap metric and σ_k are the positive real singular values. The bound (22) is also proven in [13] (and was at the same time independently established by Timo Reis). Note that our second main result of this paper, Theorem 1.2, is the expected generalisation of (22).

3 Preliminaries

In this section we collect the infinite-dimensional results from the literature that we shall require to prove our main results. Recall that we are seeking to approximate $H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ functions, which in contrast to Section 2 need not be rational, and thus state-space representations will generally be infinite-dimensional. In this work $\mathscr{U}$ and $\mathscr{Y}$ denote the input and output spaces respectively, which are always assumed to be finite-dimensional Hilbert spaces.

Transfer functions belonging to $H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ can be realised by well-posed linear systems and Section 3.1 contains the corresponding notation and key required material. Sections 3.2 and 3.3 describe the optimal control problems and spectral factor systems respectively that we will require for bounded real and positive real balanced truncation.

3.1 Well-posed linear systems

Well-posed linear systems on L^2 go back to the work of Salamon [25], [26]. The monograph of Staffans [31] is dedicated to the study of general well-posed linear systems, and we shall make use of many results from this text. We remark that there are several different but equivalent formulations in the literature of a well-posed linear system. Although we use many results from [37], we have chosen to use the formulation of [31] so as to more readily apply results from that book. The equivalence between the formulations in [37] and [31] is shown in [31, Section 2.8].

For precise definitions of the following objects we refer the reader to [31, Section 2.2]. We denote by $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathscr{Y}, \mathscr{X}, \mathscr{U})$ (respectively, the output, state and input spaces) an L^p well-posed linear system with state x and output y given by

$$\begin{aligned} x(t) &= \mathfrak{A}^t x_0 + \mathfrak{B}_0^t u, \\ y &= \mathfrak{C}_0 x_0 + \mathfrak{D}_0 u, \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{23}$$

for input $u \in L^p_{\mathrm{loc}}(\mathbb{R}^+; \mathscr{U})$. We shall mostly be using L^2 well-posed systems, though we shall also need L^1 well-posed systems. In the above $(\mathfrak{A}^t)_{t\geq 0}$ is a strongly continuous semigroup on the state-space $\mathscr{X}$, $\mathfrak{B}_0^t$ is the input map (with initial time 0 and final time t), $\mathfrak{C}_0$ the output map and $\mathfrak{D}_0$ the input-output map (both with initial time 0). We remark that the finite-dimensional input-state-output system (5) has operators $\mathfrak{A}^t, \mathfrak{B}_0^t, \mathfrak{C}_0$ and $\mathfrak{D}_0$ given by

$$\begin{gathered} \mathfrak{A}^t = \mathrm{e}^{At}, \qquad \mathfrak{B}_0^t u = \int_0^t \mathrm{e}^{A(t-s)} B u(s)\, ds, \\ (\mathfrak{C}_0 x_0)(t) = C\mathrm{e}^{At} x_0, \quad (\mathfrak{D}_0 u)(t) = Du(t) + C\int_0^t \mathrm{e}^{A(t-s)} B u(s)\, ds. \end{gathered} \tag{24}$$

Remark 3.1. As explained in [31, Definition 2.2.6] and [31, Theorem 2.2.14], the operators $\mathfrak{B}_0^t$, $\mathfrak{C}_0$ and $\mathfrak{D}_0$ can be expressed in terms of the master operators

$\mathfrak{B}, \mathfrak{C}$ and $\mathfrak{D}$ and vice versa. There is no issue, therefore, with using the master operators $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and $\mathfrak{D}$. For example, for the finite-dimensional system (5), $\mathfrak{B}, \mathfrak{C}$ and $\mathfrak{D}$ are given by

$$\mathfrak{B}u = \int_{-\infty}^{0} \mathrm{e}^{-As} B u(s)\, ds, \qquad \mathfrak{C}x = (\mathbb{R}^+ \ni t \mapsto C\mathrm{e}^{At}x),$$
$$\mathfrak{D}u = \left(\mathbb{R} \ni t \mapsto \int_{-\infty}^{t} C\mathrm{e}^{A(t-s)} u(s)\, ds + Du(t)\right).$$

Remark 3.2. We collect some notation we shall need. Let π_+ and π_- denote the projections from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$ respectively. We let τ^t denote the bilateral shift by t on $L^2(\mathbb{R})$, i.e. for $t, s \in \mathbb{R}$, $(\tau^t u)(s) = u(t+s)$.

Remark 3.3. We assume that the reader is familiar with the generators of a well-posed linear system. The control operator and observation operator of well-posed linear systems date back to Weiss, [33] and [34] respectively. We shall also require the notion of a regular transfer function, as introduced by Weiss [36], and an operator node, system node, a compatible operator node and an admissible feedback operator. All of these concepts can be found in [31] and the latter are only drawn upon in some of the proofs of our later results, and are not needed for understanding the statements of those results.

The term realisation of an input-output (linear, time-invariant, causal) map $\mathfrak{D}$ on L^p refers to an L^p well-posed linear system with input-output map $\mathfrak{D}$, see [31, Definition 2.6.3] for more details. The transfer function G of an L^p well-posed system is defined as (see [31, Definition 4.6.1]) the analytic $B(\mathscr{U}, \mathscr{Y})$ valued function

$$s \mapsto \big(u \mapsto \mathfrak{D}(\mathrm{e}^{st}u)(0)\big), \quad u \in \mathscr{U}, \tag{25}$$

defined for $\operatorname{Re} s > \omega_{\mathfrak{A}}$ (the growth bound of $\mathfrak{A}$). The transfer function G is usually understood, however, as the "Laplace transform of the input-output map", which by [31, Corollary 4.6.10] is equivalent to the above definition. We refer the reader to [31, Corollary 4.6] or Weiss [35] for more information.

The transfer function in (25) determines $\mathfrak{D}$ uniquely and hence by a realisation of a transfer function we mean a realisation of the input-output map $\mathfrak{D}$ related to G by (25).

The following result is well-known and simply states that every H^∞ function has a (stable) L^2 well-posed realisation, with Hilbert space state space.

Lemma 3.4. *Given $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$, there exists a L^2 well-posed realisation $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathscr{Y}, \mathscr{X}, \mathscr{U})$ with $\mathscr{X}$ a Hilbert space such that the following stability assumptions hold:*

$$\mathfrak{A}, \mathfrak{A}^* \text{ are strongly stable}, \tag{26}$$
$$\mathfrak{B} : L^2(\mathbb{R}^-; \mathscr{U}) \to \mathscr{X} \text{ is bounded}, \tag{27}$$
$$\mathfrak{C} : \mathscr{X} \to L^2(\mathbb{R}^+; \mathscr{Y}) \text{ is bounded}, \tag{28}$$
$$\mathfrak{D} : L^2(\mathbb{R}; \mathscr{U}) \to L^2(\mathbb{R}; \mathscr{Y}) \text{ is bounded}. \tag{29}$$

We call such a system (in particular satisfying (26)-(29)*) a stable L^2 well-posed linear system.*

Proof. This is well-known and follows from, for example, [37, Theorem 4.2]. □

We need the following notion of a dual transfer function and a dual realisation.

Definition 3.5. Given a function $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ the dual G_d is defined as

$$G_d \in H^\infty(\mathbb{C}_0^+; B(\mathscr{Y}, \mathscr{U})), \quad \mathbb{C}_0^+ \ni s \mapsto G_d(s) = [G(\overline{s})]^*. \tag{30}$$

Given an L^2 well-posed linear system $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathscr{Y}, \mathscr{X}, \mathscr{U})$ (Hilbert spaces) we call the L^2 well-posed linear system Σ_d given by

$$\Sigma_d = ({}^d\mathfrak{A}, {}^d\mathfrak{B}, {}^d\mathfrak{C}, {}^d\mathfrak{D}) = (\mathfrak{A}^*, \mathfrak{C}^* R, R\mathfrak{B}^*, R\mathfrak{D}^* R),$$

on $(\mathscr{U}, \mathscr{X}, \mathscr{Y})$ the (causal) dual of Σ. Here R is the reflection in time, i.e $(Rv)(t) = v(-t)$. The reflection R acts on $L^2(\mathbb{R})$, and we view elements of $L^2(\mathbb{R}^+)$ or $L^2(\mathbb{R}^-)$ as belonging to $L^2(\mathbb{R})$ by extending by zero. Given an input $y_d \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathscr{Y})$ the state x_d and output u_d of Σ_d are defined by

$$\begin{aligned} x_d(t) &= (\mathfrak{A}^t)^* x_0 + {}^d\mathfrak{B}_0^t y_d, \\ u_d &= {}^d\mathfrak{C} x_0 + {}^d\mathfrak{D}_0 y_d, \qquad t \geq 0, \\ x_d(0) &= x_0. \end{aligned} \tag{31}$$

Remark 3.6. It is proven in [31, Theorem 6.2.3] that Σ_d is an L^2 well-posed linear system. Furthermore, it is easy to see that a transfer function is (strictly) bounded real if and only if its dual is (dual in the sense of the above definition). Similarly, for (strictly) positive real transfer functions.

The following result describes some properties of dual systems, notably that the dual system realises the dual transfer function, and is again taken from [31].

Lemma 3.7. *Let $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ denote an L^2 well-posed linear system on $(\mathscr{Y}, \mathscr{X}, \mathscr{U})$ and let (A, B, C) and G denote the generators and transfer function of Σ respectively. Then the dual system Σ_d has generators (A^*, C^*, B^*) and transfer function G_d. If Σ is stable, then so is Σ_d.*

Proof. The claims regarding the generators and transfer function of Σ_d follow from [31, Theorem 6.2.13]. That Σ_d is stable follows from the definition of the operators ${}^d\mathfrak{A}$, ${}^d\mathfrak{B}$, ${}^d\mathfrak{C}$ and ${}^d\mathfrak{D}$ and the stability of Σ. □

3.2 Optimal control problems

Bounded real (positive real) balanced truncation makes use of the unique optimal cost operators of the scattering supply rate (impedance supply rate) optimal control problems described below. The following results are special cases of [37,

Proposition 7.2] that can also be found in Staffans [28], and are the first instances of why we restrict attention to the strictly bounded real (positive real) case.

Lemma 3.8. *Let $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ denote a stable L^2 well-posed linear system with strictly bounded real transfer function. Then the optimal control problem: for $x_0 \in \mathscr{X}$ minimise*

$$\mathcal{J}(x_0, u) = \int_{\mathbb{R}^+} \|u(s)\|_{\mathscr{U}}^2 - \|y(s)\|_{\mathscr{Y}}^2 \, ds, \tag{32}$$

over all $u \in L^2(\mathbb{R}^+; \mathscr{U})$ subject to (23), has a solution in the sense that for any $x_0 \in \mathscr{X}$

$$\inf_{u \in L^2(\mathbb{R}^+;\mathscr{U})} \mathcal{J}(x_0, u) = \mathcal{J}(x_0, u_{opt}) = -\langle P_m x_0, x_0 \rangle_{\mathscr{X}}, \tag{33}$$

where the optimal control is unique and is given by

$$u_{opt} = (I - \pi_+ \mathfrak{D}^* \mathfrak{D} \pi_+)^{-1} \pi_+ \mathfrak{D}^* \mathfrak{C} x_0, \tag{34}$$

and $P_m : \mathscr{X} \to \mathscr{X}$ is bounded and satisfies $P_m = P_m^* \geq 0$ and

$$P_m = \mathfrak{C}^* \mathfrak{C} + \mathfrak{C}^* \mathfrak{D} \pi_+ (I - \pi_+ \mathfrak{D}^* \mathfrak{D} \pi_+)^{-1} \pi_+ \mathfrak{D}^* \mathfrak{C}. \tag{35}$$

Proof. See [37, Proposition 7.2]. The assumption that the transfer function G is strictly bounded real is equivalent to

$$I - \pi_+ \mathfrak{D}^* \mathfrak{D} \pi_+ \geq \varepsilon I,$$

see [37, Section 7] and hence $I - \pi_+ \mathfrak{D}^* \mathfrak{D} \pi_+$ is boundedly invertible. Therefore the optimal control u_{opt} and optimal cost operator P_m in (34) and (35) respectively are well-defined. Furthermore, in [37] it is assumed that G is weakly regular (with zero feedthrough), but that is not needed for this proof. □

Corollary 3.9. *Using the assumptions and notation of Lemma 3.8 let Σ_d denote the dual system from Definition 3.5. The dual optimal control problem: for $x_0 \in \mathscr{X}$ minimise*

$$\mathcal{J}_d(x_0, y_d) = \int_{\mathbb{R}^+} \|y_d(s)\|_{\mathscr{Y}}^2 - \|u_d(s)\|_{\mathscr{U}}^2 \, ds,$$

over all $y_d \in L^2(\mathbb{R}^+; \mathscr{U})$ subject to Σ_d, has bounded optimal cost operator $Q_m : \mathscr{X} \to \mathscr{X}$ satisfying $Q_m = Q_m^* \geq 0$ and given by

$$Q_m := ({}^d\mathfrak{C})^{*\,d}\mathfrak{C} + ({}^d\mathfrak{C})^{*\,d}\mathfrak{D} \pi_+ (I - \pi_+ ({}^d\mathfrak{D})^{*\,d}\mathfrak{D} \pi_+)^{-1} \pi_+ ({}^d\mathfrak{D})^{*\,d}\mathfrak{C}. \tag{36}$$

Proof. The result follows immediately from Definition 3.5, Remark 3.6 and Lemma 3.8. □

