Coprime factorization and optimal control on the doubly infinite discrete time axis

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Abstract

We study the problem of strongly coprime factorization over H-infinity of the unit disc. We give a necessary and sufficient condition for the existence of such a coprime factorization in terms of an optimal control problem over the doubly infinite discrete-time axis. In particular, we show that an equivalent condition for the existence of such a coprime factorization is that both the control and filter algebraic Riccati equation (of an arbitrary realization) have a solution (in general unbounded and even non densely defined) and that a coupling condition involving these solutions is satisfied.

1 Introduction

This is the third in a series of articles dealing in a novel way with the quadratic cost minimization problem for infinite-dimensional time-invariant linear systems in discrete and continuous time. In the first article [9] we investigated the full information infinite-horizon LQ (Linear Quadratic) problem, in the second article [10] we studied a deterministic version of the discrete time infinite-horizon Kalman filtering problem. In this third article we combine the results from the previous two articles to study dynamic stabilization, coprime factorization and optimal control on the whole $\mathbb{Z}$ axis. Perhaps surprisingly, in our general setting, in addition to assuming the solvability of both the full information LQ problem and the Kalman filtering problem, a coupling condition has to be imposed.

In the remainder of this introduction we will explain our results starting with the simplest possible case (a minimal finite-dimensional system) and gradually build up to the very general case for which precise definitions and proofs are given in the main body of the article.

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1.1 Minimal systems

So at first we consider the system

$$x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n + Du_n,$$

where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is assumed to be minimal and the input, state and output spaces are finite-dimensional. A classical problem is to consider this system on the positive time axis and for a given initial state x_0 to minimize the cost functional

$$J_f(x_0, u) := \sum_{n=0}^{\infty} \|u_n\|^2 + \|y_n\|^2,$$

over all input sequences u. It is well-known that for any x_0 in the state space there exists a minimizing input $u^{\min}$, that the minimal cost is given by

$$J_f(x_0, u^{\min}) = \langle Qx_0, x_0 \rangle,$$

where Q is the unique positive self-adjoint solution of the control algebraic Riccati equation

$$A^*QA - Q + C^*C - (A^*QB + C^*D)(I + D^*D + B^*QB)^{-1}(B^*QA + D^*C) = 0,$$

and that $u^{\min}$ is of state feedback form

$$u_n^{\min} = Kx_n,$$

where

$$K = -S^{-1}(B^*QA + D^*C), \quad S = I + D^*D + B^*QB.$$

It is also interesting to consider the situation when the input is not equal to the minimizing one, but there is some disturbance: $u = u^{\min} + v$. It is then natural to consider a new system with the disturbance as input and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ as the output (the effect of the disturbance on the cost is then the same as its effect on the ℓ^2 norm of this new output). This new system is described by the equations

$$\begin{aligned} x_{n+1} &= (A + BK)x_n + Bv_n, \\ \begin{bmatrix} u \\ y \end{bmatrix}_n &= \begin{bmatrix} K \\ C + DK \end{bmatrix} x_n + \begin{bmatrix} I \\ D \end{bmatrix} v_n. \end{aligned}$$

Interestingly, the transfer function $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ of this closed-loop system provides a right factorization of the transfer function G of the original system, i.e. $\mathsf{G}(z) = \mathsf{N}(z)\mathsf{M}(z)^{-1}$ for all z in a neighbourhood of the origin.

Less well-studied than the initial state optimal control problem is the similar final state optimal control problem. In this problem for a given final state x_0 the objective is to minimize the cost functional

$$J_p(x_0, u) := \sum_{n=-\infty}^{-1} \|u_n\|^2 + \|y_n\|^2.$$

In this final state optimal control problem some care is needed in defining the trajectories of the system for a given input. The most natural choice seems to be to initially only consider compactly supported inputs, i.e. to assume that there exists a $N > 0$ such that $u_n = 0$ for $n \leq -N$, and take $x_{-N} = 0$ (i.e the system starts at rest). Within this class of inputs the cost functional J_p does in general not have a minimum, only an infimum, but the minimizing sequence converges to a unique element $u^{\min}$ of $\ell^2(\mathbb{Z}^-;\mathscr{U})$ and the corresponding state $x^{\min}$ and output $y^{\min}$ can be defined (details are given in [10]). The minimal cost is given by

$$J_p(x_0, u^{\min}) = \langle P^{-1}x_0, x_0\rangle,$$

where P is the unique positive self-adjoint solution of the filter algebraic Riccati equation

$$APA^* - P + BB^* - (APC^* + BD^*)(I + DD^* + CPC^*)^{-1}(CPA^* + DB^*) = 0.$$

We remark that when A is invertible, then this final state optimal control problem is equivalent to the initial state optimal control problem for the time-inverted system

$$x_{n+1} = A^{-1}x_n - A^{-1}Bu_n, \quad y_n = CA^{-1}x_n + (D - CA^{-1}B)u_n.$$

The final state optimal control problem is related to left factorizations of the transfer function. The output injection closed-loop system

$$\begin{aligned} x_{n+1} &= (A - HC)x_n + (B - HD)u_n - Hy_n, \\ w_n &= Cx_n + Du_n + y_n, \end{aligned}$$

with

$$H = -(APC^* + BD^*)R^{-1}, \quad R = I + DD^* + CPC^*$$

provides a left factorization $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ of the transfer function G of the original system, i.e. $\mathsf{G}(z) = \tilde{\mathsf{M}}(z)^{-1}\tilde{\mathsf{N}}(z)$ for all z in a neighbourhood of the origin.

Once the initial state and final state optimal control problems are posed, it is natural to consider the cost functional

$$J(x_0, u) := \sum_{n=-\infty}^{\infty} \|u_n\|^2 + \|y_n\|^2,$$

where the state is required to pass through the given state x_0 at time zero. It seems reasonable to call this the *intermediate state optimal control problem.* Obviously this problem splits into a final state optimal control problem and an initial state optimal control problem and the minimum cost is given by

$$J(x_0, u^{\min}) = \langle P^{-1}x_0, x_0\rangle + \langle Qx_0, x_0\rangle.$$

Using the operators associated to these optimal control problems we can define the following system whose transfer function $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}]$ is a Bézout factor for the right factorization that came from the initial state optimal control problem (i.e. $\tilde{\mathsf{X}}\tilde{\mathsf{M}} - \tilde{\mathsf{Y}}\tilde{\mathsf{N}} = I$):

$$\begin{aligned} x_{n+1} &= (A - HC)x_n + (B - HD)u_n + Hy_n, \\ w_n &= Kx_n + u_n. \end{aligned} \tag{1}$$

It follows from the factorization approach to control theory (as in e.g. [14]) that $\tilde{\mathsf{X}}^{-1}\tilde{\mathsf{Y}}$ is the transfer function of a stabilizing dynamic controller for the original system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. The realization of this controller resulting from the above realizations of the Bézout factors is

$$\begin{aligned} \hat{x}_{n+1} &= [A + BK - H(C + DF)]\hat{x}_n - Hy_n, \\ u_n &= K\hat{x}_n. \end{aligned}$$

This is the well-known LQG or H^2 controller.

1.2 Finite and coercive cost systems

A weaker condition than controllability is the following finite future cost condition: for every x_0 there exist a control u^f such that $J_f(x_0, u^f) < \infty$. All that was said about the initial state optimal control problem carries through under only this assumption with two exceptions. Uniqueness of the solution of the control Riccati equation need no longer hold. The solution that has to be chosen is the smallest nonnegative semi-definite one (since we consider optimal control problems without an internal stability constraint). The second exception is that in contrast to the minimal case, the smallest solution Q of the control Riccati equation is only nonnegative semi-definite (it need not be strictly positive). This means that certain initial states can have zero cost associated to them (these are exactly the unobservable ones).

The similarly weaker replacement for observability seems to have appeared first in [10]. It is the state coercive past cost condition: there exists a constant M such that for all x_0 and all u^p: $\|x_0\|^2 \le M^2 J_p(x_0, u^p)$. Again, everything that was said about the final state optimal control problem carries through with two exceptions. The first is again the statement about the uniqueness of the solution of the filter Riccati equation (and again the smallest nonnegative semi-definite solution is the right one). The second exception relates to the fact that certain final states may now not be reachable. This means that for these final states $J_p(x_0, u^p)$ will not be finite for any choice of control u^p. This relates to the smallest solution P of the filter Riccati equation being only nonnegative semi-definite and not necessarily strictly positive. See [10, Section 1.2] for some simple examples illustrating this.

When both the finite future cost condition and the state coercive past cost condition hold then the statements made above about Bézout factors and stabilizing dynamic controllers hold without change.

