Model Reduction for Distributed Parameter Systems: a Functional Analytic View

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***Abstract*— We study error bounds for model reduction from a functional analytic viewpoint. We give three examples of linear distributed parameter systems that have very different approximation properties. We also comment on implications of this for numerical approximation schemes.**

I. Introduction

The purpose of model reduction is to replace an elaborate model with a simpler one that is close to the original model [1]. Closeness is usually measured by the difference of the input-output maps being small in the $\mathcal{L}(L^2(0,\infty))$ norm. This is equivalent to the difference of the transfer functions being small in the H^∞ norm. In this paper we consider three examples from a functional analytic viewpoint. We show that these three examples have very different approximation properties. We also comment on implications for numerical approximation schemes.

II. The three examples

Example 1: Consider the one-dimensional heat equation on the unit interval with interior control and observation of the state:

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial \xi^2} + u(\xi,t),\\ \frac{\partial w}{\partial \xi}(t,0) &= 0,\\ w(t,1) &= 0,\\ w(0,\xi) &= 0\\ y(t,\xi) &= w(t,\xi).\end{aligned}$$

This can be written in the usual abstract state space form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

by choosing the spaces

$$\mathscr{X} = \mathscr{U} = \mathscr{Y} = L^2(0,1),$$

and the operators

$$\begin{aligned}A &= \frac{\partial^2}{\partial \xi^2},\\ D(A) &= \left\{z \in W^{2,2}(0,1) : z'(0) = z(1) = 0\right\},\\ B &= I,\\ C &= I.\end{aligned}$$

The transfer function of this system is simply the resolvent operator of A.

Example 2: Consider the one-dimensional heat equation on the unit interval with Neumann boundary control and observation of the temperature at the same boundary point:

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial \xi^2},\\ \frac{\partial w}{\partial \xi}(t,0) &= -u(t),\\ w(t,1) &= 0,\\ w(0,\xi) &= 0\\ y(t) &= w(t,0).\end{aligned}$$

This can be written in the usual abstract state space form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

by choosing the spaces

$$\begin{aligned}\mathscr{X} &= L^2(0,1),\\ \mathscr{U} &= \mathscr{Y} = \mathbb{C},\end{aligned}$$

and the operators

$$\begin{aligned}A &= \frac{\partial^2}{\partial \xi^2},\\ D(A) &= \left\{z \in W^{2,2}(0,1) : z'(0) = z(1) = 0\right\},\\ Cz &= z(0),\\ D(C) &= D(A),\\ B &= C^*.\end{aligned}$$

The transfer function of this system can be computed (as in [3, Section 4.3]) to be

$$G(s) = \sum_{k=1}^{\infty} \frac{2}{s-\lambda_k}, \tag{1}$$

where

$$\lambda_k = -\mu_k^2, \quad \mu_k = -\frac{\pi}{2} + k\pi.$$

Another expression for the transfer function is

$$G(s) = \frac{1}{\sqrt{s}}\tanh(\sqrt{s}).$$

Example 3: Consider the one-dimensional transport equation on the unit interval with boundary control at one end,

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boundary observation at the other end and actuator dynamics:

$$\begin{aligned}\frac{\partial w}{\partial t} &= -\frac{\partial w}{\partial \xi},\\ w(t,0) &= f(t),\\ w(0,\xi) &= 0\\ y(t) &= w(t,1),\\ \hat{f}(s) &= F(s)\hat{u}(s),\end{aligned}$$

where the rational transfer function F describes the actuator dynamics and is assumed to have relative degree r.

The infinite-dimensional subsystem can be written in the usual abstract state space form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bf(t),\\ y(t) &= Cx(t),\end{aligned}$$

by choosing the spaces

$$\begin{aligned}\mathscr{X} &= L^2(0,1),\\ \mathscr{U} &= \mathscr{Y} = \mathbb{C},\end{aligned}$$

and the operators

$$\begin{aligned}A &= -\frac{\partial}{\partial \xi}\\ D(A) &= \left\{z \in W^{1,2}(0,1) : z(0) = 0\right\},\\ Cz &= z(1),\\ D(C) &= D(A),\\ B &= H^*,\end{aligned}$$

where $Hz = z(0)$ and $D(H) = W^{1,2}(0,1)$.

The transfer function of the overall system is

$$F(s)\mathrm{e}^{-s}.$$

III. Error bounds

An important operator associated to a system is the Hankel operator [10], [13], [15]. Especially important for model reduction are the singular values of the Hankel operator. Recall that the singular values of an operator T (where T is an operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$) are defined as follows:

$$\sigma_n(T) := \inf_{\{T_n \in \mathcal{L}(\mathscr{H},\mathscr{K}):\ \text{rank } T_n < n\}} \|T - T_n\|_{\mathcal{L}(\mathscr{H},\mathscr{K})}.$$

If T is compact, then the nonzero singular values are precisely the squareroots of the eigenvalues of TT^* (or equivalently: of T^*T) when we repeat the eigenvalues according to their multiplicity. The McMillan degree of a transfer function equals the number of nonzero singular values of the corresponding Hankel operator.