Lemma 3.10. *Let $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ denote a stable L^2 well-posed linear system with strictly positive real transfer function. Then the optimal control problem: for $x_0 \in \mathscr{X}$ minimise*

$$\mathcal{L}(x_0, u) = \int_{\mathbb{R}^+} 2\mathrm{Re}\,\langle u(s), y(s)\rangle_{\mathscr{U}}\, ds, \tag{37}$$

over all $u \in L^2(\mathbb{R}^+; \mathscr{U})$ subject to (23), has a solution in the sense that for any $x_0 \in \mathscr{X}$

$$\inf_{u \in L^2(\mathbb{R}^+;\mathscr{U})} \mathcal{L}(x_0, u) = \mathcal{L}(x_0, \tilde{u}_{opt}) = -\langle \tilde{P}_m x_0, x_0\rangle_{\mathscr{X}}, \tag{38}$$

where the optimal control is unique and is given by

$$\tilde{u}_{opt} = -(\mathfrak{D}\pi_+ + \pi_+\mathfrak{D}^*)^{-1}\mathfrak{C}\, x_0, \tag{39}$$

and $\tilde{P}_m : \mathscr{X} \to \mathscr{X}$ is bounded and satisfies $\tilde{P}_m = \tilde{P}_m^* \geq 0$ and

$$\tilde{P}_m = \mathfrak{C}^*(\mathfrak{D}\pi_+ + \pi_+\mathfrak{D}^*)^{-1}\mathfrak{C}. \tag{40}$$

Proof. See [37, Proposition 7.2]. The assumption that the transfer function is strictly positive real is equivalent to

$$\mathfrak{D}\pi_+ + \pi_+\mathfrak{D}^* \geq \varepsilon I,$$

see [37, Section 7] and hence $(\mathfrak{D}\pi_+ + \pi_+\mathfrak{D}^*)$ is boundedly invertible. Therefore the optimal control $\tilde{u}_{opt}$ and optimal cost operator $\tilde{P}_m$ in (39) and (40) respectively are well-defined. Furthermore, in [37] it is assumed that the transfer function is weakly regular (with zero feedthrough), but that is not needed for this proof. □

Corollary 3.11. *Using the assumptions and notation of Lemma 3.10 let Σ_d denote the dual system from Definition 3.5. The dual optimal control problem: for $x_0 \in \mathscr{X}$ minimise*

$$\mathcal{L}_d(x_0, y_d) = \int_{\mathbb{R}^+} 2\mathrm{Re}\,\langle y_d(s), u_d(s)\rangle_{\mathscr{U}}\, ds,$$

over all $y_d \in L^2(\mathbb{R}^+; \mathscr{Y})$ subject to Σ_d has bounded optimal cost operator $\tilde{Q}_m : \mathscr{X} \to \mathscr{X}$ satisfying $\tilde{Q}_m = \tilde{Q}_m^* \geq 0$ and given by

$$\tilde{Q}_m := ({}^d\mathfrak{C})^*({}^d\mathfrak{D}\pi_+ + \pi_+({}^d\mathfrak{D})^*)^{-1}\,{}^d\mathfrak{C}. \tag{41}$$

Proof. The result follows immediately from Definition 3.5, Remark 3.6, and Lemma 3.10. □

3.3 Extended systems

Here we recall some results on spectral factorisations and particularly spectral factor systems developed in [37]. This is the second instance where we require strict bounded realness of G. The material here is used in the next section to construct the bounded real balanced truncation.

Lemma 3.12. *If $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ is a strictly bounded real function, then there exist functions θ satisfying $\theta, \theta^{-1} \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ and ξ satisfying $\xi, \xi^{-1} \in H^\infty(\mathbb{C}_0^+; B(\mathscr{Y}))$ such that*

$$I - [G(\mathrm{i}\omega)]^* G(\mathrm{i}\omega) = [\theta(\mathrm{i}\omega)]^* \theta(\mathrm{i}\omega), \quad \text{for almost all } \omega \in \mathbb{R}, \tag{42}$$

and

$$I - G(\mathrm{i}\omega)[G(\mathrm{i}\omega)]^* = \xi(\mathrm{i}\omega)[\xi(\mathrm{i}\omega)]^*, \quad \text{for almost all } \omega \in \mathbb{R}. \tag{43}$$

The functions θ and ξ are uniquely determined up to multiplication by a unitary operator in $B(\mathscr{U})$ and $B(\mathscr{Y})$ respectively. Specifically, if θ_0 satisfies (42) *and ξ_0 satisfies* (43) *then the sets of all spectral factors satisfying* (42) *and* (43) *are given by*

$$\{U\theta_0 \,:\, U \in B(\mathscr{U}),\, U \text{ unitary}\} \quad \text{and} \quad \{\xi_0 V \,:\, V \in B(\mathscr{Y}),\, V \text{ unitary}\}, \tag{44}$$

respectively.

Proof. The assumption that G is strictly bounded real implies that

$$I - [G(\mathrm{i}\omega)]^* G(\mathrm{i}\omega) \geq \varepsilon I, \quad \text{for almost all } \omega \in \mathbb{R}.$$

The existence of the spectral factor θ satisfying $\theta, \theta^{-1} \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$, the equality (42) and uniqueness up to a unitary transformation follows from Rosenblum & Rovnyak [24, Theorem 3.7]. The claims regarding ξ follow from the above and duality. □

Lemma 3.13. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function with stable L^2 well-posed realisation $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$. Let $\theta \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ denote a spectral factor from Lemma 3.12 satisfying* (42) *with input-output map $\mathfrak{D}_\theta$. Define*

$$\mathfrak{C}_E := \begin{bmatrix} \mathfrak{C} \\ \mathfrak{C}_\theta \end{bmatrix} : \mathscr{X} \to L^2\left(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]\right), \tag{45}$$

$$\mathfrak{D}_{E_1} := \begin{bmatrix} \mathfrak{D} \\ \mathfrak{D}_\theta \end{bmatrix} : L^2(\mathbb{R}; \mathscr{U}) \to L^2\left(\mathbb{R}; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]\right), \tag{46}$$

where

$$\mathfrak{C}_\theta := -\pi_+ \mathfrak{D}_\theta^{-*} \mathfrak{D}^* \mathfrak{C} : \mathscr{X} \to L^2(\mathbb{R}^+; \mathscr{U}). \tag{47}$$

In the above $\mathfrak{D}_\theta^{-*} = (\mathfrak{D}_\theta^{-1})^*$. *Then* $\mathfrak{C}_E$ *is bounded and* $\Sigma_{E_1} := (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}_E, \mathfrak{D}_{E_1})$ *is a stable* L^2 *well-posed linear system on* $(\left[\begin{smallmatrix}\mathscr{Y}\\ \mathscr{U}\end{smallmatrix}\right], \mathscr{X}, \mathscr{U})$, *with transfer function*

$$G_{E_1} := \begin{bmatrix} G \\ \theta \end{bmatrix} \in H^\infty\left(\mathbb{C}_0^+; B\left(\mathscr{U}, \begin{bmatrix}\mathscr{Y}\\ \mathscr{U}\end{bmatrix}\right)\right), \tag{48}$$

and observability Gramian P_m *given by* (33), *i.e. the optimal cost operator of the optimal control problem* (32).

Proof. By [31, Theorem 10.3.5] (alternatively [35, Theorem 1.3]), to the H^∞ function θ we can associate a time invariant, causal, bounded operator

$$\mathfrak{D}_\theta : L^2(\mathbb{R}; \mathscr{U}) \to L^2(\mathbb{R}; \mathscr{U}).$$

The operator $\mathfrak{D}_\theta$ is boundedly invertible since θ^{-1} exists and $\theta^{-1} \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$, $\mathfrak{D}_\theta^{-1}$ is causal and

$$I - \mathfrak{D}^*\mathfrak{D} = \mathfrak{D}_\theta^*\mathfrak{D}_\theta, \quad \Rightarrow \quad I - \pi_+\mathfrak{D}^*\mathfrak{D}\pi_+ = \pi_+\mathfrak{D}_\theta^*\mathfrak{D}_\theta\pi_+, \tag{49}$$

which follows from [37, Section 11] (see particularly (11.5) in the numbering of [37]).

That Σ_{E_1} is an L^2 well-posed linear system now follows from [37, Theorem 11.1] and [37, Theorem 11.3], only adjusted for our notation. Detailed proofs can be found in Guiver [11, Lemma 6.2.4]. It remains to see that the observability Gramian of Σ_{E_1} equals P_m. We have that this observability Gramian equals

$$\begin{aligned}
\mathfrak{C}_E^*\mathfrak{C}_E &= \begin{bmatrix}\mathfrak{C}^* & \mathfrak{C}_\theta^*\end{bmatrix}\begin{bmatrix}\mathfrak{C}\\ \mathfrak{C}_\theta\end{bmatrix} = \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}_\theta^*\mathfrak{C}_\theta \\
&= \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}^*\mathfrak{D}\mathfrak{D}_\theta^{-1}\pi_+^2\mathfrak{D}_\theta^{-*}\mathfrak{D}^*\mathfrak{C}, \quad \text{from (47)}, \\
&= \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}^*\mathfrak{D}(\pi_+ + \pi_-)\mathfrak{D}_\theta^{-1}\pi_+^2\mathfrak{D}_\theta^{-*}(\pi_+ + \pi_-)\mathfrak{D}^*\mathfrak{C} \\
&= \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}^*\mathfrak{D}\pi_+\mathfrak{D}_\theta^{-1}\pi_+^2\mathfrak{D}_\theta^{-*}\pi_+\mathfrak{D}^*\mathfrak{C},
\end{aligned}$$

since $\mathfrak{D}_\theta^{-1}$ is causal and $\mathfrak{D}_\theta^{-*}$ is anticausal. Now an elementary calculation shows that $(\mathfrak{D}_\theta\pi_+)^{-1} = \mathfrak{D}_\theta^{-1}\pi_+$ and thus $(\mathfrak{D}_\theta\pi_+)^{-*} = \pi_+\mathfrak{D}_\theta^{-*}$. Therefore

$$\begin{aligned}
\mathfrak{C}_E^*\mathfrak{C}_E &= \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}^*\mathfrak{D}\pi_+(\mathfrak{D}_\theta\pi_+)^{-1}(\mathfrak{D}_\theta\pi_+)^{-*}\pi_+\mathfrak{D}^*\mathfrak{C} \\
&= \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}^*\mathfrak{D}\pi_+[(\mathfrak{D}_\theta\pi_+)^*(\mathfrak{D}_\theta\pi_+)]^{-1}\pi_+\mathfrak{D}^*\mathfrak{C} \\
&= \mathfrak{C}^*\mathfrak{C} + \mathfrak{C}^*\mathfrak{D}\pi_+[\pi_+\mathfrak{D}_\theta^*\mathfrak{D}_\theta\pi_+]^{-1}\pi_+\mathfrak{D}^*\mathfrak{C} \\
&= P_m, \quad \text{from (49) and (35)}.
\end{aligned}$$

□

Lemma 3.14. *Let* $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ *denote a strictly bounded real function with stable* L^2 *well-posed realisation* $\Sigma = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$. *Let* $\xi \in H^\infty(\mathbb{C}_0^+; B(\mathscr{Y}))$

denote a spectral factor from Lemma 3.12 satisfying (43) *with input-output map* $\mathfrak{D}_\xi$. *Define*

$$\mathfrak{B}_E := \begin{bmatrix}\mathfrak{B} & \mathfrak{B}_\xi\end{bmatrix} : L^2\left(\mathbb{R}^-; \left[\begin{smallmatrix}\mathscr{U}\\ \mathscr{Y}\end{smallmatrix}\right]\right) \to \mathscr{X}, \tag{50}$$

$$\mathfrak{D}_{E_2} := \begin{bmatrix}\mathfrak{D} & \mathfrak{D}_\xi\end{bmatrix} : L^2(\mathbb{R}; \left[\begin{smallmatrix}\mathscr{U}\\ \mathscr{Y}\end{smallmatrix}\right]) \to L^2(\mathbb{R}; \mathscr{Y}), \tag{51}$$

where

$$\mathfrak{B}_\xi = -\mathfrak{B}\mathfrak{D}^*\mathfrak{D}_\xi^{-*}\pi_- : L^2(\mathbb{R}^-; \mathscr{Y}) \to \mathscr{X}. \tag{52}$$

Then $\mathfrak{B}_E$ *is bounded and* $\Sigma_{E_2} := (\mathfrak{A}, \mathfrak{B}_E, \mathfrak{C}, \mathfrak{D}_{E_2})$ *is a stable* L^2 *well-posed linear system on* $(\mathscr{Y}, \mathscr{X}, \left[\begin{smallmatrix}\mathscr{U}\\ \mathscr{Y}\end{smallmatrix}\right])$ *with transfer function*

$$G_{E_2} := \begin{bmatrix}G & \xi\end{bmatrix} \in H^\infty\left(\mathbb{C}_0^+; B\left(\left[\begin{smallmatrix}\mathscr{U}\\ \mathscr{Y}\end{smallmatrix}\right], \mathscr{Y}\right)\right), \tag{53}$$

and controllability Gramian Q_m *given by* (36), *which is the optimal cost operator of the dual optimal control problem.*

Proof. The claims follow immediately once we note that Σ_{E_2} is the dual of the system constructed in Lemma 3.13 applied to the dual transfer function G_d instead of G (and therefore now using the spectral factor ξ instead of θ from Lemma 3.12). □

Remark 3.15. For a fixed strictly bounded real transfer function G and stable L^2 well-posed realisation Σ of G there are many extended output systems Σ_{E_1} and many extended input systems Σ_{E_2} owing to the non-uniqueness of the spectral factors θ and ξ from Lemma 3.12. However, given any Σ_{E_1}, every other extended output system is determined by Σ_{E_1} and a unitary operator $U \in B(\mathscr{U})$. As such we say that from G and Σ we obtain a family of extended output systems, parameterised by U. Similarly for Σ_{E_2}, now parameterised by unitary $V \in B(\mathscr{Y})$.