1.3 No assumptions at all

The condition that for every x_0 there exist a control u^f such that $J_f(x_0, u^f) < \infty$ is actually also stronger than is needed. To obtain a satisfactory theory it is enough to assume this condition only for certain x_0. We will now give a Riccati equation based argument for what would be reasonable conditions to put on the set of finite cost initial conditions, we have shown in [9] that those conditions indeed lead to a satisfactory theory. Since $\langle Qx_0, x_0\rangle = \|Q^{1/2}x_0\|^2$ is the optimal cost, the set of finite cost initial conditions is exactly $D(Q^{1/2})$. So we will get a solution Q of the control Riccati equation that is only defined on some subset of the state space. At this point it makes sense to write the control Riccati equation in a slightly different form. In terms of the sesquilinear forms $q[x, z] := \langle Qx, z\rangle = \langle Q^{1/2}x, Q^{1/2}z\rangle$ and $s[x, z] := \langle Sx, z\rangle$ it (and the definitions of S and K) can equivalently be written as

$$q[Az + Bu, Az + Bu] + \|Cz + Du\|^2_{\mathscr{Y}} + \|u\|^2_{\mathscr{U}} = q[z, z] + s[Kz - u, Kz - u].$$

A glance at this equation shows that for this equation to make sense we must have that the image of B is in $D(q)$ (because the equation should make sense for $z = 0$) and that $D(q)$ must be invariant under A (since the equation should make sense for $u = 0$). This exactly means that the reachable states should be in the domain of q. As argued above, the domain of q will consist exactly of those initial states for which the initial state optimal control problem can be solved. So from the point of view of making sense of the Riccati equation we naturally arrive at the condition that for every reachable x_0 there should exist a control u^f such that $J_f(x_0, u^f) < \infty$. We called this the finite future incremental cost condition in [9] and there we showed that under this condition a satisfactory theory for the initial state optimal control problem can indeed be developed. We summarize those results (for the general possibly infinite-dimensional case) in Section 2. The upshot is that with some slight modifications (essentially boiling down to replacing the state space with the reachable subspace) everything that we said above about the initial state optimal control problem in the minimal case still holds under the finite future incremental cost condition. When the input, output and state space of the system are finite-dimensional then it turns out that the finite future incremental cost condition always holds, so that in this case in fact no assumptions at all are needed!

For the final state optimal control problem the state coercive past cost condition can be weakened to the output coercive past cost condition: there exists a constant M such that for all x_0 and all u^p: $\|Cx_0\|^2 \leq M^2 J_p(x_0, u^p)$. We studied this situation (for the general possibly infinite-dimensional case) in [10] and those results are summarized in Section 3. Again, in the finite-dimensional case this output coercive past cost condition is in fact always satisfied.

The focus of this article is the intermediate state optimal control problem. The appropriate assumption for that problem turns out to be the past cost dominance condition (Definition 4.1): there exists a M and a u^f such that for any u^p with compact support $J_f(x_0, u^f) \leq M^2 J_p(x_0, u^p)$ where x_0 is the state reached at time zero by applying the control u^p. Also this condition is always satisfied in the finite-dimensional case, so that in the next subsection we highlight its significance in the infinite-dimensional case.

1.4 Infinite-dimensional systems

We start our remarks on infinite-dimensional systems by reviewing some work on the initial state optimal control problem that predates our [9]. We first of all note that in the infinite-dimensional case minimality no longer implies the finite future cost condition. The results surveyed above for finite-dimensional systems were first obtained for infinite-dimensional systems under exponential stabilizability and detectability conditions. That the Riccati equation and the feedback operator can be obtained under the weaker finite future cost condition has been known for some time. The relation with right factorizations at this level of generality was perhaps first made in [8] (the fact that the $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ is in H^∞ is trivial under the exponential stabilizability assumption whereas it requires considerable work under the finite future cost condition). In Mikkola [6] it was shown that the obtained $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ is actually weakly right coprime (in this article we are interested in the stronger property of strong or Bézout right coprimeness). The above mentioned result on the finite future cost condition and right factorizations is optimal in the sense that not only does the fact that the finite future cost condition holds for some realization imply that the transfer function has a right factorization, but also the converse is true: if the transfer function has a right factorization, then it has a realization for which the finite future cost condition holds.

As mentioned earlier, the results from [9] are reviewed in some detail in Section 2. Here we just mention that whereas the result about the finite future cost condition states the existence of a realization, the similar statement concerning the finite future incremental cost condition is: if the transfer function has a right factorization, then for any realization the finite future incremental cost condition holds. We also remark that (even for a minimal system) the solution of the control Riccati equation of an arbitrary realization may be unbounded (and for nonminimal systems it need not even by densely defined).

In this article we are interested in the intermediate state optimal control problem and its relation to (strong or Bézout) coprime factorizations and stabilizing dynamic controllers as expounded upon above for finite-dimensional systems. We note that in the infinite-dimensional case it is also no longer true that every transfer function has a coprime factorization. We first mention some previous work in continuous-time. Under exponential stabilizability and detectability conditions the results are well-known. In [13] the concept of a jointly stabilizable/detectable system was introduced and it was shown that this implies that the transfer function has a strongly coprime factorization and that a stabilizing dynamic controller exists. Conversely, if a transfer function has a strongly coprime factorization (or equivalently: a stabilizing dynamic controller), then it has a realization that is jointly stabilizable/detectable. This result was improved upon in [1] where it was shown that 'jointly stabilizable/detectable' can be replaced here by the condition that the finite future cost condition holds for both the system and its dual system (this condition on the dual is equivalent to the coercive past cost condition holding for the system itself). Interestingly, the Bézout factors constructed in [1] are not (1) which result in the LQG controller, but rather the ones that result in a robustly stabilizing controller as in Glover and McFarlane [3]. Whether (1) provides a Bézout factor is —at this level of generality— still an open problem. The formulas obtained in [1] for the Bézout factors are however still in terms of the solutions Q and P of the control and filter Riccati equation respectively. These solutions exist and are bounded by the cost conditions assumed. The discrete-time equivalents of the results from [1] can be found in [2].

In this article we show that the above mentioned past cost dominance condition implies the existence of Bézout factors and therefore of a stabilizing dynamic controller. Conversely, if a transfer function has a strongly coprime factorization (or equivalently a stabilizing dynamic controller), then any realization satisfies the past cost dominance condition. Another equivalent condition is that the control Riccati equation and the filter Riccati equation of an arbitrary realization both have a (possibly unbounded) solution and that a coupling condition is satisfied. This coupling between the full information and filtering problems is reminiscent of H^∞ control theory.

2 The initial state optimal control problem

In this section we review and extend the relevant results from [9]. The system under study in this section is

$$x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n + Du_n, \quad n \in \mathbb{Z}^+; \qquad x_0 = z, \tag{2}$$

where $A\colon \mathscr{X} \to \mathscr{X}$, $B\colon \mathscr{U} \to \mathscr{X}$, $C\colon \mathscr{X} \to \mathscr{Y}$, and $D\colon \mathscr{U} \to \mathscr{Y}$ are bounded linear operators, $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{Y}$ are Hilbert spaces, and $\mathbb{Z}^+$ is the set of nonnegative integers. By the *node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ we mean operators as above. The associated cost function is $J_f(z,u) := \sum_{n=0}^{\infty} \|u_n\|^2 + \|y_n\|^2$. Define Ξ_f as the set of those $z \in \mathscr{X}$ for which there exists a u such that $J_f(z,u) < \infty$. Standard arguments show that for each $z \in \Xi_f$ there exists a unique optimal control. We define $\mathcal{I}_f : \Xi_f \to \ell^2(\mathbb{Z}^+; \mathscr{U} \times \mathscr{Y})$ as the map that sends z to the corresponding optimal input-output pair. We further define the sesquilinear form ξ_f with $D(\xi_f) = \Xi_f$ as $\xi_f[z_1, z_2] := \langle \mathcal{I}_f z_1, \mathcal{I}_f z_2 \rangle_{\ell^2(\mathbb{Z}^+; \mathscr{U} \times \mathscr{Y})}$. The corresponding quadratic form $\xi_f[z,z]$ gives the optimal cost.

Definition 2.1. The finite future incremental cost condition is the condition $B\mathscr{U} \subset \Xi_f$. The finite future cost condition is the condition $\Xi_f = \mathscr{X}$.

The following is the standard control algebraic Riccati equation re-written in a way (using sesquilinear forms) that easily allows for unbounded solutions.

Definition 2.2. The triple (q, s, K) is called a (nonnegative) solution of the *control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. q is a closed nonnegative symmetric sesquilinear form in $\mathscr{X}$ whose domain satisfies $AD(q) \subset D(q)$, $B\mathscr{U} \subset D(q)$.

2. s is a bounded nonnegative symmetric sesquilinear form on $\mathscr{U}$.

3. $K : D(q) \to \mathscr{U}$ is a linear operator.

4. For all $z \in D(q)$, $u \in \mathscr{U}$ we have

$$q[Az+Bu, Az+Bu]+\|Cz+Du\|_{\mathscr{Y}}^2+\|u\|_{\mathscr{U}}^2 = q[z,z]+s[Kz-u, Kz-u]. \quad (3)$$

The solution is called *classical* when $D(q) = \mathscr{X}$.

Remark 2.3. In Part 1 [9] we gave several equivalent formulations of the control Riccati equation. Among them is the following in terms of operators instead of sesquilinear forms. The triple (Q, S, K) is called a (nonnegative) solution of the *operator control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if:

1. Q is a closed nonnegative self-adjoint operator in $\mathscr{X}$ whose domain satisfies $AD(Q^{1/2}) \subset D(Q^{1/2})$, $B\mathscr{U} \subset D(Q^{1/2})$.

2. S is a bounded nonnegative self-adjoint operator on $\mathscr{U}$.

3. $K : D(Q^{1/2}) \to \mathscr{U}$ is a linear operator.