We also need the (non-standard) notion of *characteristic values* of an operator. We will denote the sequence of characteristic values by $(\kappa_k)_{k=1}^{\infty}$. The characteristic values are exactly the singular values where we however omit repeated singular values, i.e. as sets $\{\kappa_k : k \in \mathbb{N}\} = \{\sigma_k : k \in \mathbb{N}\}$, but $\kappa_i \neq \kappa_j$ for $i \neq j$. We still order the characteristic values so that $(\kappa_k)_{k=1}^{\infty}$ is a decreasing sequence.

The following theorem is trivial to prove, but important none the less. For the rational case it appears in e.g. [8, Section 10.2.3].

Theorem 4: Suppose that $G \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U},\mathscr{Y}))$ has Hankel singular values $(\sigma_k)_{k=1}^{\infty}$. For any $G_n \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U},\mathscr{Y}))$ with McMillan degree smaller than or equal to n we have

$$\sigma_{n+1} \leq \|G - G_n\|_\infty.$$

The next theorem, which gives an upper-bound on the error for a particular approximation, is far from easy to prove in the non-rational case. In the rational case it is due to Enns [4] and Glover [6] (see also [8, Section 9.4.3]). The general case under stronger assumptions then those mentioned here is due to Glover, Curtain and Partington [7] (the assumptions imposed by [7] for example exclude our Example 2). The improvement given here was proven in [9].

Theorem 5: Suppose that $G \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U},\mathscr{Y}))$ has Hankel characteristic values $(\kappa_k)_{k=1}^{\infty}$ and that $\mathscr{U}$ and $\mathscr{Y}$ are finite-dimensional. For the transfer function $G_n \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U},\mathscr{Y}))$ with McMillan degree equal to n obtained by Lyapunov balanced truncation we have

$$\|G - G_n\|_\infty \leq 2 \sum_{k=n+1}^{\infty} \kappa_k.$$

By using an optimal Hankel norm approximation with an appropriate feedthrough instead of a Lyapunov balanced truncation, the error bound becomes

$$\|G - G_n\|_\infty \leq \sum_{k=n+1}^{\infty} \kappa_k.$$

See again Glover [6] (and also [8, Section 10.5.2]) for the rational case and [7] and [9] for the general case.

For non-rational transfer functions, balanced truncations and optimal Hankel norm approximations are generally impossible to compute exactly. Numerical approximations are hampered by the need to do expensive very large scale computations (but see e.g. [2]). However, the above upper-bounds are extremely important for another reason: they show that approximations with an error that is close to the lowerbound obtained in Theorem 4 exist. This provides a benchmark for other (perhaps computationally more feasible) methods.

IV. Decay of Hankel singular values

It follows from the error-bounds mentioned in Section III that the Hankel singular values are important objects when considering model reduction error. In this section we consider the asymptotic behavior of the Hankel singular values.

A result by Ober [11] and Treil [17] shows that any decreasing sequence of nonnegative numbers is the sequence of singular values of some Hankel operator. So in general, nothing can be said about the asymptotic behavior of the Hankel singular values. However, for our three examples (and cases like them) much can be said.

A. Interior control

We continue Example 1. The Hankel operator is in this case self-adjoint and is given by

$$\int_0^\infty e^{A(t+s)}u(s)\,ds.$$

It follows that the singular vectors of the Hankel operator are $v_n(t,\xi) = e^{\lambda_n t}\phi_n(\xi)$ and the singular values are $\sigma_n = \frac{-1}{2\lambda_n}$, where ϕ_n are the eigenvectors and λ_n the eigenvalues of the self-adjoint operator A (which has a compact inverse). These are given by

$$\lambda_n = -\mu_n^2, \quad \mu_n = -\frac{\pi}{2} + n\pi, \quad \phi_n(\xi) = \sqrt{2}\cos(\mu_n \xi).$$

It follows from the lower-bound discussed in Section III that for this example for any approximation G_n with McMillan degree smaller than or equal to n

$$\frac{1}{2\mu_{n+1}^2} = \frac{1}{2\left(\frac{\pi}{2} + n\pi\right)^2} \le \|G - G_n\|_\infty. \tag{2}$$

A balanced truncation can be exactly computed in this case. The observability and controllability gramian are both equal to $-\frac{1}{2}A^{-1}$ so that the balancing modes are equal to the eigenvectors of A. It follows that balanced truncation is in this case the same as 'modal approximation'. With respect to the basis of eigenvectors $(\phi_k)_{k=1}^\infty$, the full order transfer function has a diagonal matrix representation with diagonal entries

$$[G(s)]_{ii} = \frac{1}{s - \lambda_i}.$$

The transfer function of McMillan degree n obtained by balanced truncation also has a diagonal matrix representation with respect to this basis, namely that with diagonal entries

$$[G_n(s)]_{ii} = \begin{cases} \frac{1}{s-\lambda_i} & i \le n, \\ 0 & i > n. \end{cases}$$

It is then easily directly computed that

$$\|G - G_n\|_\infty = \frac{1}{-\lambda_{n+1}} = \frac{1}{\mu_{n+1}^2} = \frac{1}{\left(\frac{\pi}{2} + n\pi\right)^2}.$$

We see that this differs a factor two from the lower-bound (2) and is much better than the upper-bound from Theorem 5 (note that Theorem 5 is however not applicable to this example since the input and output spaces in this example are both infinite-dimensional).