4 Bounded real balanced truncation

In this section we construct the bounded real balanced truncation of a strictly bounded real function and prove Theorem 1.1. We note that existence of bounded real balanced realisations in the infinite-dimensional case is shown in [31, Theorem 11.8.14], however bounded real balanced truncation is not addressed there.

We construct the bounded real balanced truncation by relating it to the Lyapunov balanced truncation of a certain extended system, as outlined (for the finite-dimensional case) in Section 2.1. The details of the construction of this extended system are given in Section 4.1. Subsequently in Section 4.2 we define the bounded real balanced singular values and the bounded real balanced truncation and prove Theorem 1.1.

4.1 Extended Hankel operators and transfer functions

Given a stable L^2 well-posed realisation of the strictly bounded real function G we seek to combine an extended output system Σ_{E_1} and an extended input system Σ_{E_2} from Lemmas 3.13 and Lemma 3.14 respectively into one (jointly) extended system with transfer function of the form

$$G_E = \begin{bmatrix} G & \xi \\ \theta & \chi \end{bmatrix},$$

where χ is yet to be determined. Towards this end, we first consider a (extended) Hankel operator constructed from an (extended) output map $\mathfrak{C}_E$ and (extended) input map $\mathfrak{B}_E$ from Lemmas 3.13 and 3.14 above respectively. The extended transfer function G_E will subsequently be defined in terms of this extended Hankel operator.

Since the extended systems from Lemmas 3.13 and Lemma 3.14 are not unique, we in fact obtain a family of extended systems, parameterised by two unitary operators. Compare this construction with that in the finite-dimensional case, described in Remark 2.4.

Hankel operators are well-studied objects, with unfortunately many different conventions being used in the literature. We say that an operator

$$H : L^2(\mathbb{R}^+; \mathscr{Z}_1) \to L^2(\mathbb{R}^+; \mathscr{Z}_2),$$

is Hankel if

$$\tau_2^t H = H(\tau_1^t)^*, \quad \forall\, t \geq 0, \tag{54}$$

where τ_i^t is the usual left shift by $t \geq 0$ on $L^2(\mathbb{R}^+; \mathscr{Z}_i)$ with adjoint $(\tau_i^t)^*$ the corresponding right shift.

Remark 4.1. We adopt the convention of Hankel operators mapping forwards time to forwards time. Therefore it is necessary to include a reflection operator R (as in Definition 3.5) in our definition of Hankel operator of a well-posed linear system when compared to [31].

Lemma 4.2. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function with stable L^2 well-posed realisation Σ. Let θ, ξ denote spectral factors as in Lemma 3.12 and let $\mathfrak{C}_E$ and $\mathfrak{B}_E$ denote the output map and input map from Lemma 3.13 and 3.14 respectively. Define the bounded operator H_E by*

$$H_E := \mathfrak{C}_E \mathfrak{B}_E R : L^2\left(\mathbb{R}^+; \begin{bmatrix} \mathscr{U} \\ \mathscr{Y} \end{bmatrix}\right) \to L^2\left(\mathbb{R}^+; \begin{bmatrix} \mathscr{Y} \\ \mathscr{U} \end{bmatrix}\right), \tag{55}$$

where R is the reflection from Definition 3.5. Then H_E is a Hankel operator. The operator H_E is independent of the choice of realisation Σ of G and depends on the spectral factors chosen as follows. If $H_E(\theta_0, \xi_0)$ is the Hankel operator for the choice of spectral factors θ_0, ξ_0 and $H_E(\theta, \xi)$ is the Hankel operator for spectral factors θ, ξ related to θ_0, ξ_0 by (44), then the Hankel operators are related by

$$H_E(\theta, \xi) = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} H_E(\theta_0, \xi_0) \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}. \tag{56}$$

Remark 4.3. In equation (56), $\left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right]$ is understood as an operator

$$L^2(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{U} \\ \mathscr{Y} \end{smallmatrix}\right]) \to L^2(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{U} \\ \mathscr{Y} \end{smallmatrix}\right]),$$

acting by (pointwise) multiplication. The same is true for $\left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]$, only now acting on $L^2(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$.

Proof of Lemma 4.2: A proof that that the operator H_E is Hankel in the sense of (54) can be found in [11, Lemma 6.3.1.]. The result follows readily from the intertwining properties of input maps and output maps of well-posed linear systems.

To see that H_E is independent of the choice of Σ we calculate

$$H_E = \mathfrak{C}_E \mathfrak{B}_E R = \begin{bmatrix} \mathfrak{C} \\ \mathfrak{C}_\theta \end{bmatrix} \begin{bmatrix} \mathfrak{B} & \mathfrak{B}_\xi \end{bmatrix} R = \begin{bmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathfrak{B}_\xi \\ \mathfrak{C}_\theta \mathfrak{B} & \mathfrak{C}_\theta \mathfrak{B}_\xi \end{bmatrix} R,$$

and using the formulae (45) for $\mathfrak{C}_\theta$ and (52) for $\mathfrak{B}_\xi$ gives that this equals

$$\begin{bmatrix} \pi_+ \mathfrak{D} \pi_- & \pi_+ \mathfrak{D}_\xi \pi_- \\ \pi_+ \mathfrak{D}_\theta \pi_- & \pi_+ \mathfrak{D}_\theta^{-*} \mathfrak{D}^* \pi_+ \mathfrak{D} \pi_- \mathfrak{D}^* \mathfrak{D}_\xi^{-*} \pi_- \end{bmatrix} R. \tag{57}$$

By inspection of (57), for given spectral factors θ and ξ, H_E depends only on the terms $\mathfrak{D}, \mathfrak{D}_\theta, \mathfrak{D}_\xi$ and their adjoints and inverses where applicable. We recall that an input-output map is completely determined by its transfer function (and vice versa). Therefore, (57) depends only on G and the spectral factors θ and ξ. By their construction in Lemma 3.12 the spectral factors are certainly independent of the stable L^2 well-posed realisation of G and hence so is H_E.

Equation (56) follows from (57) and the (easily established) relations

$$\mathfrak{D}_\theta = \mathfrak{D}_{U\theta_0} = U \mathfrak{D}_{\theta_0} \quad \text{and} \quad \mathfrak{D}_\xi = \mathfrak{D}_{\xi_0 V} = \mathfrak{D}_{\xi_0} V.$$

Again U and V are here understood as operators respectively acting on $L^2(\mathbb{R}^+; \mathscr{U})$ and $L^2(\mathbb{R}^+; \mathscr{Y})$ by pointwise multiplication, and certainly commute with π_+, π_- and R. □

Definition 4.4. Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function and for a choice of spectral factors θ_0, ξ_0 as in Lemma 3.12 let H_E^0 denote the corresponding Hankel operator from Lemma 4.2. The set of Hankel operators given by

$$\left\{ \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} H_E^0 \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \,:\, U \in B(\mathscr{U}) \text{ unitary}, V \in B(\mathscr{Y}) \text{ unitary.} \right\}, \tag{58}$$

is called the family of extended Hankel operators of G.

Remark 4.5. It follows from the above definition and the relationships (44) and (56) that there is a one-to-one correspondence between pairs of spectral factors of G and members of the family of extended Hankel operators of G.

We recall the definition of singular values, nuclear operators and what we mean by the transfer function of a Hankel operator.

Definition 4.6. For a bounded linear operator $T : \mathscr{Z}_1 \to \mathscr{Z}_2$ between Banach spaces, the k^{th} singular value s_k is defined as

$$s_k := \inf \{\|T - T_k\| \,:\, \operatorname{rank} T_k \leq k-1\}.$$

The operator T is nuclear if its singular values $(s_k)_{k\in\mathbb{N}}$ are summable, i.e.

$$\sum_{k\in\mathbb{N}} s_k < \infty.$$

Remark 4.7. In this work we use the term singular value in a non-standard manner. Given the above definition, we call σ_k the k^{th} singular value of T, but counted with multiplicities, so that if $s_1 = s_2 = \cdots = s_{p_1}$ and $s_{p_1} > s_{p_1+1}$, for some $p_1 \in \mathbb{N}$ then we set

$$\sigma_1 = s_1 = s_2 = \cdots = s_{p_1}, \quad \sigma_2 = s_{p_1+1} = \ldots,$$

and so on. As such, our k^{th} singular value σ_k has multiplicity $p_k \in \mathbb{N}$ and satisfies $\sigma_k > \sigma_{k+1}$, however note that σ_k need not necessarily be the distance of T to rank $k-1$ operators. Using this convention the operator T is nuclear if

$$\sum_{k\in\mathbb{N}} p_k\sigma_k < \infty.$$

We remark that if all the singular values are simple, then our convention and the usual convention coincide.

The following facts about Hankel operators are well-known, and proofs of these assertions are included in, for example, [11]. Given a bounded Hankel operator $H : L^2(\mathbb{R}^+;\mathscr{Z}_1) \to L^2(\mathbb{R}^+;\mathscr{Z}_2)$ a function ψ satisfying

$$\mathcal{L}H\mathcal{L}^{-1} = \mathcal{P}_+ M_\psi R_C : H^2(\mathbb{C}_0^+;\mathscr{Z}_1) \to H^2(\mathbb{C}_0^+;\mathscr{Z}_2),$$

is called a symbol for H, where $\mathcal{L}$ is the unilateral Laplace transform, R_C is the reflection $(R_C\phi)(s) = \phi(-s)$ and M_ψ is multiplication by ψ. In general H may have many symbols but every bounded Hankel operator has a symbol $\phi \in L^\infty(\mathrm{i}\mathbb{R}, B(\mathscr{Z}_1,\mathscr{Z}_2))$. Functions $\phi \in L^\infty(\mathrm{i}\mathbb{R}, B(\mathscr{Z}_1,\mathscr{Z}_2))$ have a decomposition

$$\phi = \phi_1 + \phi_2,$$

where ϕ_1 can be extended analytically to the right-half complex plane and ϕ_2 can be extended analytically to the left-half complex plane. The components ϕ_1 and ϕ_2 are unique up to an additive constant. We call ϕ_1 the analytic part of ϕ in $\mathbb{C}_0^+$. We define a transfer function corresponding to H as the analytic part of a symbol in $\mathbb{C}_0^+$ of H (which up to an additive constant is uniquely determined by H), plus an arbitrary constant operator.

The following result is based on [12, Proposition 3.4] for nuclear Hankel operators $L^2(\mathbb{R}^+;\mathscr{Z}_1) \to L^2(\mathbb{R}^+;\mathscr{Z}_2)$, which in turn is based on the Coifman & Rochberg decompositions [4].

Proposition 4.8. *A nuclear Hankel operator $H : L^2(\mathbb{R}^+; \mathscr{Z}_1) \to L^2(\mathbb{R}^+; \mathscr{Z}_2)$ (where both $\mathscr{Z}_1$ and $\mathscr{Z}_2$ are finite-dimensional Hilbert spaces) has a regular transfer function that belongs to $H^\infty(\mathbb{C}_0^+; B(\mathscr{Z}_1, \mathscr{Z}_2))$.*

Proof. This is a condensed version of [12, Proposition 3.4]. □

Lemma 4.9. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function. Then any two members of the family of extended Hankel operators of G from Definition 4.4 have the same singular values. In particular, if one member of this family is nuclear, then all are.*

Proof. Let H_E^0 and H_E denote two members of the family of extended Hankel operators of G which by definition are related by (56) for some unitary operators $U \in B(\mathscr{U})$ and $V \in B(\mathscr{Y})$. For notational convenience set $\tilde{U} := \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]$ and $\tilde{V} := \left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right]$, so that (56) becomes

$$H_E = \tilde{U} H_E^0 \tilde{V}.$$

The operators $\tilde{U}$ and $\tilde{V}$ are unitary and from this an easy calculation shows that for bounded $T : L^2(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{U} \\ \mathscr{Y} \end{smallmatrix}\right]) \to L^2(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$

$$\|H_E^0 - T\| = \|\tilde{U} H_E^0 \tilde{V} - \tilde{U} T \tilde{V}\| = \|H_E - \tilde{U} T \tilde{V}\|.$$

It is also easy to see that for any $n \in \mathbb{N}$ the map $T \mapsto \tilde{U} T \tilde{V}$ is a bijection of rank n operators to rank n operators. Therefore for $n \in \mathbb{N}$

$$\begin{aligned} s_n(H_E^0) &= \inf \left\{ \|H_E^0 - T\| \ : \ \operatorname{rank} T < n \right\} \\ &= \inf \left\{ \|H_E - \tilde{U} T \tilde{V}\| \ : \ \operatorname{rank} T < n \right\} = s_n(H_E). \end{aligned}$$

By counting with multiplicities it follows that $\sigma_k(H_E^0) = \sigma_k(H_E)$ for every $k \in \mathbb{N}$, which completes the proof. □

Definition 4.10. Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function. We say that G has a nuclear family of extended Hankel operators if some member of the family of extended Hankel operators of G from Definition 4.4 is nuclear.

We are now able to construct our desired extended transfer function.