4. For all $z \in D(Q^{1/2})$, $u \in \mathscr{U}$ we have

$$\|Q^{1/2}(Az+Bu)\|_{\mathscr{X}}^2 + \|Cz+Du\|_{\mathscr{Y}}^2 + \|u\|_{\mathscr{U}}^2 = \|Q^{1/2}z\|_{\mathscr{X}}^2 + \|S(Kz-u)\|_{\mathscr{U}}^2.$$

The solution is called *classical* when $D(Q) = \mathscr{X}$.

We note that if Q is not densely defined, then what it means for it to be self-adjoint is not immediately obvious. We defined this in an ad-hoc way in [9]. A better way (which amounts to the same as what we did in [9]) is to say that Q is the operator part of a nonnegative self-adjoint multi-valued operator, which is always unambiguously defined (see Appendix A, especially example A.2).

To discuss transfer functions, we use the following notation: H^∞ denotes the Hardy space of uniformly bounded holomorphic functions and $\mathbb{D}$ denotes the unit disc. The transfer function of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is defined in a neighbourhood of zero by $zC(I - zA)^{-1}B + D$. A node is called a *realization* of a holomorphic function defined in a neighbourhood of zero if that function is the transfer function of the node. We note that any holomorphic function defined in a neighbourhood of zero has a realization (in fact, it has infinitely many).

Definition 2.4. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic at the origin. A function $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ is called a *right factorization* of G if $\mathsf{M}(z)$ is invertible for all z in a neighbourhood of the origin and $\mathsf{G}(z) = \mathsf{N}(z)\mathsf{M}(z)^{-1}$ in a neighbourhood of the origin.

Theorem 2.5. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic at the origin and let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a realization of G. The following are equivalent conditions.*

- $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future incremental cost condition.*
- *The control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *has a (nonnegative self-adjoint) solution.*
- G *has a right factorization.*

Under these equivalent conditions, the triple (q_f, s_f, K_f) *defined by*

$$\begin{aligned} q_f[z_1, z_2] &:= \langle \mathcal{I}_f z_1, \mathcal{I}_f z_2 \rangle_{\ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y})}, \\ s_f[u, v] &:= \langle u, v\rangle_{\mathscr{U}} + \langle Du, Dv\rangle_{\mathscr{Y}} + q_f[Bu, Bv], \\ K_f z &= P_{\mathscr{U}}(\mathcal{I}_f z)_0, \end{aligned}$$

is the smallest nonnegative self-adjoint solution of the control Riccati equation. Here $P_{\mathscr{U}}$ *is the canonical projection* $\mathscr{U} \times \mathscr{Y} \to \mathscr{U}$*.*

Proof. This follows from [9, Theorem 6.3] combined with [9, Theorem 3.14]. $\square$

In this article we are mainly interested in strongly coprime factorizations for whose existence more assumptions are needed than those made in the above theorem. Under the assumptions of the above theorem weakly right coprime factorizations however do already exist (Corollary 2.7). We first recall the relevant definition (note that by [5, Theorem 2.19] this definition is equivalent to other definitions of weakly right coprime factorization in the literature).

Definition 2.6. A right factorization $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ of G is called *weakly right coprime* if the range of the operator on H^2 of multiplication with $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ equals the graph of the operator on H^2 of multiplication with G. It is called *normalized* if multiplication with $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ is an isometry on H^2.

Corollary 2.7. *Assume that the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future incremental cost condition. Then its transfer function has a normalized weakly right coprime factorization* $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ *and with*

$$\begin{aligned} \mathcal{M}(z) &= zK_f \left[I - z(A + BK_f)\right]^{-1} BS_f^{-1/2} + S_f^{-1/2}, \\ \mathcal{N}(z) &= z\left[C + DK_f\right]\left[I - z(A + BK_f)\right]^{-1} BS_f^{-1/2} + DS_f^{-1/2}, \\ \mathcal{P}(z) &= -zS_f^{1/2} K_f (I - zA)^{-1} B + S_f^{1/2}, \end{aligned}$$

where S_f *is the nonnegative self-adjoint operator corresponding to the sesquilinear form* s_f*, we have* $\rho(A+BK_f)\cap\mathbb{D} \subset \rho(A)\cap\mathbb{D}$ *and on the connected component of* $\rho(A)\cap\mathbb{D}$ *that contains zero we have*

$$\mathcal{M}(z) = \mathsf{M}(z), \quad \mathcal{N}(z) = \mathsf{N}(z), \quad \mathcal{P}(z) = \mathsf{M}(z)^{-1}.$$

Proof. Consider the graph node associated to q_f of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as defined in [9, Definition 4.10]. It follows from part 1 of [9, Lemma 4.11] that the transfer function of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and that of its graph node are the same. It follows from [9, Lemma 4.11] that this graph node has (q_f, K_f, s_f) as the smallest nonnegative self-adjoint solution of its control Riccati equation and that this solution is classical (that it is the smallest solution is not stated in [9, Lemma 4.11], but in this special case where $q = q_f$, it follows from [9, Theorem 3.14]). From [6, Theorem 1.2] it then follows that the transfer function of the graph node has a normalized weakly coprime factorization given in a neighbourhood of zero by the above formulas $\mathcal{M}$ and $\mathcal{N}$. The formula $\mathcal{P}$ for M^{-1} follows from applying [12, (12.1.7) on page 701] to the formula $\mathcal{M}$. The claims about the resolvent sets and the validity of the above formulas on the connected component of $\rho(A) \cap \mathbb{D}$ that contains zero follow using the argument from the proof of [5, Theorem 2.21]. The proof is then complete. $\square$

We recall that a node is called *minimal* if it is both approximately controllable and approximately observable [12, Definition 9.1.2].

Definition 2.8. The node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is called *LQ future normalized* if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is minimal, satisfies the finite future cost condition and $\mathcal{I}_f : \mathscr{X} \to \ell^2(\mathbb{Z}^+; \mathscr{U} \times \mathscr{Y})$ is an isometry.

An LQ future normalized realization can be constructed from a node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ that satisfies the finite future incremental cost condition by (see [9]) compressing the system onto its reachable subspace, then factoring out the unobservable subspace, subsequently taking $\xi_f[z, z]$ —or more accurately the quadratic form that it induces on the quotient space— as the new norm and finally completing the so obtained state space with respect to this norm.

Remark 2.9. Like input and output normalized systems [12, Section 9.5] and optimal and $*$-optimal systems [12, Section 11.8], LQ future normalized realizations of a transfer function are unique up to a unitary similarity transformation in the state space (Lemma 2.11). Just as optimal and $*$-optimal realizations are natural for contraction valued transfer functions and input and output normalized realizations are natural for transfer functions which induce a bounded Hankel operator, LQ future normalized realizations are natural realizations for transfer functions that have a right factorization over H^∞ (Theorem 2.12).

Before we state and prove the next lemmas, we recall the operator (introduced in [10])

$$\Gamma_f : D(\Gamma_f) \subset \ell^2(\mathbb{Z}^-; \mathscr{U}) \to \ell^2(\mathbb{Z}^+; \mathscr{U} \times \mathscr{Y}),$$

that sends a compactly supported input to the corresponding optimal future input-output trajectory. We note that this operator depends only on the transfer function and not on the realization.

The following lemma shows that in some sense a LQ future normalized realization has the largest state space among minimal realizations that satisfy the finite future cost condition.

Lemma 2.10. *Let* $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ *be minimal and satisfy the finite future cost condition. Let* $\left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$ *be a LQ future normalized realization of the same transfer* *function. Then there exists an injective operator $U \in \mathcal{L}(\mathscr{X}_1, \mathscr{X}_2)$ such that*

$$\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}.$$

Proof. We first show that $\overline{R(\mathcal{I}_f^1)} = R(\mathcal{I}_f^2)$. By controllability, the closure of $R(\mathcal{I}_f^i)$ (for $i = 1, 2$) equals the closure of $R(\Gamma_f)$. Since $\mathcal{I}_f^2$ is an isometry, its range is closed, which proves the assertion.

Since $\mathcal{I}_f^2$ is an isometry, as an operator $\mathscr{X}_2 \to R(\mathcal{I}_f^2)$ it is invertible. It follows from this and the above established $\overline{R(\mathcal{I}_f^1)} = R(\mathcal{I}_f^2)$ that the operator $U := \left(\mathcal{I}_f^2\right)^{-1} \mathcal{I}_f^1$ is well-defined. Since $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ is observable, $\mathcal{I}_f^1$ is injective [9, Lemma 4.4] so that U is injective.

It remains to show the intertwining conditions. We first show that $UB_1 = B_2$, or equivalently that $\mathcal{I}_f^1 B_1 v = \mathcal{I}_f^2 B_2 v$ for all $v \in \mathscr{U}$. This follows since both sides equal $\Gamma_f u$ where u is given by $u_k = 0$ for $k \neq -1$ and $u_{-1} = v$. It similarly follows that $\mathcal{I}_f^1 A_1^n B_1 = \mathcal{I}_f^2 A_2^n B_2$ for all $n \in \mathbb{Z}^+$, i.e. $UA_1^n B_1 = A_2^n B_2$ for all $n \in \mathbb{Z}^+$. Now consider the equality $\mathcal{I}_f^1 A_1 A_1^n B_1 = \mathcal{I}_f^2 A_2 A_2^n B_2$ (which is just a re-writing of the just obtained one) and use the just established $UA_1^n B_1 = A_2^n B_2$ to re-write this as $\mathcal{I}_f^1 A_1 A_1^n B_1 = \mathcal{I}_f^2 A_2 U A_1^n B_1$. By controllability it follows that $\mathcal{I}_f^1 A_1 = \mathcal{I}_f^2 A_2 U$ on a dense set, which by continuity extends to all of $\mathscr{X}_1$. We then note that this equality is nothing else than $UA_1 = A_2 U$, which is one of the other desired intertwining conditions. The last intertwining condition $C_1 = C_2 U$ is proven somewhat differently. By definition of U we have $\mathcal{I}_f^2 U = \mathcal{I}_f^1$. Projecting onto the zero-th component shows that

$$\begin{bmatrix} K_f^2 \\ C_2 + DK_f^2 \end{bmatrix} U = \begin{bmatrix} K_f^1 \\ C_1 + DK_f^1 \end{bmatrix},$$

from which it follows that $C_2 U = C_1$. □

The next lemma shows that LQ future normalized realizations are essentially unique.

Lemma 2.11. *Let $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$ be two LQ future normalized realizations of the same transfer function. Then there exists a unitary operator $U : \mathscr{X}_1 \to \mathscr{X}_2$ such that*

$$\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}.$$

Proof. By Lemma 2.10 it only remains to show that U is unitary. By the proof of Lemma 2.10 we have $U = \left(\mathcal{I}_f^2\right)^{-1} \mathcal{I}_f^1$ and $R(\mathcal{I}_f^1) = R(\mathcal{I}_f^2)$. As the composition of two isometries, U is an isometry and as the composition of two surjective operators —where the intermediate space is $R(\mathcal{I}_f^1)$—, it is surjective. It follows that U is unitary. □

Theorem 2.12. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic at the origin. The following are equivalent:*

- G *has a right factorization,*
- G *has a LQ future normalized realization.*

Proof. We first prove 1 implies 2. The completed q_f^--compression from [9], which is there shown to exist under the condition that G has a right factorization, is a LQ future normalized realization.

We now show 2 implies 1. A LQ future normalized realization by definition satisfies the finite future cost condition, so it follows from Theorem 2.5 that its transfer function G has a right factorization. □

3 The final state optimal control problem

In this section we review and extend the relevant results from [10]. The system under study in this section is

$$\begin{aligned} x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n + Du_n, \quad n \in \mathbb{Z}^-; \qquad x_0 = z, \\ \exists N \in \mathbb{Z}^+ : x_n = 0 = u_n \quad \forall n \leq -N, \end{aligned} \tag{4}$$

where $A\colon \mathscr{X} \to \mathscr{X}$, $B\colon \mathscr{U} \to \mathscr{X}$, $C\colon \mathscr{X} \to \mathscr{Y}$, and $D\colon \mathscr{U} \to \mathscr{Y}$ are bounded linear operators, $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{Y}$ are Hilbert spaces, and $\mathbb{Z}^-$ is the set of negative integers. The associated cost function is $J_p(z,u) := \sum_{n=-\infty}^{-1} \|u_n\|^2 + \|y_n\|^2$. In [10], the set Ξ_p of finite cost final states and the map $\mathcal{I}_p : \Xi_p \to \ell^2(\mathbb{Z}^+; \mathscr{U} \times \mathscr{Y})$ that sends a finite cost final state to the optimal input-output trajectory that reaches it are defined. We define the (closed nonnegative) sesquilinear form ξ_p with domain $D(\xi_p) = \Xi_p$ by $\xi_p[z_1, z_2] = \langle \mathcal{I}_p z_1, \mathcal{I}_p z_2 \rangle_{\ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y})}$. The corresponding quadratic form $\xi_p[z,z]$ then gives the optimal cost.

We define Ξ_- as the subset of $\mathscr{X}$ of states that are reachable in a finite time. For Ξ_- define

$$\mathscr{W}_c(z) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \ell^2_c(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) : \exists x \text{ such that (4) holds} \right\},$$

the set of compactly supported input-output trajectories with z as final state.

Definition 3.1. A node satisfies the *output coercive past cost condition* if there exists a $M > 0$ such that for all $z \in \Xi_-$ and all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{W}_c(z)$

$$\|Cz\|_{\mathscr{Y}} \leq M \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})}.$$

The node satisfies the *state coercive past cost condition* if there exists a $M > 0$ such that for all $z \in \Xi_-$ and all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{W}_c(z)$

$$\|z\|_{\mathscr{X}} \leq M \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})}.$$

Definition 3.2. The triple (p, r, T) is called a (nonnegative) solution of the *filter Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. p is a closed nonnegative symmetric sesquilinear form in $\mathscr{X}$ whose domain satisfies $A^* D(p) \subset D(p)$, $C^* \mathscr{Y} \subset D(p)$.
2. r is a bounded nonnegative symmetric sesquilinear form on $\mathscr{Y}$.
3. $T : D(p) \to \mathscr{Y}$ is a linear operator.
4. For all $z \in D(p)$, $y \in \mathscr{Y}$ we have

$$p[A^*z+C^*y, A^*z+C^*y]+\|B^*z+D^*y\|^2_{\mathscr{U}}+\|y\|^2_{\mathscr{Y}} = p[z,z]+r[Tz-y, Tz-y]. \tag{5}$$

The solution is called *classical* when $D(p) = \mathscr{X}$.

Just as for the control Riccati equation (Remark 2.3) an equivalent operator version of the filter Riccati equation can be defined.

Definition 3.3. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic at the origin. A function $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y} \times \mathscr{U}, \mathscr{Y}))$ is called a *left factorization* of G if $\tilde{\mathsf{M}}(z)$ is invertible for all z in a neighbourhood of the origin and $\mathsf{G}(z) = \tilde{\mathsf{M}}(z)^{-1}\tilde{\mathsf{N}}(z)$ in a neighbourhood of the origin.

Theorem 3.4. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic at the origin and let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a realization of G. The following are equivalent conditions.*

- *$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the output coercive past cost condition.*
- *The filter Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has a (nonnegative self-adjoint) solution.*
- *G has a left factorization.*

Under these equivalent conditions, the filter Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has a smallest (nonnegative self-adjoint) solution (p_p, T_p, r_p) with $p_p = \xi_p^{-1}$.

Proof. This is [10, Theorem 6.10], where the precise description $p_p = \xi_p^{-1}$ has now been made explicit. □

In the above theorem we used the inverse of the closed nonnegative sesquilinear form ξ_p. When a sesquilinear form is given by $\langle Tx, y\rangle$ for some bounded nonnegative self-adjoint operator T with a bounded inverse, then its inverse sesquilinear form is simply $\langle T^{-1}x, y\rangle$. When dealing with sesquilinear forms corresponding to unbounded or non-invertible operators, some more care is needed. We deal with that case in Appendix A by relating sesquilinear forms to nonnegative self-adjoint multi-valued operators.

Definition 3.5. A left factorization $\left[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}\right]$ is called *weakly left coprime* if $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ defined by $\mathsf{M}(z) = \tilde{\mathsf{M}}(\overline{z})^*$, $\mathsf{N}(z) = \tilde{\mathsf{N}}(\overline{z})^*$ is weakly right coprime and it is called *normalized* if multiplication with $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ is an isometry on H^2.

Corollary 3.6. *Assume that the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the output coercive past cost condition. Then its transfer function has a normalized weakly left coprime factorization* $\left[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}\right]$ *and with*

$$\begin{aligned}
\tilde{\mathcal{M}}(z) &= zR_p^{-1/2}C\left[I - z(A+T_pC)\right]^{-1}T_p + R_p^{-1/2},\\
\tilde{\mathcal{N}}(z) &= zR_p^{-1/2}C\left[I - z(A+T_pC)\right]^{-1}\left[B+T_pD\right] + R_p^{-1/2}D,\\
\tilde{\mathcal{P}}(z) &= -zC(I-zA)^{-1}T_pR_p^{1/2} + R_p^{1/2},
\end{aligned}$$

where R_p *is the nonnegative self-adjoint operator corresponding to the sesquilinear form* r_p*, we have* $\rho(A+T_pC)\cap\mathbb{D} \subset \rho(A)\cap\mathbb{D}$ *and on the connected component of* $\rho(A)\cap\mathbb{D}$ *that contains zero we have*

$$\tilde{\mathcal{M}}(z) = \tilde{\mathsf{M}}(z), \quad \tilde{\mathcal{N}}(z) = \tilde{\mathsf{N}}(z), \quad \tilde{\mathcal{P}}(z) = \tilde{\mathsf{M}}(z)^{-1}.$$

Proof. This is the dual of Corollary 2.7. □

Definition 3.7. The node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is called *LQ past normalized* if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is minimal, satisfies the state coercive past cost condition and $\mathcal{I}_p : \mathscr{X} \to \ell^2(\mathbb{Z}^-; \mathscr{U}\times\mathscr{Y})$ is an isometry.

An LQ past normalized realization can be constructed from a node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ that satisfies the output coercive past cost condition by [10] factoring out the unobservable subspace, restricting the result to its reachable subspace, taking the new norm in this subspace to be $\xi_p[z,z]$ —or more accurately the quadratic form that it induces on the quotient space— and completing this space.

The following lemma shows that in some sense a LQ past normalized realization has the smallest state space among minimal realizations that satisfy the coercive past cost condition.