Lemma 4.11. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function and assume that G has a nuclear family of extended Hankel operators. Let H_E denote a member of this family corresponding to the spectral factors θ, ξ. Then there exists $\chi \in H^\infty(\mathbb{C}_0^+; B(\mathscr{Y}, \mathscr{U}))$ such that*

$$G_E = \begin{bmatrix} G & \xi \\ \theta & \chi \end{bmatrix} \in H^\infty(\mathbb{C}_0^+; B\left(\left[\begin{smallmatrix} \mathscr{U} \\ \mathscr{Y} \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]\right)), \tag{59}$$

is regular and is a transfer function of H_E. The feedthrough of χ can without loss of generality be taken equal to zero. Therefore we let

$$D_E = \begin{bmatrix} D & D_\xi \\ D_\theta & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{U} \\ \mathscr{Y} \end{bmatrix} \to \begin{bmatrix} \mathscr{Y} \\ \mathscr{U} \end{bmatrix}, \tag{60}$$

denote the bounded operator such that

$$\lim_{\substack{s \to +\infty \\ s \in \mathbb{R}^+}} G_E(s) = D_E.$$

The components D, D_θ and D_ξ of D_E are the feedthroughs of G, θ and ξ respectively. By always fixing the feedthrough of χ as zero, for each G, θ and ξ there is a one-to-one correspondence between Hankel operators H_E and transfer functions G_E given by (59).

Proof. The existence of χ and the regularity of G_E (and hence G and the spectral factors θ and ξ) follows from Proposition 4.8. By that result the Hankel operator H_E determines G_E uniquely up to an additive constant, which is determined by the feedthroughs of G, θ and ξ and the choice of χ having feedthrough zero. □

Definition 4.12. Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function with nuclear family of extended Hankel operators. By Lemma 4.11, each member of this family has a unique transfer function G_E given by (59). We call the set of transfer functions G_E the family of extended transfer functions of G.

From its construction, we see that the original transfer function G and the spectral factors θ and ξ are components of the extended transfer function G_E. The next lemma describes how we can obtain L^p well-posed realisations of G and θ from L^p well-posed realisations of G_E. We shall need those later in this work.

Lemma 4.13. *Given strictly bounded real $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ with nuclear family of extended Hankel operators, let G_E denote a member of the family of extended transfer functions of G. If $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is an L^p well-posed realisation on $(\left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right], \mathscr{X}, \left[\begin{smallmatrix} \mathscr{U} \\ \mathscr{Y} \end{smallmatrix}\right])$ of G_E with $1 \leq p < \infty$, then*

$$(\mathfrak{A}, \mathfrak{B}|_{\mathscr{U}}, P_{\mathscr{Y}}\mathfrak{C}, P_{\mathscr{Y}}\mathfrak{D}|_{\mathscr{U}}), \quad (\mathfrak{A}, \mathfrak{B}|_{\mathscr{U}}, P_{\mathscr{U}}\mathfrak{C}, P_{\mathscr{U}}\mathfrak{D}|_{\mathscr{U}}), \tag{61}$$

are L^p well-posed realisations of G and θ respectively. Here $P_{\mathscr{U}}$ denotes the orthogonal projection of $\left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]$ onto $\mathscr{U}$ and $P_{\mathscr{Y}}$ denotes the orthogonal projection of $\left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]$ onto $\mathscr{Y}$. If A, B, C and D denote the generators of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$, then $A, B|_{\mathscr{U}}, P_{\mathscr{Y}}C$ and $P_{\mathscr{Y}}D|_{\mathscr{U}}$ and $A, B|_{\mathscr{U}}, P_{\mathscr{U}}C$ and $P_{\mathscr{U}}D|_{\mathscr{U}}$ are the generators of the above realisations of G and θ respectively. Furthermore, if $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is a stable L^2 well-posed realisation of G_E then the realisations in (61) *are stable L^2 well-posed realisations of G and θ respectively.*

Proof. It is routine to verify that the two systems in (61) satisfy the conditions of [31, Definition 2.2.1], and hence are L^p well-posed. Since the generators are unique, a short calculation demonstrates that the formulae given are indeed the generators. The final claim is immediate from the definition of a stable L^2 well-posed realisation since restriction and projection are bounded operations. □

4.2 Bounded real balanced truncation

We now define the bounded real singular values and bounded real balanced truncation for strictly bounded real functions. These in principle depend on the choice of extended system, but Remark 4.15 and Lemma 4.20 will demonstrate that this dependence is trivial.

Definition 4.14. Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function. We define the bounded real singular values of G as the singular values (using the convention of Remark 4.7) of some member of the family of extended Hankel operators of G from Definition 4.4.

Remark 4.15.

1. By Lemma 4.9 all members of the family of extended Hankel operators of G have the same singular values, so the bounded real singular values depend only on G.

2. Our next result shows that the above definition is consistent with the finite-dimensional version in Section 2.1. There the bounded real singular values were defined as the square roots of the eigenvalues of the product of the bounded real optimal cost operators. An analogous approach in the infinite-dimensional case is trickier because although the product of the optimal cost operators is bounded, it is not *a priori* clear why it should have (nonnegative, real) eigenvalues. However, we prove that when the bounded real singular values are summable (which is always true in the finite-dimensional case) then the definitions coincide.

Lemma 4.16. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function and let P_m and Q_m denote the optimal cost operators of the optimal control problems from Lemma 3.8 and Corollary 3.9 respectively corresponding to a given stable L^2 well-posed realisation of G. Then the bounded real singular values are summable if and only if $P_m Q_m$ is compact and the square roots of its eigenvalues are summable. If these conditions hold then apart from possibly zero the bounded real singular values are precisely the square roots of the eigenvalues of $P_m Q_m$ (which therefore depend only on G).*

Proof. For brevity we give an outline of the proof. For all the details see [11, Lemma 6.3.12]. Choose a stable L^2 well-posed realisation of G so that P_m and Q_m are given by (35) and (36) respectively. For some choice of spectral factors as in Lemma 3.12, let $\mathfrak{C}_E$ and $\mathfrak{B}_E$ denote the extended output operator and extended input operator from Lemmas 3.13 and 3.14 respectively. By those results it follows that $P_m = \mathfrak{C}_E^* \mathfrak{C}_E$ and $Q_m = \mathfrak{B}_E \mathfrak{B}_E^*$. Let H_E denote the Hankel operator given by (55), which is a member of the family of extended

Hankel operators of G. By Definition 4.14 the bounded real singular values are the singular values of H_E.

Combining the following two facts for bounded operators $T, S : \mathscr{X} \to \mathscr{X}$ on a Hilbert space

- if T is compact then TS and ST are compact,
- T is compact if and only is T^* is ,

it is not difficult to prove that H_E is compact if and only if $P_m Q_m$ (equivalently $Q_m P_m$) is. Using the fact that for $\lambda \neq 0$

$$\lambda \in \sigma(TS) \iff \lambda \in \sigma(ST),$$

it follows that when H_E is compact, $H_E^* H_E$ and $Q_m P_m$ have the same non-zero eigenvalues and, arguing carefully with eigenvectors, we see that these eigenvalues have the same multiplicities. We recall that if H_E is compact then its singular values are precisely the (nonnegative) square roots of the eigenvalues of $H_E^* H_E$. These observations combined prove the first assertion.

From Lemma 4.2 it follows that H_E is independent of the stable L^2 well-posed realisation of G chosen, hence so are its singular values and thus when the bounded real singular values are summable, we see that the non-zero eigenvalues of $P_m Q_m$ also depend only on G. □

We very briefly recap some of the results from [12] on Lyapunov balanced truncation in the infinite-dimensional case.

A nuclear Hankel operator $H : L^2(\mathbb{R}^+; \mathscr{Z}_1) \to L^2(\mathbb{R}^+; \mathscr{Z}_2)$ (with $\mathscr{Z}_1$, $\mathscr{Z}_2$ finite-dimensional) is necessarily given by an integral operator of the form

$$(Hf)(t) = \int_0^\infty h(t+s) f(s)\, ds, \quad \forall\, f \in L^2(\mathbb{R}^+; \mathscr{Z}_1),\ \text{a.a } t \geq 0, \tag{62}$$

with $h \in L^1(\mathbb{R}^+; B(\mathscr{Z}_1, \mathscr{Z}_2))$ satisfying

$$h(t) := \sum_{n \in \mathbb{N}} \lambda_n (\operatorname{Re} a_n) \mathrm{e}^{a_n t}, \quad t > 0, \tag{63}$$

for sequences $(\lambda_n)_{n \in \mathbb{N}} \in \ell^1(B(\mathscr{Z}_1, \mathscr{Z}_2))$ and $(a_n)_{n \in \mathbb{N}} \subset \mathbb{C}_0^-$. The series in (63) converges absolutely and uniformly on $t > 0$. This result follows from the decompositions of [4], and a proof can be found in [12, Corollary 4.4, Lemma 4.6 and Corollary 4.7]. Operators of the form (62) (with L^1 kernel h) are compact

$$\begin{aligned} L^1(\mathbb{R}^+; \mathscr{Z}_1) &\to L^1(\mathbb{R}^+; \mathscr{Z}_2), \\ L^2(\mathbb{R}^+; \mathscr{Z}_1) &\to L^2(\mathbb{R}^+; \mathscr{Z}_2), \\ \text{and} \quad W^{1,1}(\mathbb{R}^+; \mathscr{Z}_1) &\to W^{1,1}(\mathbb{R}^+; \mathscr{Z}_2), \end{aligned}$$

where $W^{m,p}$ denotes the usual Sobolev space. Recall that the Schmidt pairs of a (compact) operator $T : L^2(\mathbb{R}^+; \mathscr{Z}_1) \to L^2(\mathbb{R}^+; \mathscr{Z}_2)$ are the eigenvectors

of T^*T and TT^* respectively, with respect to the eigenvalue σ_i^2. Here $(\sigma_i)_{i\in\mathbb{N}}$ are the singular values of T. The Schmidt pairs of the operator H given by (62), denoted by $(v_{i,k}, w_{i,k})$ where $i \in \mathbb{N}$, $1 \leq k \leq p_i$ with p_i the geometric multiplicity of σ_i^2, satisfy

$$v_{i,k} \in L^1 \cap L^2 \cap W^{1,1}(\mathbb{R}^+; \mathscr{Z}_1), \quad w_{i,k} \in L^1 \cap L^2 \cap W^{1,1}(\mathbb{R}^+; \mathscr{Z}_2). \tag{64}$$

We recall the well-posed realisations and their generators that are a crucial ingredient for the Lyapunov balanced truncation of [12].

Lemma 4.17. *For a linear, time-invariant, causal operator $\mathfrak{D} : L^p(\mathbb{R}; \mathscr{Z}_1) \to L^p(\mathbb{R}; \mathscr{Z}_2)$ with $1 \leq p < \infty$ and $\mathscr{Z}_1$, $\mathscr{Z}_2$ Banach spaces the system*

$$^{sr}\Sigma^p = (\tau, HR, I, \mathfrak{D}), \quad \text{on } (\mathscr{Z}_2, L^p(\mathbb{R}^+; \mathscr{Z}_2), \mathscr{Z}_1),$$

is an L^p well-posed linear system. Here τ and I are the left-shift and identity on $L^p(\mathbb{R}^+; \mathscr{Z}_2)$ respectively, and $H = \pi_+ \mathfrak{D} \pi_- R$ is the Hankel operator. We call $^{sr}\Sigma^1$ the exactly observable shift realisation on L^1 of $\mathfrak{D}$ and $^{sr}\Sigma^2$ the output-normal shift realisation on L^2 of $\mathfrak{D}$.

Proof. That $^{sr}\Sigma^p$ is an L^p well-posed linear system follows from [31, Example 2.6.5 (ii)] (noting our convention in Remark 4.1 for Hankel operators). Note that the left shift τ is a strongly continuous semigroup on $L^p(\mathbb{R}^+; \mathscr{Z}_2)$ by [31, Example 2.3.2 (ii)]. □

Lemma 4.18. *Let $H : L^2(\mathbb{R}^+; \mathscr{Z}_1) \to L^2(\mathbb{R}^+; \mathscr{Z}_2)$, $h \in L^1(\mathbb{R}^+; B(\mathscr{Z}_1, \mathscr{Z}_2))$ and $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{Z}_1, \mathscr{Z}_2))$ denote a Hankel operator given by (62), its kernel and transfer function respectively. Then for $1 \leq p < \infty$ the shift realisation $^{sr}\Sigma^p$ of G from Lemma 4.17 is an L^p well-posed linear realisation of G and has generators A, B and C given by*

$$A : D(A) \to L^p(\mathbb{R}^+; \mathscr{Z}_2), \qquad A = \frac{d}{dt}, \quad D(A) = W^{1,p}(\mathbb{R}^+; \mathscr{Z}_2), \tag{65}$$

$$B : \mathscr{Z}_1 \to W^{-1,p}(\mathbb{R}^+; \mathscr{Z}_2), \quad (Bu)(t) = h(t)u, \quad p > 1, \tag{66}$$

$$C : D(A) \to \mathscr{Z}_2 \qquad Cx = x(0). \tag{67}$$

Here $W^{-1,p}(\mathbb{R}^+; \mathscr{Z}_2)$ is the dual space of $W_0^{1,p}(\mathbb{R}^+; \mathscr{Z}_2)$. When $p = 1$ the control operator B is bounded and is defined by

$$B : \mathscr{Z}_1 \to L^1(\mathbb{R}^+; \mathscr{Z}_2), \qquad (Bu)(t) = h(t)u. \tag{68}$$

Proof. The main operator A is the generator of the left shift semigroup on $\mathbb{R}^+$, see [31, Example 3.2.3 (ii)]. By [31, Example 4.4.6] the operator C in (67) is the observation operator of $^{sr}\Sigma^p$. For a proof of the control operator in the L^1 case see [12, Lemma 5.1] and for a proof for general $p > 1$ see [11, Lemma 5.3.1]. □