Lemma 3.8. *Let* $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ *be minimal and satisfy the state coercive past cost condition. Let* $\left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$ *be a LQ past normalized realization of the same transfer function. Then there exists an injective operator* $U \in \mathcal{L}(\mathscr{X}_2, \mathscr{X}_1)$ *such that*

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.$$

Proof. We use the notation from [10]. We recall the operator Γ_p that maps a compactly supported input-output trajectory on $\mathbb{Z}^-$ to the corresponding output on $\mathbb{Z}^+$ when the input is chosen to be zero on $\mathbb{Z}^+$. We note that this operator depends only on the transfer function and not on the realization. The closure of Γ_p (which exists due to the output coercive past cost condition being satisfied, see [10, Remark 3.3]) is denoted by $\overline{\Gamma}_p$. We further note that for observable realizations the space $\mathscr{G}_{\mathrm{opt}}$ of optimal past input-output trajectories with a well-defined final state is equal to $D(\overline{\Gamma}_p) \ominus N(\overline{\Gamma}_p)$ since $D(\overline{\Gamma}_p) = D(\mathcal{J})$ and $N(\overline{\Gamma}_p) = N(\mathcal{C}\mathcal{J}) = N(\mathcal{J})$, where observability is used in the last equality. It follows that $\mathscr{G}_{\mathrm{opt}}$ depends only on the transfer function and not on the realization (as long as the realization is observable). The output coercive past cost condition implies that $\overline{\Gamma}_p$ is a bounded operator, so that $\mathscr{G}_{\mathrm{opt}}$ is closed. Moreover, $\overline{\Gamma}_p$ restricted to $\mathscr{G}_{\mathrm{opt}}$ is injective.

By the definition of $\mathcal{I}_p$ in [10] it follows that $R(\mathcal{I}_p) = \mathscr{G}_{\mathrm{opt}}$. So $R(\mathcal{I}_p^1) = R(\mathcal{I}_p^2)$ and this set is closed. It follows that the operator $U := \left(\mathcal{I}_p^1\right)^{-1} \mathcal{I}_p^2$ is well-defined. By the state coercive past cost condition $\left(\mathcal{I}_p^1\right)^{-1}$ is bounded —with domain $R(\mathcal{I}_p^1)$— and since $\mathcal{I}_p^2$ is an isometry it is also bounded. It follows that U is a bounded and injective.

It remains to show the intertwining conditions. We first show that $C_1 U = C_2$. By definition of U we have $\mathcal{I}_p^1 U = \mathcal{I}_p^2$. Since both sides belong to $\mathscr{G}_{\mathrm{opt}}$ and Γ_p is injective on that set, this is equivalent to $\Gamma_p \mathcal{I}_p^1 U = \Gamma_p \mathcal{I}_p^2$. We have $\Gamma_p \mathcal{I}_p = \mathcal{C}$, the initial state to output map, so that equivalently $\mathcal{C}_1 U = \mathcal{C}_2$. Projecting onto the zero-th component shows that $C_1 U = C_2$ as desired.

Next we show that $B_1 = U B_2$, or equivalently that $\mathcal{I}_p^1 B_1 = \mathcal{I}_p^2 B_2$. Since both sides belong to $\mathscr{G}_{\mathrm{opt}}$ and Γ_p is injective on that set, this is equivalent to $\Gamma_p \mathcal{I}_p^1 B_1 = \Gamma_p \mathcal{I}_p^2 B_2$. We have $\Gamma_p \mathcal{I}_p = \mathcal{C}$, the initial state to output map, so that equivalently $\mathcal{C}_1 B_1 = \mathcal{C}_2 B_2$. This equality indeed holds since the Hankel operators of the two systems are the same.

Entirely similarly it follows that $A_1^n B_1 = U A_2^n B_2$ for all $n \in \mathbb{Z}^+$. Now consider the equality $A_1 A_1^n B_1 = U A_2 A_2^n B_2$ (which is just a re-writing of the just obtained one) and use the just established $A_1^n B_1 = U A_2^n B_2$ to re-write this as $A_1 U A_2^n B_2 = U A_2 A_2^n B_2$. By controllability it follows that $A_1 U = U A_2$ on a dense set, which by continuity extends to all of $\mathscr{X}_2$. Thus the last remaining intertwining conditions is established. □

The next lemma shows that LQ past normalized realizations are essentially unique.

Lemma 3.9. *Let* $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$ *be two LQ past normalized realizations of the same transfer function. Then there exists a unitary operator* $U : \mathscr{X}_2 \to \mathscr{X}_1$ *such that*

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.$$

Proof. By Lemma 3.8 it only remains to show that U is unitary. By the proof of Lemma 3.8 we have $U = \left(\mathcal{I}_p^1\right)^{-1} \mathcal{I}_p^2$ and $R(\mathcal{I}_p^1) = R(\mathcal{I}_p^2)$. As the composition of two isometries, U is an isometry. The operator $\mathcal{I}_p^2 : \mathscr{X}_2 \to R(\mathcal{I}_p^2)$ is obviously surjective. The range of $\left(\mathcal{I}_p^1\right)^{-1}$ equals the domain of $\mathcal{I}_p^1$, which is dense by controllability and closed since $\mathcal{I}_p^1$ is an isometry. It follows that the range of $\left(\mathcal{I}_p^1\right)^{-1}$ equals $\mathscr{X}_1$. So U as the composition of two surjective operators is surjective. As a surjective isometry it is unitary. □

Theorem 3.10. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic at the origin. The following are equivalent:*

- G *has a left factorization,*

- G *has a LQ past normalized realization.*

Proof. We first prove 1. implies 2. The completed $\mathcal{I}_{p,-}$ compression from [10, Remark 3.12] is a LQ past normalized factorization. It is shown in [10] to exist under the condition that G has a left factorization.

We now show that 2. implies 1. A LQ past normalized factorization by definition satisfies the state coercive past cost condition, so it follows from Theorem 3.4 that its transfer function G has a left factorization. $\square$

4 The intermediate state optimal control problem

The following is the fundamental new definition in this article. Its significance becomes clear from Theorem 4.8.

Definition 4.1. The node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the *past cost dominance condition* if there exists an $M > 0$ such that for every $z \in \Xi_-$ there exists a $u^f \in \ell^2(\mathbb{Z}^+;\mathscr{U})$ such that

$$\|u^f\|^2_{\ell^2(\mathbb{Z}^+;\mathscr{U})} + \|y^f\|^2_{\ell^2(\mathbb{Z}^+;\mathscr{Y})} \leq M\left(\|u^p\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{U})} + \|y^p\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{Y})}\right), \qquad (6)$$

for all $u^p : \mathbb{Z}^- \to \mathscr{U}$ with compact support that reach z. Here y is the output for the input u defined by $u_n = u^p_n$ for $n \in \mathbb{Z}^-$ and $u_n = u^f_n$ for $n \in \mathbb{Z}^+$; y^f is the projection of y onto $\ell^2(\mathbb{Z}^+;\mathscr{Y})$ and y^p the projection of y onto $\ell^2(\mathbb{Z}^-;\mathscr{Y})$.

Note that with the earlier introduced cost functions (6) reads

$$J_f(z,u^f) \leq M J_p(z,u^p).$$

The next lemma relates the just introduced cost condition to the cost conditions introduced earlier for the initial state and final state optimal control problems, respectively.

Lemma 4.2. *If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the past cost dominance condition, then it satisfies both the finite future incremental cost condition and the output coercive past cost condition.*

Proof. Obviously the past cost dominance condition implies that each $z \in \Xi_-$ has a finite future cost, so the finite future incremental cost condition holds.

We now show that the output coercive past cost condition holds. Let $z \in \Xi_-$ and let u^p be an input that reaches z. By the past cost dominance condition there exists a control u^f such that

$$\|u^f_0\|^2_{\mathscr{U}} + \|Cz + Du^f_0\|^2 \leq M\left(\|u^p\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{U})} + \|y^p\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{Y})}\right). \qquad (7)$$

We further have

$$\|Cz\|^2_{\mathscr{Y}} \leq \|Cz+Du^f_0\|^2_{\mathscr{Y}} + \|Du^f_0\|^2_{\mathscr{Y}} \leq \max\{1, \|D\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})}\}\left(\|Cz + Du^f_0\|^2_{\mathscr{Y}} + \|u^f_0\|^2_{\mathscr{U}}\right),$$

so that by combining this with (7) we have

$$\|Cz\|_{\mathscr{Y}}^2 \le \tilde{M} \left(\|u^p\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{U})} + \|y^p\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{Y})} \right),$$

with $\tilde{M} := M \max\{1, \|D\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})}\}$. Hence the output coercive past cost condition holds. $\square$

Remark 4.3. Using the notation of [10], for a node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, introduce the following operator: $\Gamma_{p,f} = \mathcal{I}_f \mathcal{J}_c$, which maps a compactly supported input-output pair on $\mathbb{Z}^-$ with final state in Ξ_f to the corresponding optimal input-output pair on $\mathbb{Z}^+$. Note that if the future incremental cost condition holds then the assumption that the final state is in Ξ_f is superfluous.

The past cost dominance condition is equivalent to $R(\mathcal{J}_c) \subset D(\mathcal{I}_f)$ and $\Gamma_{p,f}$ extending to a bounded operator $\mathscr{G} \to \ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y})$, where $\mathscr{G}$ is the closure in $\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})$ of the set of compactly supported input-output pairs. Using that $\Gamma_{p,f} = \mathcal{I}_f \mathcal{J}_c$ it follows that the past cost dominance condition is also equivalent to $R(\mathcal{J}_c) \subset \Xi_f$ and $\mathcal{J}_c$ extending to a bounded operator $\tilde{\mathcal{J}}_c : \mathscr{G} \to \Xi_f$, where Ξ_f is equipped with the semi-norm induced by the sesquilinear form ξ_f.

Lemma 4.4. *If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the past cost dominance condition, then there exists a $M > 0$ such that for all $z \in \Xi_p$ there holds $\xi_f[z,z] \le M\xi_p[z,z]$.*

Proof. We use the notation from Remark 4.3. The operator $\tilde{\mathcal{J}}_c$ mentioned there is an extension of the operator $\mathcal{J}$ from [10]. Since $\Xi_p = R(\mathcal{J})$ it then follows that $\Xi_p \subset \Xi_f$. The restriction of $\tilde{\mathcal{J}}_c$ to $\tilde{\mathscr{G}}_{\mathrm{opt}} := \mathscr{G} \ominus \mathcal{N}(\tilde{\mathcal{J}}_c)$ is an extension of the operator $\mathcal{J}_r$ considered in [10]. It follows that we have $\xi_p[z,z] = \|\tilde{\mathcal{J}}_r^{-1} z\|^2_{\ell^2(\mathbb{Z}^+;\mathscr{Y}\times\mathscr{U})}$ for $z \in \Xi_p$. Let $z \in \Xi_p$ and define $w := \tilde{\mathcal{J}}_r^{-1} z$. We then have (with $\tilde{\Gamma}_{p,f}$ the continuous extension of $\Gamma_{p,f}$, which we know exists by the past cost dominance condition):

$$\xi_f[z,z] = \xi_f[\tilde{\mathcal{J}}_c w, \tilde{\mathcal{J}}_c w] = \|\mathcal{I}_f \tilde{\mathcal{J}}_c w\|^2 = \|\tilde{\Gamma}_{p,f} w\|^2 \le M\|w\|^2 = M\|\tilde{\mathcal{J}}_r^{-1} z\|^2 = M\xi_p[z,z].$$

$\square$

The following lemma will be used in the proof of Theorem 4.8.

Lemma 4.5. *If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the past cost dominance condition, then the LQ past normalized realizations of its transfer function satisfy the finite future cost condition.*

Proof. Since by Lemma 4.2 the past cost dominance condition implies the output coercive past cost condition, it follows from Theorems 3.4 and 3.10 that the transfer function of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ indeed has a LQ past normalized realization.

From Lemma 4.4 It follows that ξ_f is bounded on the dense set Ξ_- of the state space of the LQ past normalized realization constructed from $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Hence this LQ past normalized realization satisfies the finite future cost condition. Since all LQ past normalized realizations of the same transfer function are unitarily equivalent (Lemma 3.9), the result follows. $\square$

In the following theorem we again use the notion of inverse of a closed nonnegative sesquilinear form as in Appendix A. The theorem relates the introduced cost condition to solutions of Riccati equations. Recall that for symmetric sesquilinear forms $t_1 \leq t_2$ means that $D(t_1) \supset D(t_2)$ and $t_1[z,z] \leq t_2[z,z]$ for all $z \in D(t_2)$.

Theorem 4.6. *The node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the past cost dominance condition if and only if there exists a (nonnegative self-adjoint) solution q of its control Riccati equation and a (nonnegative self-adjoint) solution p of its filter Riccati equation such that there exists a $M > 0$ such that $q \leq Mp^{-1}$.*

Proof. First assume that the past cost dominance condition holds. Lemma 4.2 together with Theorems 2.5 and 3.4 implies the existence of solutions to the Riccati equations. In fact, ξ_f and ξ_p^{-1} are the smallest solutions. It is therefore obviously sufficient to prove that $\Xi_f \supset \Xi_p$ and $\xi_f[z,z] \leq M\xi_p[z,z]$ for $z \in \Xi_p$. This was already proven in Lemma 4.4.

Now assume the condition on the Riccati equations. Since ξ_f and ξ_p are the smallest solutions, this condition obviously implies that $\xi_f[z,z] \leq M\xi_p[z,z]$ for $z \in \Xi_-$. It follows that we can take u^f in the past cost dominance condition to be the optimal control starting at z. $\square$

Definition 4.7. A right factorization $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ is called *strongly right coprime* if there exists a $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}] \in H^\infty$ such that $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}]\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right] = I$. Similarly, a left factorization $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ is called *strongly left coprime* if there exist a $\left[\begin{smallmatrix} \mathsf{X} \\ -\mathsf{Y} \end{smallmatrix}\right]$ such that $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]\left[\begin{smallmatrix} \mathsf{X} \\ -\mathsf{Y} \end{smallmatrix}\right] = I$.

It is known that existence of a strongly right coprime factorization and a strongly left coprime factorization (over H^∞) are equivalent [4], so we will just speak about a function having a strongly coprime factorization.

The notion of strong coprimeness is stronger than the notion of weak coprimeness from Mikkola [6]. Whereas every function that has a right factorization has a weakly right coprime factorization (and similarly with right replaced by left), it is not true that every function that has a right and a left factorization has a strongly coprime factorization, see e.g [6, Example 7.2]. Note that this example also shows that the converse of Lemma 4.2 is not true.

The following theorem is the main result of this article. In the remarks that follow, some further equivalent conditions are pointed out.

Theorem 4.8. *The following are equivalent for a $\mathcal{L}(\mathscr{U}, \mathscr{Y})$ valued function* G *that is holomorphic in zero:*

1. *G has a strongly coprime factorization;*

2. *For all realizations the past cost dominance condition holds;*

3. *There exists a realization for which the past cost dominance condition holds;*

4. *There exists a realization for which both the finite future cost condition and the state coercive past cost condition hold;*

5. *There exists a realization* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *for which the finite future cost condition holds for both* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$.

Proof. We show 1 implies 2, 3 implies 4 and 4 implies 5. The implication 5 implies 1 is proven in [1] for continuous-time systems and in [2, 7] for discrete-time systems. The implication 2 implies 3 is trivial.

1 implies 2. Consider an arbitrary realization of G. Let $z \in \Xi_-$. Let $u^p \in \ell^2(\mathbb{Z}^-; \mathscr{U})$ with compact support be an input such that z is reached and let $y^p \in \ell^2(\mathbb{Z}^-; \mathscr{Y})$ denote the corresponding output restricted to $\mathbb{Z}^-$. Let $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ be a right coprime factorization of G. Since by assumption M is invertible in a neighbourhood of zero, we may define $r : \mathbb{Z} \to \mathscr{U}$ through its Z-transform as: $\hat{r}(s) = \mathsf{M}(s)^{-1}\hat{u}^p(s)$. Define r^p as the restriction of r to $\mathbb{Z}^-$. Since u^p has compact support, so does r^p. Define $[u; y] \in \ell^2(\mathbb{Z}; \mathscr{U} \times \mathscr{Y})$ by $[\hat{u}(s); \hat{y}(s)] = [\mathsf{M}(s); \mathsf{N}(s)]\hat{r}^p(s)$. Then y is the output for the input u for the system defined by the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ since $\hat{y}(s) = \mathsf{N}(s)\hat{r}^p(s) = \mathsf{N}(s)\mathsf{M}(s)^{-1}\hat{u}(s)$, where we then use that $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right]$ is a right factorization. Define $[u^f; y^f]$ as the restriction to $\mathbb{Z}^+$ of $[u; y]$. By the Bézout equation we have $\hat{r}^p(s) = [\tilde{\mathsf{X}}(s), -\tilde{\mathsf{Y}}(s)][\hat{u}(s); \hat{y}(s)]$ and by causality $[\hat{u}(s); \hat{y}(s)]$ can be replaced here by $[\hat{u}^p(s); \hat{y}^p(s)]$. This implies

$$\left\| \begin{bmatrix} u^f \\ y^f \end{bmatrix} \right\|^2_{\ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})} \leq \left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|^2_{\ell^2(\mathbb{Z}, \mathscr{U} \times \mathscr{Y})} \leq \left\| \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} \right\|^2_{H^\infty(\mathbb{D}; \mathscr{U} \times \mathscr{Y})} \|r^p\|^2_{\ell^2(\mathbb{Z}^-, \mathscr{U})}$$
$$\leq \left\| \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} \right\|^2_{H^\infty(\mathbb{D}; \mathscr{U} \times \mathscr{Y})} \left\| [\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}] \right\|^2_{H^\infty(\mathbb{D}; \mathscr{U} \times \mathscr{Y})} \left\| \begin{bmatrix} u^p \\ y^p \end{bmatrix} \right\|^2_{\ell^2(\mathbb{Z}^-, \mathscr{U} \times \mathscr{Y})}.$$

Hence with

$$M := \left\| \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} \right\|^2_{H^\infty(\mathbb{D}; \mathscr{U} \times \mathscr{Y})} \left\| [\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}] \right\|^2_{H^\infty(\mathbb{D}; \mathscr{U} \times \mathscr{Y})},$$

which is clearly independent of z, the past cost dominance condition is satisfied.