Define the truncation space

$$\mathscr{X}_n := \langle w_{i,k} \mid 1 \leq i \leq n,\ 1 \leq k \leq p_i \rangle, \tag{69}$$

which by (64) is a subspace of L^1, L^2 and $W^{1,1}(\mathbb{R}^+; \mathscr{Z}_2)$, and also the projection

$$\mathcal{P}_n : L^1(\mathbb{R}^+; \mathscr{Z}_2) \to \mathscr{X}_n, \quad x \mapsto \mathcal{P}_n x := \sum_{i=1}^{n} \sum_{k=1}^{p_i} \langle w_{i,k}, x \rangle_{L^2} w_{i,k}. \tag{70}$$

The operator $\mathcal{P}_n$ is clearly linear and well-defined (bounded even) by the Hölder inequality

$$|\langle w_{i,k}, x \rangle_{L^2}| \leq \|w_{i,k}\|_\infty \cdot \|x\|_1 \leq \|w_{i,k}\|_{1,1} \cdot \|x\|_1.$$

Definition 4.19. Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function with summable bounded real singular values and let G_E denote a member of the family of extended transfer functions of G from Definition 4.12. Let (A_E, B_E, C_E, D_E) denote the generators of the exactly observable shift realisation ${}^{sr}\Sigma_E^1$ of G_E and for $n \in \mathbb{N}$, let $\mathscr{X}_n$ and $\mathcal{P}_n$ denote the space and projection from (69) and (70) respectively. Define the operators

$$\begin{aligned} &(A_E)_n := \mathcal{P}_n A|_{\mathscr{X}_n} : \mathscr{X}_n \to \mathscr{X}_n, \quad (B_E)_n^{\mathscr{U}} := \mathcal{P}_n B_E|_{\mathscr{U}} : \mathscr{U} \to \mathscr{X}_n, \\ &(C_E)_n^{\mathscr{Y}} := P_{\mathscr{Y}} C_E|_{\mathscr{X}_n} : \mathscr{X}_n \to \mathscr{Y}. \end{aligned} \tag{71}$$

Here $P_{\mathscr{Y}}$ is the orthogonal projection of $\left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]$ onto $\mathscr{Y}$. We call the finite-dimensional system on $(\mathscr{Y}, \mathscr{X}_n, \mathscr{U})$ generated by $\left[\begin{smallmatrix} (A_E)_n & (B_E)_n^{\mathscr{U}} \\ (C_E)_n^{\mathscr{Y}} & D \end{smallmatrix}\right]$ the reduced order system obtained by bounded real balanced truncation (determined by G_E), where $D = P_{\mathscr{Y}} D_E|_{\mathscr{U}}$ is the feedthrough of G. The function G_n defined by

$$G_n(s) := (C_E)_n^{\mathscr{Y}} (sI - (A_E)_n)^{-1} (B_E)_n^{\mathscr{U}} + D, \tag{72}$$

is called the reduced order transfer function obtained from G by bounded real balanced truncation.

Note that the bounded real balanced truncation depends on the choice of extended transfer function G_E. The next lemma shows that different choices of G_E give rise to bounded real balanced truncations of G that are unitarily equivalent to one another. In particular, they all give rise to the same reduced order transfer function in (72).

Lemma 4.20. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function with summable bounded real singular values. Then the bounded real balanced truncation is unique up to a unitary transformation, determined by the choice of extended transfer function G_E. Every bounded real balanced truncation gives rise to the same reduced order transfer function obtained by bounded real balanced truncation, which is therefore independent of the above choice.*

Proof. For the choice of spectral factors θ_0 and ξ_0, let G_E^0 denote the resulting member of the family of extended transfer functions of G. If the spectral factors θ and ξ are related to θ_0 and ξ_0 by (44) and G_E is the corresponding extended transfer function, then G_E^0 and G_E are related by

$$G_E = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} G_E^0 \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}, \tag{73}$$

where $U \in B(\mathscr{U})$, $V \in B(\mathscr{Y})$ are unitary. The relation (73) readily follows from the version for the corresponding extended Hankel operators (56) and our definition of the feedthrough D_E of G_E in (60).

Let $(A_E^0, B_E^0, C_E^0, D_E^0)$ denote the generators of the exactly observable shift realisation on L^1 of G_E^0. It is readily seen that

$$(A_E, B_E, C_E, D_E) := (A_E^0, \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] B_E^0 \left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right], C_E^0, \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] D_E^0 \left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right]),$$

generate the exactly observable shift realisation on L^1 of G_E. A simple calculation shows that if $(v_{i,k}^0, w_{i,k}^0)$ are Schmidt pairs of H_E^0 then

$$(v_{i,k}, w_{i,k}) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right]^{-1} v_{i,k}^0, \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] w_{i,k}^0 \right), \tag{74}$$

are Schmidt pairs for H_E. Therefore for $n \in \mathbb{N}$

$$\mathscr{X}_n := \langle w_{i,k} \,|\, 1 \le i \le n,\ 1 \le k \le p_i \rangle = \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] \mathscr{X}_n^0,$$

and in fact $\left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] : \mathscr{X}_n^0 \to \mathscr{X}_n$ is an isomorphism. Furthermore if $\mathcal{P}_n^0$ denotes the projection of $L^1(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$ onto $\mathscr{X}_n^0$, defined analogously to $\mathcal{P}_n$ in (70) then

$$\mathcal{P}_n \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] \mathcal{P}_n^0. \tag{75}$$

With these observations we are able to describe how the bounded real balanced truncations of G_E^0 and G_E are related. By definition of the projections $\mathcal{P}_n$, $\mathcal{P}_n^0$ and the operators A_E and A_E^0 we see that for $1 \le i \le n$ and $1 \le k \le p_i$

$$(A_E)_n w_{i,k} = \mathcal{P}_n A_E |_{\mathscr{X}_n} w_{i,k} = \sum_{j=1}^{n} \sum_{r=1}^{p_j} \langle w_{j,r}, \dot{w}_{i,k} \rangle_{L^2} w_{j,r}, \tag{76}$$

and

$$\begin{aligned} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] (A_E^0)_n \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1} w_{i,k} &= \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1} \mathcal{P}_n A_E^0 |_{\mathscr{X}_n^0} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1} w_{i,k} \\ &= \sum_{j=1}^{n} \sum_{r=1}^{p_j} \langle w_{j,r}^0, \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1} \dot{w}_{i,k} \rangle_{L^2} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] w_{j,r}^0. \end{aligned} \tag{77}$$

The Schmidt pair relations (74) and the fact that $\left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]$ is unitary imply that the expressions in (76) and (77) are equal. Since this equality holds on a basis for $\mathscr{X}_n$ we infer that

$$(A_E)_n = \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] (A_E^0)_n \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1}. \tag{78}$$

Similarly, using the projection relation (75) yields

$$\mathcal{P}_n B_E = \mathcal{P}_n \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] B_E^0 \left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] \mathcal{P}_n^0 B_E^0 \left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right],$$

which implies that

$$(B_E)_n^{\mathscr{U}} = \mathcal{P}_n B_E |_{\mathscr{U}} = \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] \mathcal{P}_n^0 B_E^0 |_{\mathscr{U}} = \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] (B_E^0)_n^{\mathscr{U}}. \tag{79}$$

As with A_E^0 and A_E, the operators C_E^0 and C_E are the same and we see that

$$\begin{aligned}(C_E)_n^{\mathscr{Y}} &= P_{\mathscr{Y}} C_E^0|_{\mathscr{X}_n} = P_{\mathscr{Y}} C_E^0|_{\mathscr{X}_n} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1} = P_{\mathscr{Y}} C_E^0|_{\mathscr{X}_n^0} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1} \\ &= (C_E^0)_n^{\mathscr{Y}} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]^{-1}. \end{aligned} \tag{80}$$

Finally,

$$P_{\mathscr{Y}} D_E|_{\mathscr{U}} = P_{\mathscr{Y}} \left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right] D_E^0 \left[\begin{smallmatrix} I & 0 \\ 0 & V \end{smallmatrix}\right]|_{\mathscr{U}} = D = P_{\mathscr{Y}} D_E^0|_{\mathscr{U}}. \tag{81}$$

We see from (78)-(81) that the bounded real balanced truncations of $G_E^{U,V}$ and G_E are similar, with unitary similarity transformation $\left[\begin{smallmatrix} I & 0 \\ 0 & U \end{smallmatrix}\right]$. In particular, they both give rise to the same transfer function G_n. □

Having defined the bounded real balanced truncation, we now seek to prove Theorem 1.1. The next lemma describes some of the properties of the reduced order system obtained by bounded real balanced truncation, and is the key ingredient in proving Theorem 1.1.

Lemma 4.21. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly bounded real function with summable bounded real singular values and for $n \in \mathbb{N}$ let G_n denote the reduced order transfer function obtained by bounded real balanced truncation. Then G_n is rational, bounded real and for every choice of extended transfer function the resulting bounded real balanced truncation from Definition 4.19 is a stable realisation of G_n.*

Proof. By Lemma 4.20 the bounded real balanced truncations are all unitarily equivalent to one another. Since the stability of A of the realisation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is invariant under unitary transformation, it does not matter which bounded real balanced truncation we pick. Therefore, for this proof we pick a member G_E of the family of extended transfer functions arbitrarily and for notational convenience we denote the bounded real balanced truncation by $\left[\begin{smallmatrix} A_n & B_n^{\mathscr{U}} \\ C_n^{\mathscr{Y}} & D \end{smallmatrix}\right]$.

That G_n is rational is clear, as $\left[\begin{smallmatrix} A_n & B_n^{\mathscr{U}} \\ C_n^{\mathscr{Y}} & D \end{smallmatrix}\right]$ is a realisation on a finite-dimensional state-space. It was proven in [12, Proposition 6.12] that A_n is stable (see also [12, Definition 5.5]).

It remains to see that G_n is bounded real and for this we use the Bounded Real Lemma, Lemma 2.3. We seek a solution (P, K, W), with $P : \mathscr{X}_n \to \mathscr{X}_n$ self-adjoint and positive, of the bounded real Lur'e equations (7) (subject to the realisation $\left[\begin{smallmatrix} A_n & B_n^{\mathscr{U}} \\ C_n^{\mathscr{Y}} & D \end{smallmatrix}\right]$). Noting that

$$C_n = C|_{\mathscr{X}_n} = \begin{bmatrix} P_{\mathscr{Y}} C|_{\mathscr{X}_n} \\ P_{\mathscr{U}} C|_{\mathscr{X}_n} \end{bmatrix} = \begin{bmatrix} C_n^{\mathscr{Y}} \\ C_n^{\mathscr{U}} \end{bmatrix} : \mathscr{X}_n \to \begin{bmatrix} \mathscr{Y} \\ \mathscr{U} \end{bmatrix}, \tag{82}$$

we claim that

$$P := I : \mathscr{X}_n \to \mathscr{X}_n, \qquad K := C_n^{\mathscr{U}} : \mathscr{X}_n \to \mathscr{U}, \qquad W := D_\theta : \mathscr{U} \to \mathscr{U},$$

solve (7) and we proceed to verify equations (7a), (7b) and (7c).

In [11, Lemma 5.3.16] (see also [12, Proposition 6.12]), it is proven that the Lyapunov equation

$$A_n^* + A_n + C_n^* C_n = 0, \tag{83}$$

holds as an operator equation on $\mathscr{X}_n$ equipped with the L^2 inner product. Using (82) we can rewrite (83) as

$$A_n^* + A_n + (C_n^{\mathscr{Y}})^* C_n^{\mathscr{Y}} = -(C_n^{\mathscr{U}})^* C_n^{\mathscr{U}}, \tag{84}$$

which is (7a). We now verify the second equation (7b), i.e. we demonstrate that

$$B_n^{\mathscr{U}} + (C_n^{\mathscr{Y}})^* D = -(C_n^{\mathscr{U}})^* D_\theta. \tag{85}$$

Firstly, by applying Lemma 4.13 we obtain L^2 well-posed realisations of G and θ from the output normal realisation (Lemma 4.17) of G_E. We denote the generators of these realisations of G and θ by (A, B, C, D) and $(A, B, C_\theta, D_\theta)$ respectively. Applying [37, Theorem 12.4] to these realisations (noting that $P_m = I$) gives

$$B_\Lambda^* + D^* C = -D_\theta^* C_\theta \quad \text{on } D(A), \tag{86}$$

and also shows that $W^{1,2}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right]) = D(A) \subset D(B_\Lambda^*)$, where B_Λ^* is the Λ-extension of B^*, given by

$$B_\Lambda^* x = \lim_{\substack{\alpha \to +\infty \\ \alpha \in \mathbb{R}^+}} B^* \alpha (\alpha I - A^*)^{-1} x,$$

with domain consisting of the $x \in \mathscr{X}$ such that the above limit exists.

If for the truncation space $\mathscr{X}_n$ we have $\mathscr{X}_n \subset W^{1,2}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$, then we can restrict the equality (86) to $\mathscr{X}_n$. Noting that the generators of the output normal realisation on L^2 and those of the exactly observable shift realisation on L^1 agree on the intersection of their domains, we thus obtain

$$(B_n^{\mathscr{U}})^* + D^* C_n^{\mathscr{Y}} = -D_\theta^* C_n^{\mathscr{U}}, \tag{87}$$

as an operator equation from $\mathscr{X}_n$ to $\mathscr{Y}$ (both finite-dimensional), which when adjointed gives (85), as required.

In general we however do not have $\mathscr{X}_n \subset W^{1,2}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$ so that a somewhat more involved argument is needed. The essential argument is as follows (a more detailed argument can be found in [11, Lemma 6.3.16.]). We first restrict (86) to the intersection $D(A) \cap W^{1,2}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$. We then use that the generators of the output normal realisation on L^2 and those of the exactly observable shift realisation on L^1 agree on the intersection of their domains to replace the operators in (86) by their L^1 well-posed equivalents. By continuity and density, that version of (86) in fact holds on $W^{1,1}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$. We can then restrict to $\mathscr{X}_n$, which is always a subset of $W^{1,1}(\mathbb{R}^+; \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{U} \end{smallmatrix}\right])$, to obtain (87), which as above implies (85).