3 implies 4. This follows immediately from Lemma 4.5: a LQG past normalized realization satisfies the state coercive past cost condition by definition and Lemma 4.5 shows that under assumption 3 it also satisfies the finite future cost condition.

4 implies 5. The proof is similar to that of [10, Theorem 6.4]. By [10, Lemma 6.3], we have (with the notation from that article) that the adjoint of the input-output trajectory to final state map $\mathcal{J}_c$ equals $\mathcal{R}^*$ times the future minimizing operator $\mathcal{I}_{f,d}$ of the dual node. Since the state coercive past cost condition is equivalent to $\mathcal{J}_c$ extending to a bounded operator and the finite future cost condition of the dual system is equivalent to $\mathcal{I}_{f,d}$ extending to a bounded operator, it follows that the state coercive past cost condition is equivalent to the finite future cost condition of the dual system. This gives the result. □

Remark 4.9. In the proof of Theorem 4.8 we chose to prove the equivalences in a certain order. Some other implications are however also easy to see. The argument which showed that 4 implies 5 in fact shows that 4 and 5 are equivalent. From 4 it follows that both the control and the filter Riccati equations of the mentioned realization have classical solutions. Theorem 4.6 then shows that the past cost dominance condition holds for that particular realization (i.e. 3 holds). We only referred to [7] for the implication 5 implies 1, but in fact in [7, Chapter 7] equivalence is shown.

Remark 4.10. Note that by Theorem 4.6 we could in Theorem 4.8 have replaced conditions 2 and 3 with conditions on solutions of Riccati equations.

Remark 4.11. Since the existence of a strongly coprime factorization is equivalent to the existence of a dynamic stabilizing controller [4], the conditions of Theorem 4.8 are also equivalent to the existence of a dynamic stabilizing controller.

Remark 4.12. Since all weakly coprime factorizations are strongly coprime if strongly coprime factorizations exist, under the past cost dominance condition, the formulas given in Corollary 2.7 provide a strongly right coprime factorization and those given in Corollary 3.6 provide a strongly left coprime factorization. Formulas for corresponding Bézout factors appear in [2].

A Multi-valued operators and the inverse of a sesquilinear form

In this appendix we define the inverse of a closed nonnegative symmetric sesquilinear form by relating them to multi-valued operators.

Definition A.1. A multi-valued operator (or relation) $T : \mathscr{H} \to \mathscr{K}$ is a subspace $\mathscr{V}_T$ of $\mathscr{H} \times \mathscr{K}$. The operator T is called closed when the subspace $\mathscr{V}_T$ is closed. We have for the domain, kernel, range and multi-valued part of T:

$$\begin{aligned}
D(T) &= \{x \in \mathscr{H} : \exists y \text{ such that } (x, y) \in \mathscr{V}_T\},\\
N(T) &= \{x \in \mathscr{H} : (x, 0) \in \mathscr{V}_T\},\\
R(T) &= \{y \in \mathscr{K} : \exists x \text{ such that } (x, y) \in \mathscr{V}_T\},\\
M(T) &= \{y \in \mathscr{K} : (0, y) \in \mathscr{V}_T\}.
\end{aligned}$$

A multi-valued operator T is called *single-valued* if $M(T) = \{0\}$. In that case T is the graph of an operator in the usual sense and since there is no possibility of confusion we will not distinguish between an operator and its graph.

The inverse of a multi-valued operator $T : \mathscr{H} \to \mathscr{K}$ is the multi-valued operator

$$T^{-1} := \{(y, x) \in \mathscr{K} \times \mathscr{H} : (x, y) \in T\}.$$

Clearly,

$$D(T^{-1}) = R(T),\ R(T^{-1}) = D(T),\ N(T^{-1}) = M(T),\ M(T^{-1}) = N(T).$$

In the sequel we shall only deal with *closed* multi-valued operators. In the case of a closed multi-valued operator T both $N(T)$ and $M(T)$ are closed. Let us denote the orthogonal projections onto $N(T)^\perp$ and $M(T)^\perp$ by $P_{N(T)^\perp}$ and $P_{M(T)^\perp}$, respectively. It is easy to see that $D(T)$ is invariant under $P_{N(T)^\perp}$ and $R(T)$ is invariant under $P_{M(T)^\perp}$, i.e., if $x \in D(T)$ then $P_{N(T)^\perp} x \in D(T) \cap N(T)^\perp$, and if $y \in R(T)$, then $P_{M(T)^\perp} y \in R(T) \cap M(T)^\perp$.

If T is a multi-valued operator, then $T_s := P_{M(T)^\perp} T$ is a single-valued operator. This single-value operator is called the operator part of the multi-valued operator T, and it is closed whenever T is closed. Note that $D(T_s) = D(T)$, $N(T_s) = N(T)$, and $R(T_s) = R(T) \cap M(T)^\perp$. By restricting T_s to $D(T) \cap N(T)^\perp$ we get the injective single-valued operator $T_i := P_{M(T)^\perp} T|_{D(T)\cap N(T)^\perp}$. This operator has the same range as T_s, so that $D(T_i) = D(T) \cap N(T)^\perp$ and $R(T_i) = R(T) \cap M(T)^\perp$. It is easy to see that also T_i is closed as an operator $\overline{D(T)} \cap N(T)^\perp \to \overline{R(T)} \cap M(T)^\perp$ whenever T is closed.

The structure of a closed multi-valued operator T can be seen most easily by decomposing the domain space $\mathscr{H}$ and the range space $\mathscr{K}$ orthogonally into

$$\begin{aligned} \mathscr{H} &= N(T) \oplus \left(\overline{D(T)} \cap N(T)^\perp\right) \oplus D(T)^\perp, \\ \mathscr{K} &= R(T)^\perp \oplus \left(\overline{R(T)} \cap M(T)^\perp\right) \oplus M(T). \end{aligned} \tag{8}$$

With respect to these two decompositions the multi-valued operator T is given by

$$T = \left\{ \left(\begin{bmatrix} x_0 \\ x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ T_i x_1 \\ y_0 \end{bmatrix} \right) : \begin{bmatrix} x_0 \\ x_1 \\ y_0 \end{bmatrix} \in \begin{bmatrix} N(T) \\ D(T_i) \\ M(T) \end{bmatrix} \right\}, \tag{9}$$

and T^{-1} is given by

$$T^{-1} = \left\{ \left(\begin{bmatrix} 0 \\ y_1 \\ y_0 \end{bmatrix}, \begin{bmatrix} x_0 \\ T_i^{-1} y_1 \\ 0 \end{bmatrix} \right) : \begin{bmatrix} x_0 \\ y_1 \\ y_0 \end{bmatrix} \in \begin{bmatrix} N(T) \\ R(T_i) \\ M(T) \end{bmatrix} \right\}. \tag{10}$$

The adjoint of a multi-valued operator T is the multi-valued operator $T^* : \mathscr{K} \to \mathscr{H}$ defined by

$$T^* = \{(y^*, x^*) \in \mathscr{K} \times \mathscr{H} : \langle x^*, x\rangle_{\mathscr{H}} = \langle y^*, y\rangle_{\mathscr{K}} \quad \forall h = (x, y) \in T\}.$$

Note that the adjoint is always closed and that the adjoint of the adjoint of T is equal to T whenever T is closed (otherwise it is the closure of T). It is easy to check that $M(T^*) = D(T)^\perp$, $N(T^*) = R(T)^\perp$, and if T is closed (so that $(T^*)^* = T$) we also have $M(T) = D(T^*)^\perp$ and $N(T) = R(T^*)^\perp$. Moreover, if we use the same decomposition of $\mathscr{H}$ and $\mathscr{K}$ as in (8), then T^* is decomposed into

$$T^* = \left\{ \left(\begin{bmatrix} y_0 \\ y_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ T_i^* y_1 \\ x_0 \end{bmatrix} \right) : \begin{bmatrix} y_0 \\ y_1 \\ x_0 \end{bmatrix} \in \begin{bmatrix} R(T)^\perp \\ D(T_i^*) \\ D(T)^\perp \end{bmatrix} \right\}, \tag{11}$$

where T_i^* is the adjoint of T_i as a closed unbounded operator $\overline{D(T)} \cap N(T)^\perp \to \overline{R(T)} \cap M(T)^\perp$

A multi-valued operator T is called self-adjoint when the domain space $\mathscr{H}$ and the range space $\mathscr{K}$ coincide and $T = T^*$. In particular, every self-adjoint multi-valued operator is closed. Note that in this case $N(T) = R(T^*)^\perp = R(T)^\perp$ and $M(T) = D(T^*)^\perp = D(T)^\perp$, so in the case of a closed self-adjoint multi-valued operator, the two different decompositions of $\mathscr{H} = \mathscr{K}$ in (8) conicide. In particular, we also have $\overline{D(T)} \cap N(T)^\perp = \overline{R(T)} \cap M(T)^\perp$.

Example A.2. An (single-valued) operator A whose domain is not dense in the domain space $\mathscr{H}$ cannot be self-adjoint, since A^* always contains a nontrivial multi-dimensional part $M(A^*) = D(A)^\perp$. However, *a self-adjoint multi-valued operator is uniquely determined by its operator part T_s*. This can be seen as follows. If T is a self-adjoint multi-valued operator, then $T_s := P_{M(T)^\perp} T = P_{\overline{D(T)}} T$ has the following characteristic properties: a) $R(T_s) \subset \overline{D(T_s)} = \overline{D(T)}$, and b) T_s is self-adjoint when regarded as a densly defined operator in $\overline{D(T_s)}$. Conversely, let A be a (closed) operator with properties a) and b) above, i.e., suppose that $R(A) \subset \overline{D(A)}$ and that A is self-adjoint regarded as an operator in $\overline{D(A)}$. Let A_i be the injective part of A, i.e., $A_i = A|_{D(A) \cap N(A)^\perp}$. Then A_i is self-adjoint as an operator in $\overline{D(A)} \cap N(A)^\perp$. Define

$$T = \left\{ \left(x, \begin{bmatrix} Ax \\ y \end{bmatrix} \right) : x \in D(A), y \in D(A)^\perp \right\}.$$

Then T is self-adjoint with

$$\begin{aligned} &D(T) = D(A),\ N(T) = N(A),\ M(T) = D(A)^\perp,\ R(T) = R(A) + M(T), \\ &\quad T_s = A,\ T_i = A_i, \\ &D(T_i) = D(A_i) = D(A) \cap N(A)^\perp,\ R(T_s) = R(T_i) = R(A) = R(A_i), \end{aligned} \tag{12}$$

and (9) holds with the above substitutions. That T is determined uniquely by A follows from the fact that both T_s and $M(T) = D(A)^\perp$ are determined uniquely by A.

A multi-valued self-adjoint operator $T : \mathscr{H} \to \mathscr{H}$ is called nonnegative if $\langle x, y \rangle_{\mathscr{H}} \geq 0$ for all $(x, y) \in T$. This condition is equivalent to the condition that the injective single-valued part T_i of T is nonnegative. Note that a multi-valued operator is self-adjoint and nonnegative if and only if its inverse is.

Two multi-valued operators $T : \mathscr{H} \to \mathscr{K}$ and $S : \mathscr{K} \to \mathscr{L}$ can be multiplied as follows

$$ST = \{(x, z) \in \mathscr{H} \times \mathscr{L} : \exists y \in \mathscr{K}, (x, y) \in T, (y, z) \in S\}.$$

As for single-valued operators, it can be shown that for a nonnegative self-adjoint multi-valued operator T, there exists a unique nonnegative self-adjoint multi-valued operator, denoted $T^{1/2}$, such that $(T^{1/2})^2 = T$. The operator T

has the same decomposition (9) as T, except that T_i has been replaced by $T_i^{1/2}$. In particular,

$$N(T^{1/2}) = N(T),\ \overline{D(T^{1/2})} = \overline{D(T)},\ \overline{R(T^{1/2})} = \overline{R(T)},\ M(T^{1/2}) = M(T).$$

See [11, Section 4.2] for details.

There is the following one-to-one correspondence between closed nonnegative symmetric sesquilinear forms t and nonnegative self-adjoint multi-valued operators T on $\mathscr{H}$:

$$D(t) = D(T^{1/2}), \quad t(x,y) = \langle (T^{1/2})_s x, (T^{1/2})_s y\rangle_{\mathscr{H}}, \quad x,y \in D(T^{1/2}),$$

see [11, Lemma 4.4.2]. In particular, this implies that

$$D(t) = D(T^{1/2}) = D((T^{1/2})_s) = N(T) + D((T^{1/2})_i).$$

Also note that $t[x,x] = 0$ if and only if $x \in N(T^{1/2}) = N((T^{1/2})_s)$.

This correspondence allows as to define the inverse of a closed nonnegative symmetric sesquilinear form t as the closed nonnegative symmetric sesquilinear form corresponding to the nonnegative self-adjoint multi-valued operator T^{-1}.

Example A.3. We continue example A.2. Assume A satisfies not only the conditions listed in Example A.2, but also that A is nonnegative. Let A_i be the injective part of A, i.e., $A_i = A_s|_{D(A)\cap N(A)^\perp}$. Then A_i is densely defined, self-adjoint and positive in the space $\overline{D(A)} \cap N(A)^\perp$. We denote the nonnegative self-adjoint square roots of A and A_i, regarded as a densely defined operator in $\overline{D(A)}$ and $\overline{D(A)} \cap N(A)^\perp$, by $A^{1/2}$ and $A_i^{1/2}$, respectively. In the same way as in Example A.2 we associate a self-adjoint multi-valued operator T with A. Then (12) holds, and so does (9) with the substitutions listed in (12).

Let $T^{1/2}$ be the nonnegative self-adjoint square root of T. Then

$$\begin{aligned} D(T^{1/2}) &= D(A^{1/2}),\ N(T^{1/2}) = N(A),\ M(T^{1/2}) = D(A)^\perp, \\ R(T^{1/2}) &= R(A^{1/2}) + M(T), \\ (T^{1/2})_s &= A^{1/2},\ (T^{1/2})_i = A_i^{1/2}, \\ D(T^{1/2})_i &= D(A_i^{1/2}) = D(A^{1/2}) \cap N(A)^\perp, \\ R(T_s^{1/2}) &= R(T_i^{1/2}) = R(A^{1/2}) = R(A_i^{1/2}), \end{aligned} \tag{13}$$

and (9) holds with the above substitutions.

Next we associate a closed nonnegative sesquilinear form t with T and $T^{1/2}$ as described above. Then

$$D(t) = D(T^{1/2}) = D(A^{1/2}) = N(A) + D(A_i^{1/2}),$$

and with respect to the decomposition $N(A) \oplus \overline{D(A_i)} = N(A) \oplus \overline{D(A_i^{1/2})}$ the form t has the representation

$$\begin{aligned} t\left[\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}\right] &= \left\langle A^{1/2}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, A^{1/2}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}\right\rangle \\ &= \left\langle A_i^{1/2}x_1, A_i^{1/2}y_1\right\rangle, \qquad \begin{bmatrix} x_0 \\ x_1 \end{bmatrix},\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in D(t). \end{aligned}$$

The inverse sesquilinear form t^{-1} has

$$D(t^{-1}) = R(T^{1/2}) = R(A^{1/2}) + M(T^{1/2}) = R(A_i^{1/2}) + D(A)^{\perp},$$

and with respect to the decomposition $\overline{R(A_i)} \oplus D(A)^{\perp}$ the form t^{-1} has the representation

$$t^{-1}\left[\begin{bmatrix} x_1 \\ x_\infty \end{bmatrix}, \begin{bmatrix} y_1 \\ y_\infty \end{bmatrix}\right] = \left\langle A_i^{-1/2} x_1, A_i^{-1/2} y_1 \right\rangle, \qquad \begin{bmatrix} x_1 \\ x_\infty \end{bmatrix}, \begin{bmatrix} y_1 \\ y_\infty \end{bmatrix} \in D(t).$$

Note, in particular, that the kernel of t^{-1} is the orthogonal complement to $D(t)$, and that the kernel of t is the orthogonal complement to $D(t^{-1})$. The intuitive explanation for this phenomenon is that a nonnegative quadratic form "takes the value $+\infty$" in the complement of its domain, and this forces the inverse form to vanish there.

References

[1] Ruth F. Curtain and Mark R. Opmeer. Normalized doubly coprime factorizations for infinite-dimensional linear systems. *Math. Control Signals Systems*, 18(1):1–31, 2006.

[2] Ruth F. Curtain and Mark R. Opmeer. Coprime factorization and robust stabilization for discrete-time infinite-dimensional systems. Submitted, 2009.

[3] Keith Glover and Duncan McFarlane. Robust stabilization of normalized coprime factor plant descriptions with H_∞-bounded uncertainty. *IEEE Trans. Automat. Control*, 34(8):821–830, 1989.

[4] Kalle M. Mikkola. Coprime factorization and dynamic stabilization of transfer functions. *SIAM J. Control Optim.*, 45(6):1988–2010 (electronic), 2007.

[5] Kalle M. Mikkola. Weakly coprime factorization and continuous-time systems. *IMA J. Math. Control Inform.*, 25(4):515–546, 2008.

[6] Kalle M. Mikkola. Weakly coprime factorization and state-feedback stabilization of discrete-time systems. *Math. Control Signals Systems*, 20(4):321–350, 2008.

[7] Mark R. Opmeer. *Model reduction for controller design for infinite-dimensional systems.* PhD thesis, University of Groningen, 2006.

[8] Mark R. Opmeer and Ruth F. Curtain. Linear quadratic Gaussian balancing for discrete-time infinite-dimensional linear systems. *SIAM J. Control Optim.*, 43(4):1196–1221, 2004/05.

[9] Mark R. Opmeer and Olof J. Staffans. Optimal state feedback input-output stabilization of infinite-dimensional discrete time-invariant linear systems. *Complex Anal. Oper. Theory*, 2(3):479–510, 2008.

[10] Mark R. Opmeer and Olof J. Staffans. Optimal input-output stabilization of infinite-dimensional discrete time-invariant linear systems by output injection. *SIAM J. Control Optim.*, 48(8):5084–5107, 2010.

[11] Adrian Sandovici. *Contributions to the extension theory in Hilbert spaces.* PhD thesis, University of Groningen, 2006.

[12] Olof Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[13] Olof J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM J. Control Optim.*, 36(4):1268–1292, 1998.

[14] M. Vidyasagar. *Control system synthesis: A factorization approach.* MIT Press Series in Signal Processing, Optimization, and Control, 7. MIT Press, Cambridge, MA, 1985.