It remains to prove that the third equation (7c) of the bounded real Lur'e equations holds, i.e.

$$I - D^* D = D_\theta^* D_\theta, \tag{88}$$

In Staffans [29, Corollary 7.2] (see also [37, Remark 12.9]) the following formula is given relating the feedthroughs of the original transfer function G and the spectral factor θ (both of which are regular by Lemma 4.11):

$$D_\theta^* D_\theta = I - D^* D + \lim_{\substack{s \to \infty \\ s \in \mathbb{R}^+}} B_\Lambda^* P_m (sI - A)^{-1} B. \tag{89}$$

That the limit on the right hand side of (89) is zero for strongly stable realisations of transfer functions with an impulse response in L^1 has been proven in the PhD thesis of Mikkola [17, Theorem 9.1.15]. Our realisation of G derived from ${}^{sr}\Sigma_E^2$ satisfies these hypotheses, thus establishing (88). An alternative proof that the limit on the right hand side of (89) is zero, using the Coifman-Rochberg decomposition, is also given in [11, Lemma 6.3.16.].

Therefore, we have proven that

$$\begin{aligned} A_n^* + A_n + (C_n^{\mathscr{Y}})^* C_n^{\mathscr{Y}} &= -(C_n^{\mathscr{U}})^* C_n^{\mathscr{U}}, \\ B_n^{\mathscr{U}} + (C_n^{\mathscr{Y}})^* D &= -(C_n^{\mathscr{U}})^* D_\theta, \\ I - D^* D &= D_\theta^* D_\theta, \end{aligned}$$

which states that $(I_{\mathscr{X}_n}, C_n^{\mathscr{U}}, D_\theta)$ is a (self-adjoint, positive) solution of the bounded real Lur'e equations and hence G_n is bounded real. □

We now have all the ingredients to prove Theorem 1.1.

Proof of Theorem 1.1: From Lemma 4.21 we have that the reduced order transfer function obtained by bounded real balanced truncation G_n is rational and bounded real. It remains to prove the error bound. By Lemma 4.9, every Hankel operator H_E of an extended system Σ_E with transfer function G_E, is nuclear. So [12, Theorem 2.3] applied to G_E yields

$$\|G_E - (G_E)_n\|_{H^\infty} \leq 2 \sum_{k=n+1}^{\infty} \sigma_k, \tag{90}$$

where $(G_E)_n$ is the Lyapunov balanced truncation of G_E (*not* the bounded real balanced truncation), and σ_k are the Lyapunov singular values of G_E and so are also the bounded real singular values of G, by Definition 4.14. By construction of G_E in (59) we have that

$$G(s) = P_{\mathscr{Y}} G_E(s)|_{\mathscr{U}}.$$

Moreover, by construction of the bounded real balanced truncation and Lyapunov balanced truncation (see (72) and [12, Definition 5.5] respectively)

$$G_n(s) = P_{\mathscr{Y}} (G_E)_n(s)|_{\mathscr{U}}.$$

Together these yield

$$\|G - G_n\|_{H^\infty} = \|P_{\mathscr{Y}}(G_E - (G_E)_n)|_{\mathscr{U}}\|_{H^\infty} \leq \|G_E - (G_E)_n\|_{H^\infty}. \tag{91}$$

Combining (90) and (91) gives the result. □

5 The Cayley transform

As is well-known, bounded real and positive real systems are related by the Cayley transform (also known as the diagonal transform and as the Möbius transform). Here we collect the material we shall need in order to be able to convert bounded real balanced truncation to positive real balanced truncation.

Definition 5.1. For $\mathscr{Z}$ a Hilbert space define the set

$$D(\mathcal{S}_{\mathscr{Z}}) := \{T \in B(\mathscr{Z}) \ : \ -1 \in \rho(T)\}\,.$$

The map $\mathcal{S}_{\mathscr{Z}} : D(\mathcal{S}_{\mathscr{Z}}) \to D(\mathcal{S}_{\mathscr{Z}})$ given by

$$D(\mathcal{S}_{\mathscr{Z}}) \ni T \mapsto \mathcal{S}_{\mathscr{Z}}(T) := (I-T)(I+T)^{-1} \in B(\mathscr{Z}),$$

is the Cayley transform. It is self-inverse.

Remark 5.2. For notational convenience we define for $\mathscr{U}$ a Hilbert space

$$\mathcal{S} = \mathcal{S}_{\mathscr{U}}\,, \quad \check{\mathcal{S}} = \mathcal{S}_{L^2(\mathbb{R}^+;\mathscr{U})}.$$

Definition 5.3. For $\mathscr{U}$ a Hilbert space define the set

$$D(\tilde{\mathcal{S}}) := \left\{G : \mathbb{C}_0^+ \to B(\mathscr{U}) \ : \ -1 \in \rho(G(s)),\ \forall\, s \in \mathbb{C}_0^+\right\}.$$

We also call the map $\tilde{\mathcal{S}} : D(\tilde{\mathcal{S}}) \to D(\tilde{\mathcal{S}})$ defined by

$$D(\tilde{\mathcal{S}}) \ni G \mapsto \left(\mathbb{C}_0^+ \ni s \mapsto [\tilde{\mathcal{S}}(G)](s) := \mathcal{S}(G(s))\right),$$

the Cayley transform. It is also self-inverse.

Remark 5.4. We note that the Cayley transform as defined above is the *external* Cayley transform and should not be confused with the *internal* Cayley transform often used to obtain a discrete-time transfer function from a continuous-time transfer function.

Lemma 5.5. *Given the Cayley transform $\tilde{\mathcal{S}}$ of Definition 5.3, and $\mathscr{U}$ a finite-dimensional Hilbert space, let BR, PR, SBR and SPR denote the sets of functions $\mathbb{C}_0^+ \to B(\mathscr{U})$ that are bounded real, positive real, strictly bounded real or strictly positive real respectively. Then*

(i) $BR \not\subseteq D(\tilde{\mathcal{S}})$.

(ii) $PR \subseteq D(\tilde{\mathcal{S}})$.

(iii) $SBR \subseteq D(\tilde{\mathcal{S}})$.

(iv) $\tilde{\mathcal{S}} : SBR \to H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$.

(v) $\tilde{S} : BR \cap D(\tilde{S}) \to PR$ *is a bijection.*

(vi) $\tilde{\mathcal{S}} : SBR \to SPR$.

(vii) $\tilde{\mathcal{S}} : SPR \cap H^\infty \to SBR$ *is a bijection.*

Proof. The proofs of these assertions are elementary and are not given here. See [11, Lemma 7.1.8.] for detailed proofs. The arguments used are very similar to those in, for example, Belevitch [3, p.160, p.189]. □

Corollary 5.6. *Let $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ be strictly bounded real, where $\mathscr{U}$ is finite-dimensional. Then G is regular if and only if $\tilde{\mathcal{S}}(G)$ is.*

Proof. The proof is elementary, and can be found in [11, Corollary 7.1.10.]. □

The next result is contained within [30, Theorem 5.2], although the formulae (92) are not given there, and demonstrates that given a well-posed realisation of a strictly bounded real function G, we can obtain a well-posed realisation of (the strictly positive real function) $\tilde{S}(G)$ with the same state.

Lemma 5.7. *If for strictly bounded real $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$, $\Sigma_G = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathscr{U}, \mathscr{X}, \mathscr{U})$ is an L^p $(1 \leq p < \infty)$ well-posed realisation of G then*

$$\begin{aligned} \Sigma_{\tilde{\mathcal{S}}(G)} &= \begin{bmatrix} \mathfrak{A} - \mathfrak{B}(I+\mathfrak{D})^{-1}\mathfrak{C} & \sqrt{2}\mathfrak{B}(I+\mathfrak{D})^{-1} \\ -\sqrt{2}(I+\mathfrak{D})^{-1}\mathfrak{C} & (I-\mathfrak{D})(I+\mathfrak{D})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{A} - \mathfrak{B}(I+\mathfrak{D})^{-1}\mathfrak{C} & \sqrt{2}\mathfrak{B}(I+\mathfrak{D})^{-1} \\ -\sqrt{2}(I+\mathfrak{D})^{-1}\mathfrak{C} & \tilde{\mathcal{S}}(\mathfrak{D}) \end{bmatrix}, \end{aligned} \tag{92}$$

is an L^p well-posed realisation of $\tilde{\mathcal{S}}(G)$ on $(\mathscr{U}, \mathscr{X}, \mathscr{U})$. Moreover the state trajectories of Σ_G with input u and output y and $\Sigma_{\tilde{\mathcal{S}}(G)}$ with input v and output w given by

$$v = \frac{u+y}{\sqrt{2}}, \quad w = \frac{u-y}{\sqrt{2}}, \tag{93}$$

are the same.

Proof. See [30, Theorem 5.2]. As mentioned in the proof of that result, the relationship

$$v = \frac{u+y}{\sqrt{2}} \quad \Rightarrow \quad u = \sqrt{2}v - y,$$

can be seen as (negative identity) static output feedback with external control v. The relationship

$$w = \frac{u-y}{\sqrt{2}} \quad \Rightarrow \quad w = v - \sqrt{2}y,$$

corresponds to adding an extra feedthrough term. From these observations and the formulae for the closed loop (well-posed) linear system from [31, Theorem 7.1.2] the formulae in (92) follow. □

Remark 5.8. The above result also has a natural converse. Given an L^p $(1 \leq p < \infty)$ well-posed realisation $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathscr{U}, \mathscr{X}, \mathscr{U})$ of a strictly positive real $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ then the realisation in (92) is a L^p well-posed realisation of $\tilde{\mathcal{S}}(J)$. The proof is exactly the same.

6 Positive real balanced truncation

In this section we define the positive real balanced truncation of a strictly positive real function $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ with summable positive real singular values, and prove the gap metric error bound Theorem 1.2. To do so we make use of the material gathered in Section 5. The next result is crucial for linking positive real balanced truncation to bounded real balanced truncation.

Lemma 6.1. *Let Σ_J denote a stable L^2 well-posed linear system with strictly positive real transfer function $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$. Let $\tilde{P}_m$ and $\tilde{Q}_m$ denote the optimal cost operators of the positive real optimal control problems from Lemma 3.10 and Corollary 3.11 subject to Σ_J respectively. Let $\Sigma_{\tilde{\mathcal{S}}(J)}$ denote the L^2 well-posed realisation given by* (92)*. Then the optimal cost operators of the bounded real optimal control problems from Lemma 3.8 and Corollary 3.9 subject to $\Sigma_{\tilde{\mathcal{S}}(J)}$ are $\tilde{P}_m$ and $\tilde{Q}_m$ respectively.*

Proof. Let $\Sigma_J = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ so that by equation (92) the output map and input-output map of $\Sigma_{\tilde{\mathcal{S}}(J)}$ are given by

$$\mathfrak{C}_{\tilde{\mathcal{S}}(J)} = -\sqrt{2}(I + \mathfrak{D})^{-1}\mathfrak{C}, \quad \check{\mathcal{S}}(\mathfrak{D}) = (I - \mathfrak{D})(I + \mathfrak{D})^{-1}, \tag{94}$$

respectively. Let P_m denote the optimal cost operator of the bounded real optimal control problem (32) subject to the realisation $\Sigma_{\tilde{\mathcal{S}}(J)}$. A long, but elementary, calculation using (94) shows that

$$\begin{aligned}
\tilde{P}_m &= \mathfrak{C}^*((\mathfrak{D}\pi_+ + \pi_+\mathfrak{D}^*))^{-1}\mathfrak{C}, \quad \text{from (40)}, \\
&= \mathfrak{C}^*_{\tilde{\mathcal{S}}(J)}\mathfrak{C}_{\tilde{\mathcal{S}}(J)} + \mathfrak{C}^*_{\tilde{\mathcal{S}}(J)}\check{\mathcal{S}}(\mathfrak{D})\pi_+(I - \pi_+\check{\mathcal{S}}(\mathfrak{D})^*\check{\mathcal{S}}(\mathfrak{D})\pi_+)^{-1}\pi_+\check{\mathcal{S}}(\mathfrak{D})^*\mathfrak{C}_{\tilde{\mathcal{S}}(J)} \\
&= P_m, \quad \text{from (35)},
\end{aligned}$$

as required. The dual argument is exactly the same, using instead the dual L^2 well-posed linear systems, which are also related by Lemma 5.7. □

Definition 6.2. Let $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}, \mathscr{Y}))$ denote a strictly positive real function. We define the positive real singular values of J as the bounded real singular values of the strictly bounded real function $G := \tilde{\mathcal{S}}(J)$.

The next result shows that the above definition of positive real singular values is consistent with the finite-dimensional case stated in Section 2.2. It is the positive real version of Lemma 4.16.

Lemma 6.3. *Let $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ denote a strictly positive real function and let $\tilde{P}_m$ and $\tilde{Q}_m$ denote the optimal cost operators of the optimal control problems from Lemma 3.10 and Corollary 3.11 respectively corresponding to a given stable L^2 well-posed realisation of J. Then the positive real singular values are summable if and only if $\tilde{P}_m\tilde{Q}_m$ is compact and the square roots of its eigenvalues are summable. If these conditions hold then apart from possibly zero the positive real singular values are precisely the square roots of the eigenvalues of $\tilde{P}_m\tilde{Q}_m$ (which therefore depend only on J).*

Proof. This follows immediately from the definition of positive real singular values, Lemma 4.16 and Lemma 6.1. □

Corollary 6.4. *If $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ is strictly positive real with summable positive real singular values, then J is regular.*

Proof. Set $G := \tilde{\mathcal{S}}(J)$, which by Lemma 5.5 is strictly bounded real and has summable bounded real singular values by Definition 6.2. From Lemma 4.11 it follows that G is regular, and hence so is J by Corollary 5.6. □

The next lemma prepares the positive real balanced truncation of strictly positive real functions with summable positive real singular values. We obtain a family of L^1 well-posed realisations of J, using the Cayley transform, that we shall truncate in Definition 6.7 to give a family of positive real balanced truncations.

Lemma 6.5. *Given $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ a strictly positive real function with summable positive real singular values, set $G := \tilde{\mathcal{S}}(J)$, which is strictly bounded real and has summable bounded real singular values. Let G_E denote a member of the family of extended transfer functions of G and let (A_E, B_E, C_E, D_E) denote the generators of the exactly observable shift realisation on L^1 of G_E. Let A, B, C and D denote the generators of the L^1 well-posed realisation of G obtained from (A_E, B_E, C_E, D_E) by Lemma 4.13. The operators*

$$\begin{aligned} &\tilde{A} = A - B(I+D)^{-1}C : D(A) \to \mathscr{X}, \quad \tilde{B} = \sqrt{2}B(I+D)^{-1} : \mathscr{U} \to \mathscr{X}, \\ &\tilde{C} = -\sqrt{2}(I+D)^{-1}C : D(A) \to \mathscr{U}, \qquad \tilde{D} = (I-D)(I+D)^{-1} : \mathscr{U} \to \mathscr{U}, \end{aligned} \tag{95}$$

are well-defined and are the generators of an L^1 well-posed realisation for J. In particular,

$$J(s) = \tilde{D} + \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}, \quad s \in \mathbb{C}_0^+. \tag{96}$$

Proof. The function G is strictly bounded real by Lemma 5.5, and has summable bounded real singular values by Definition 6.2. Therefore, we can choose an extended transfer function G_E, exactly observable shift realisation on L^1 of G_E and the resulting generators of an L^1 well-posed realisation of G according to the statement of the lemma.

We transform the L^1 well-posed realisation of G generated by A, B, C and D as in Lemma 5.7, to give an L^1 well-posed realisation of J. The generators $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ of this realisation are given by [31, Theorem 7.5.1 (ii)] and [31, Lemma 5.1.2 (ii)], where we have used the boundedness of B to infer that A, B, C and D generate a compatible system node with $W = D(A)$. Note that there are changes from our (92) and [31, (7.1.5)] because we combined a feedback with an extra feedthrough term. As such the generators have also changed accordingly. The formula (96) follows from [31, Theorem 4.6.3 (ii)]. □

Remark 6.6. The result of Lemma 6.5 is an infinite-dimensional version of [19, Lemma 3]. We remark, however, that the transformation (15) in [19] is *not* the

same transformation as (5.3). As such the formulae in (95) are slightly different to those in [19, Lemma 3]; namely there is a difference in signs.

Definition 6.7. Let $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ denote a strictly positive real function with summable positive real singular values, and let G_E denote a member of the family of extended transfer functions of $G := \tilde{\mathcal{S}}(J)$. Let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ denote the generators of the L^1 well-posed realisation of J from Lemma 6.5. We define the operators $\tilde{A}_n$, $\tilde{B}_n$ and $\tilde{C}_n$ by

$$\begin{aligned} &\tilde{A}_n := \mathcal{P}_{\mathscr{X}_n} \tilde{A}|_{\mathscr{X}_n} : \mathscr{X}_n \to \mathscr{X}_n, \quad \tilde{B}_n := \mathcal{P}_{\mathscr{X}_n} \tilde{B} : \mathscr{U} \to \mathscr{X}_n, \\ &\tilde{C}_n := \tilde{C}|_{\mathscr{X}_n} : \mathscr{X}_n \to \mathscr{U}, \end{aligned} \tag{97}$$

where $\mathscr{X}_n$ is the truncation space (69). The input-state-output system generated by $\left[\begin{smallmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D} \end{smallmatrix}\right]$ is called the reduced order system obtained by positive real balanced truncation (determined by G_E). We call J_n given by

$$J_n(s) := \tilde{C}_n(sI - \tilde{A}_n)^{-1}\tilde{B}_n + \tilde{D},$$

the reduced order transfer function obtained by positive real balanced truncation.

The next lemma demonstrates that the positive real balanced truncation is determined by J up to a unitary transformation and thus that the reduced order transfer function J_n is uniquely determined by J.

Lemma 6.8. *Let $J \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$ denote a strictly positive real function with summable positive real singular values, and let G_E denote a member of the family of extended transfer functions of $G := \tilde{\mathcal{S}}(J)$. For $n \in \mathbb{N}$ let $\left[\begin{smallmatrix} A_n & B_n \\ C_n & D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D} \end{smallmatrix}\right]$ denote the bounded real and positive real balanced truncations (determined by G_E) of G and J respectively, with respective transfer functions G_n and J_n. Then*

(i) We have the following relations between the positive real and bounded real balanced truncations

$$\begin{aligned} &\tilde{A}_n = A_n - B_n(I+D)^{-1}C_n, \quad \tilde{B}_n = \sqrt{2}B_n(I+D)^{-1}, \\ &\tilde{C}_n = -\sqrt{2}(I+D)^{-1}C_n, \qquad \tilde{D} = (I-D)(I+D)^{-1}. \end{aligned} \tag{98}$$

(ii) J_n is proper rational and positive real.

(iii) Different choices of G_E gives rise to positive real balanced truncations that are unitarily equivalent, so that every choice of G_E gives rise to the same J_n.

(iv) The following commutative diagram holds

$$\begin{array}{ccc} J & \xrightarrow{\tilde{\mathcal{S}}} & \tilde{\mathcal{S}}(J) \\ {\scriptstyle \mathrm{prbt}}\downarrow & & \downarrow{\scriptstyle \mathrm{brbt}} \\ J_n & \xrightarrow{\tilde{\mathcal{S}}} & \tilde{\mathcal{S}}(J)_n \end{array}$$

As such, $G_n \in D(\tilde{\mathcal{S}})$ and $J_n = \tilde{\mathcal{S}}(G_n)$.

Proof. That (98) holds follows from the definition of $\left[\begin{smallmatrix} A_n & B_n \\ C_n & D \end{smallmatrix}\right]$ in Definition 4.19, that of $\left[\begin{smallmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D} \end{smallmatrix}\right]$ in Definition 6.7 and the fact that restriction and projection are linear operations. That different choices of G_E give rise to unitarily equivalent positive real balanced truncations now follows from the relations (98) and Lemma 4.20. In particular, every choice of G_E gives rise to the same reduced order transfer function J_n obtained by positive real balanced truncation.

An elementary, but tedious, calculation demonstrates that if (P, K, W) solve the bounded real Lur'e equations (7) subject to the realisation $\left[\begin{smallmatrix} A_n & B_n \\ C_n & D \end{smallmatrix}\right]$ then (P, K', W') solve the positive real Lur'e equations (19) subject to $\left[\begin{smallmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D} \end{smallmatrix}\right]$ where

$$K' = K - W(I + D)^{-1}C_n, \quad W' = \sqrt{2}W(I + D)^{-1}.$$

From the Positive Real Lemma it follows that J_n is positive real and it is clearly rational since it has a realisation with finite-dimensional state-space. Therefore by Lemma 5.5 (ii), $J_n \in D(\tilde{\mathcal{S}})$ and another elementary calculation using (98) shows that $\tilde{\mathcal{S}}(J_n) = G_n$. Therefore by Lemma 5.5 (v), $G_n \in D(\tilde{\mathcal{S}})$ and $\tilde{\mathcal{S}}(J_n) = G_n$.

We note that the commutative diagram is well defined in the sense that it is independent of G_E. Furthermore, the above observations have demonstrated that it does indeed commute. □

We now gather the ingredients required to prove the gap metric error bound for positive real balanced truncation, which we formulated as Theorem 1.2.

Lemma 6.9. *The map F given by*

$$F : \begin{bmatrix} L^2(\mathbb{R}^+; \mathscr{U}) \\ L^2(\mathbb{R}^+; \mathscr{U}) \end{bmatrix} \to \begin{bmatrix} L^2(\mathbb{R}^+; \mathscr{U}) \\ L^2(\mathbb{R}^+; \mathscr{U}) \end{bmatrix}, \quad F = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}, \tag{99}$$

is an isometric isomorphism. With $\check{\mathcal{S}}$ the Cayley transform of Remark 5.2 and $\mathfrak{D} \in D(\check{\mathcal{S}})$ we have

$$\mathcal{G}(\check{\mathcal{S}}(\mathfrak{D})) = F\mathcal{G}(\mathfrak{D}),$$

where $\mathcal{G}(\mathfrak{D})$ denotes the graph of $\mathfrak{D}$.

Proof. The simple proof is left to the reader. □

We remind the reader of the definition of the gap metric, for closed subspaces of a Hilbert space and for closed operators, see also Kato [16, p. 197, p.201].

Definition 6.10. For $\mathscr{M}, \mathscr{N}$ non-empty closed subspaces of a Hilbert space $\mathscr{X}$, the gap is defined as

$$\hat{\delta}(\mathscr{M}, \mathscr{N}) = \|P_{\mathscr{M}} - P_{\mathscr{N}}\|, \tag{100}$$

where $P_{\mathscr{M}}, P_{\mathscr{N}}$ are the orthogonal projections of $\mathscr{X}$ onto $\mathscr{M}$ and $\mathscr{N}$ respectively. For closed linear operators $S, T : \mathscr{X} \to \mathscr{Z}$, where $\mathscr{Z}$ is a Hilbert space, the gap between S and T is defined as

$$\hat{\delta}(S, T) := \hat{\delta}(\mathcal{G}(S), \mathcal{G}(T)), \tag{101}$$

where $\mathcal{G}(S)$ and $\mathcal{G}(T)$ denote the graphs of S and T respectively.

The following elementary lemma shows that the gap metric is invariant under isometries.

Lemma 6.11. *For $\mathscr{M}, \mathscr{N} \subseteq \mathscr{Z}$ closed subspaces of a Hilbert space $\mathscr{Z}$ and $T : \mathscr{Z} \to \mathscr{Z}$ an isometry we have*

$$\hat{\delta}(T\mathscr{M}, T\mathscr{N}) = \hat{\delta}(\mathscr{M}, \mathscr{N}).$$

Proof. This is elementary and a proof can be found in [11, Lemma 7.2.11.], for example. □

We now have all of the ingredients to prove the gap metric error bound Theorem 1.2.

Proof of Theorem 1.2: Since J is strictly positive real with summable bounded real singular values, the hypotheses of Lemma 6.5 are satisfied and thus the positive real balanced truncation J_n of Definition 6.7 is well-defined. From Lemmas 5.5 and 6.3 the transfer function $G := \tilde{\mathcal{S}}(J)$ is strictly bounded real with summable bounded real singular values. Therefore all the assumptions of Theorem 1.1 are satisfied and so the error bound (2) holds for G and its bounded real balanced truncation G_n.

Let $\mathfrak{D}_G$ and $\mathfrak{D}_{G_n}$ denote the input-output maps of G and G_n respectively. From the commuting diagram in Lemma 6.8 it follows that $\check{\mathcal{S}}(\mathfrak{D}_{G_n}) = \mathfrak{D}_{J_n}$. Therefore we compute

$$\begin{aligned}
\hat{\delta}(\mathcal{G}(\mathfrak{D}_J), \mathcal{G}(\mathfrak{D}_{J_n})) &= \hat{\delta}(\mathcal{G}(\check{\mathcal{S}}(\mathfrak{D}_G)), \mathcal{G}(\check{\mathcal{S}}(\mathfrak{D}_{G_n}))), \quad \text{by Lemma 5.7,} \\
&= \hat{\delta}(F\mathcal{G}(\mathfrak{D}_G), F\mathcal{G}(\mathfrak{D}_{G_n})), \quad \text{by Lemma 6.9,} \\
&= \hat{\delta}(\mathcal{G}(\mathfrak{D}_G), \mathcal{G}(\mathfrak{D}_{G_n})), \quad \text{by Lemma 6.11.}
\end{aligned} \tag{102}$$

From [16, Theorem 2.14] it follows that

$$\hat{\delta}(\mathcal{G}(\mathfrak{D}_G), \mathcal{G}(\mathfrak{D}_{G_n})) \leq \|\mathfrak{D}_G - \mathfrak{D}_{G_n}\|, \tag{103}$$

and it is well-known that

$$\|\mathfrak{D}_G - \mathfrak{D}_{G_n}\| = \|G - G_n\|_{H^\infty}, \tag{104}$$

(see for example [35]). Combining (102), (103), (104) and (2) yields

$$\begin{aligned}
\hat{\delta}(\mathcal{G}(\mathfrak{D}_J), \mathcal{G}(\mathfrak{D}_{J_n})) = \hat{\delta}(\mathcal{G}(\mathfrak{D}_G), \mathcal{G}(\mathfrak{D}_{G_n})) &\leq \|\mathfrak{D}_G - \mathfrak{D}_{G_n}\| \\
&= \|G - G_n\|_{H^\infty} \leq 2 \sum_{k \geq n+1} \sigma_k,
\end{aligned}$$

which is (3). Finally we note that $(\sigma_k)_{k \in \mathbb{N}}$ are the bounded real singular values of G which by definition are the positive real singular values of J. □

7 Asymptotic behavior of bounded real and positive real singular values

Our main results, Theorems 1.1 and 1.2, each have two key assumptions. We require that the transfer function is strictly bounded real (respectively strictly positive real) and has summable bounded real singular values (respectively summable positive real singular values). Here we provide a large class of examples where the latter condition is satisfied.

As we have seen in Lemma 4.16, the bounded real singular values are summable precisely when the Hankel singular values of a (equivalently every) member of the family of extended Hankel operators of G are summable. Therefore, we seek conditions which ensure that a Hankel operator is nuclear. The next result is taken from [21]. In what follows $\mathscr{X}_\alpha$ denote interpolation spaces, see for example, [31, Section 3.9] and S_p is the Schatten class; the linear operators whose singular values form a sequence in ℓ^p. In particular, S_1 is the class of nuclear operators.

Theorem 7.1. *Assume that A generates an exponentially stable analytic semigroup, $B \in B(\mathscr{U}, \mathscr{X}_\beta)$, $C \in B(\mathscr{X}_\alpha, \mathscr{Y})$ and $D \in B(\mathscr{U}, \mathscr{Y})$, with $\alpha - \beta < 1$ and that at least one of $\mathscr{U}$ and $\mathscr{Y}$ is finite-dimensional. Then the Hankel operator of this system is in S_p for all $p > 0$.*

Given a stable L^2 well-posed realisation of the strictly bounded real function G with generators (A, B, C, D), and choice of spectral factors θ and ξ as in Lemma 3.12 it follows from Lemma 4.11 that (A, B_E, C_E, D_E) generate a stable L^2 well-posed realisation of the extended transfer function G_E. Here

$$B_E = \begin{bmatrix} B & B_\xi \end{bmatrix}, \quad C_E = \begin{bmatrix} C \\ C_\theta \end{bmatrix},$$

are the generators of $\mathfrak{B}_E$ and $\mathfrak{C}_E$ from (50) and (45) respectively and D_E is as in (60). It is not *a priori* clear how unbounded C_E and B_E are because it is not presently clear how unbounded the components C_θ and B_ξ are. However, under the assumption of strict bounded realness, we are able to formulate the next result which provides checkable conditions for the summability of the bounded real singular values.

Proposition 7.2. *Assume that A generates an exponentially stable analytic semigroup on $\mathscr{X}$, $B \in B(\mathscr{U}, \mathscr{X}_\beta)$, $C \in B(\mathscr{X}_\alpha, \mathscr{Y})$ and $D \in B(\mathscr{U}, \mathscr{Y})$, with $\alpha - \beta < 1$ and that both $\mathscr{U}$ and $\mathscr{Y}$ are finite-dimensional. Then (A, B, C, D) are the generators of a stable L^2 well-posed linear system. If the transfer function of this system is strictly bounded real, then the bounded real singular values belong to ℓ^p for every $p > 0$. In particular, they are summable and moreover decay faster than any polynomial rate.*

Proof. That (A, B, C, D) are the generators of a stable L^2 well-posed linear system follows from [31, Theorem 5.7.3]. In Staffans [27, Theorem 1] it is proven

that under our assumptions the operator C_θ from (86) is bounded $\mathscr{X}_\alpha \to \mathscr{U}$. Hence C_E is bounded $\mathscr{X}_\alpha \to \left[\begin{smallmatrix}\mathscr{Y}\\ \mathscr{U}\end{smallmatrix}\right]$. Arguing analogously in the dual case we deduce that B_ξ satisfies

$$B_\xi \in B(\mathscr{Y}, (\mathscr{X}^*)_{-\beta}) = B(\mathscr{Y}, \mathscr{X}_\beta),$$

where we have identified $(\mathscr{X}^*)_\gamma$ with $\mathscr{X}_{-\gamma}$ as $\mathscr{X}$ is a Hilbert space.

We conclude that B_E is bounded $\left[\begin{smallmatrix}\mathscr{U}\\ \mathscr{Y}\end{smallmatrix}\right] \to \mathscr{X}_\beta$. Therefore from Theorem 7.1, the Hankel operator of the extended system belongs to S_p and hence the bounded real singular values belong to ℓ^p. □

The next result is a corresponding version of the above for positive real systems.

Corollary 7.3. *Assume that A generates an exponentially stable analytic semigroup on $\mathscr{X}$, $B \in B(\mathscr{U}, \mathscr{X}_\beta)$, $C \in B(\mathscr{X}_\alpha, \mathscr{Y})$ and $D \in B(\mathscr{U}, \mathscr{Y})$, with $\alpha \in [0,1]$ and $\alpha - \beta < 1$ and that $\mathscr{U}$ is finite-dimensional. Then (A, B, C, D) are the generators of a stable L^2 well-posed linear system. If the transfer function of this system is strictly positive real, then the positive real singular values belong to ℓ^p for every $p > 0$. In particular, they are summable and moreover decay faster than any polynomial rate.*

Proof. Denote the transfer function associated to (A, B, C, D) by J. From Lemma 5.5, the function $G := \tilde{S}(J)$ is strictly bounded real and from Lemma 6.3 the bounded real singular values of G are the positive real singular values of J. We seek therefore to apply Proposition 7.2, and in order to do so we require a state-space realisation of G. As argued in the proof of Lemma 6.5, the Cayley transform of operators

$$\begin{aligned} \tilde{A} &= A|_{\mathscr{X}_\alpha} - B(I+D)^{-1}C : \mathscr{X}_\alpha \to \mathscr{X}_{\alpha-1}, & \tilde{B} &= \sqrt{2}B(I+D)^{-1} : \mathscr{U} \to \mathscr{X}_\beta, \\ \tilde{C} &= -\sqrt{2}(I+D)^{-1}C : \mathscr{X}_\alpha \to \mathscr{U}, & \tilde{D} &= (I-D)(I+D)^{-1} : \mathscr{U} \to \mathscr{U}, \end{aligned}$$

is well-defined and $\left[\begin{smallmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{smallmatrix}\right]$ is a realisation of G. This follows again from [31, Theorem 7.5.1 (ii)], here using that $W = \mathscr{X}_\alpha$ is a compatible extension of $\mathscr{X}_1$ (see also [31, Lemma 5.1.2 (iii)]). From Curtain *et al.* [5, Proposition 4.5] the operator $\tilde{A}$ (where $-B(I+D)^{-1}C = \Delta$ in the notation of [5]) generates an analytic semigroup on $\mathscr{X}$ and the interpolation spaces $\mathscr{X}_\delta$ and $\tilde{\mathscr{X}}_\delta$ corresponding to A and $\tilde{A}$ respectively are equal for all $\delta \in [\alpha - 1, \beta + 1]$.

Thus

$$\tilde{B} \in \mathscr{B}(\mathscr{U}, \mathscr{X}_\beta) = \mathscr{B}(\mathscr{U}, \tilde{\mathscr{X}}_\beta), \quad \text{and} \quad \tilde{C} \in \mathscr{B}(\mathscr{X}_\alpha, \mathscr{U}) = \mathscr{B}(\tilde{\mathscr{X}}_\alpha, \mathscr{U}),$$

since trivially $\alpha, \beta \in [\alpha - 1, \beta + 1]$. It remains to see that $\tilde{A}$ generates an exponentially stable semigroup. By the same results from [31] above we can "go back again", and recover the realisation for J from that of G, namely

$$\begin{aligned} A|_{\mathscr{X}_\alpha} &= \tilde{A} - \tilde{B}(I+\tilde{D})^{-1}\tilde{C} : \mathscr{X}_\alpha \to \mathscr{X}_{\alpha-1}, & B &= \sqrt{2}\tilde{B}(I+\tilde{D})^{-1} : \mathscr{U} \to \mathscr{X}_\beta, \\ C &= -\sqrt{2}(I+\tilde{D})^{-1}\tilde{C} : \mathscr{X}_\alpha \to \mathscr{U}, & D &= (I-\tilde{D})(I+\tilde{D})^{-1} : \mathscr{U} \to \mathscr{U}. \end{aligned}$$

We now see that $\tilde{A}$ is exponentially stabilisable and detectable since

$$A|_{\mathscr{X}_\alpha} = \tilde{A} + \tilde{B}F_1, \quad F_1 = -(I + \tilde{D})^{-1}\tilde{C},$$
$$A|_{\mathscr{X}_\alpha} = \tilde{A} + F_2\tilde{C}, \quad F_2 = -\tilde{B}(I + \tilde{D})^{-1},$$

and A is exponentially stable. The system with generators $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is input-output stable, since the transfer function $G \in H^\infty(\mathbb{C}_0^+; B(\mathscr{U}))$, and so by Rebarber [23, Corollary 1.8], $\tilde{A}$ generates an exponentially stable semigroup.

All the hypotheses of Proposition 7.2 are satisfied for the realisation $\left[\begin{smallmatrix} \hat{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{smallmatrix}\right]$ of G, and thus the bounded real singular values of G are in ℓ^p for all $p > 0$. Since the bounded real singular values of G and the positive real singular values of J are the same, this completes the proof. □

Remark 7.4. It is easily seen that the transfer function in Corollary 7.3 is strictly positive real provided that, in addition to the assumptions on (A, B, C, D) in Corollary 7.3, the following conditions hold: A is dissipative, $B = C^*$ and $D + D^* > 0$.

8 Example

Consider the 1D heat equation

$$w_t = w_{xx}, \quad t \geq 0,\ x \in [0, 1], \tag{105a}$$

with Dirichlet boundary condition

$$w(t, 1) = 0, \quad \forall\, t \geq 0, \tag{105b}$$

and with input u and output y given by

$$u(t) = -w_x(t, 0), \tag{105c}$$
$$y(t) = w(t, 0) - w_x(t, 0). \tag{105d}$$

The PDE (105) can be written in the form (5) (e.g. as in [21]), with A generating an analytic, exponentially stable contraction semigroup on $\mathscr{X} = L^2(0, 1)$. Here C is the trace operator, which is bounded $\mathscr{X}_\alpha \to \mathbb{C}$ for all $\alpha > \frac{1}{4}$. Furthermore, $B = C^*$, and hence B is bounded $\mathbb{C} \to \mathscr{X}_\beta$ for all $\beta < -\frac{1}{4}$. Finally, $D = 1$. Therefore, using Remark 7.4, we see that the conditions on the operators in Corollary 7.3 are satisfied and hence (105) has summable positive real singular values (belonging to ℓ^p for all $p > 0$, in fact).

We have approximated the heat equation (105) using several standard numerical discretisation methods. Unfortunately, computing the distance in the gap metric between these discretisations and the infinite-dimensional system is intractable. Therefore we have used a piecewise linear finite element (FE)

approximation with $N = 50$ degrees of freedom as a substitute for the infinite-dimensional system. The relevant gap metric distances can then be computed using the `gapmetric` function in MATLAB. The log of the gap metric error versus the number of degrees of freedom in the numerical discretisation is plotted in Figure 1.

Computing the positive real balanced truncation of the infinite-dimensional system is also intractable. Therefore we again take the piecewise linear FE approximation with $N = 50$ degrees of freedom as a substitute for the infinite-dimensional system and compute the positive real balanced truncation of this system. We note that this is the usual procedure for approximating balanced truncations of PDEs. Again, Figure 1 contains the log of the gap metric error between the positive real balanced truncation and the piecewise linear FE approximation with $N = 50$ versus the number of degrees of freedom in the positive real balanced truncation. It can be observed that positive real balanced truncation is vastly superior to the other numerical discretisation methods.

Figure 1 also contains the gap metric error bound for the positive real balanced truncation based on the positive real singular values of the piecewise linear FE approximation with $N = 50$ degrees of freedom. It can be seen that for $n \geq 8$ this error bound is in fact smaller than the error as computed by the `gapmetric` function in MATLAB. This is due to the inaccuracy of the `gapmetric` function in MATLAB which has a maximal tolerance of 10^{-5}, which for $n \geq 8$ is larger than the actual error. With this in mind, it is clear that for this example our gap metric error bound is tight and for $n \geq 8$ it is in fact a better approximation of the actual error than the error computed by the `gapmetric` function in MATLAB.

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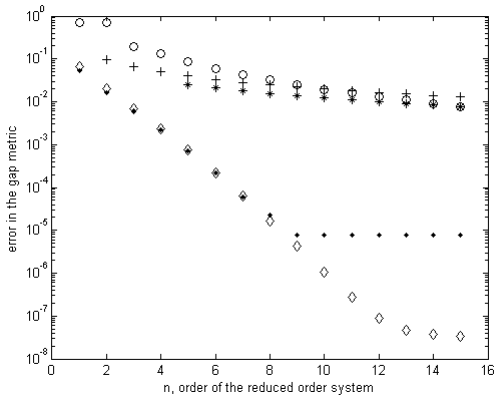
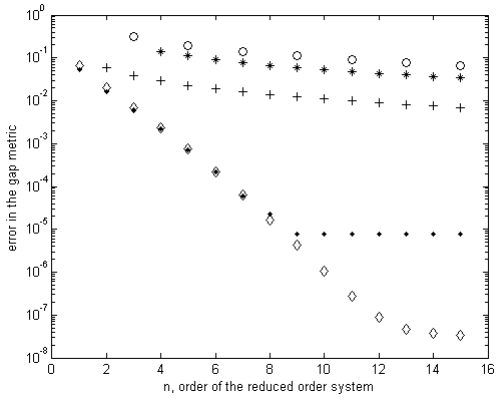


(a)



(b)

Figure 1: Approximation of heat equation (105). Both figures contain the positive real balanced truncation ($\cdot$) and the gap metric error bound ($\diamond$). Figure 1(a) in addition contains finite difference approximations of order two ($+$) and four ($*$) and the Chebyshev collocation method ($\circ$). Figure 1(b) in addition contains finite element (FE) approximations using piecewise linear ($+$), quadratic ($*$) and cubic ($\circ$) elements.

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