B. Boundary control

We now turn to Example 2. It follows from [12] that for this example the sequence of Hankel singular values is in $\ell^p(\mathbb{N})$ for all $p > 0$. It follows that for all $q > 0$ there exists a $C_q > 0$ such that

$$\sigma_n \le \frac{C_q}{n^q},$$

i.e. the Hankel singular values converge to zero faster than any polynomial rate. This rapid decay has been often observed numerically for examples like our Example 2, but to the best of our knowledge, [12] provides the first rigorous proof of this rapid decay. From this decay of the singular values and the error-bound for balanced truncations from Theorem 5, it follows that the balanced truncation error converges to zero faster than any polynomial rate as well: for all $q > 0$ there exists a $C_q > 0$ such that

$$\|G - G_n\|_\infty \le \frac{C_q}{n^q}.$$

Remark 6: The crucial assumptions in [12] are analyticity of the semigroup and that at least one of $\mathscr{U}$ or $\mathscr{Y}$ is finite-dimensional. The main result used in the proof is a theorem by Peller [14] and Semmes [16] that characterizes Schatten class Hankel operators. From [15, Corollary 6.9.4] we see that the Hankel operator is in the Schatten class S_p (i.e. its singular values are in ℓ^p) if and only if the transfer function is in the Besov space $B_{pp}^{1/p}(\mathbb{C}_0^+; S_p(\mathscr{U}, \mathscr{Y}))$. Both in Example 1 and Example 2, we have that the transfer function is in $B_{pp}^{1/p}(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. This follows from analyticity of the semigroup. In the case of Example 2 this implies that the transfer function is in $B_{pp}^{1/p}(\mathbb{C}_0^+; S_p(\mathscr{U}, \mathscr{Y}))$. This is because the input and output spaces are finite-dimensional. In Example 1 this implication is no longer true. In fact the condition that the transfer function is in $B_{pp}^{1/p}(\mathbb{C}_0^+; S_p(\mathscr{U}, \mathscr{Y}))$ implies that for every $s \in \mathbb{C}_0^+$ we have $G(s) \in S_p(\mathscr{U}, \mathscr{Y})$. In the case of Example 1 this becomes $(sI - A)^{-1} \in S_p(L^2(0,1))$ and this is only true for $p > \frac{1}{2}$.

We note that we can also perform modal truncation in this case using the explicit formula (1) for the transfer function. This gives the approximant

$$G_n(s) = \sum_{k=1}^{n} \frac{2}{s - \lambda_k},$$

and the error can be explicitly computed as

$$\begin{aligned} \|G - G_n\|_\infty &= \sum_{k=n+1}^{\infty} \frac{2}{-\lambda_k} \\ &= \sum_{k=n+1}^{\infty} \frac{2}{\mu_k^2} = \sum_{k=n+1}^{\infty} \frac{2}{\left(-\frac{\pi}{2} + k\pi\right)^2}. \end{aligned}$$

It follows that asymptotically $\|G - G_n\|_\infty \approx \frac{C}{n}$ so that modal truncation for this example is substantially inferior to balanced truncation.

C. The transport equation

We continue Example 3. It follows from results in [5] that

$$n^r \sigma_n \to C_r \neq 0,$$

where we recall that r is the relative degree of the transfer function describing the actuator dynamics. It follows that the approximation properties of this system becomes better the higher this relative degree is, but are never as good as those of Example 2.

D. Exponential decay for bounded generators

In this section we show that if A is bounded and generates an exponentially stable strongly continuous semigroup and B and C are bounded with at least one of $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional, then the corresponding Hankel singular values decay exponentially. We note that the assumption that A is bounded is unrealistic for systems described by partial differential equations.

We first consider the discrete-time case.

Theorem 7: Assume that $A \in \mathcal{L}(\mathscr{X})$ satisfies $r(A) < 1$ (spectral radius) with $\mathscr{X}$ a Hilbert space, $B \in \mathcal{L}(\mathscr{U}, \mathscr{X})$ and $C \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ with at least one of $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Then the Hankel operator $H : \ell^2(\mathbb{N}; \mathscr{U}) \to \ell^2(\mathbb{N}; \mathscr{Y})$ defined through the block operator matrix $H_{ij} = CA^{i+j-2}B$ has singular values that satisfy

$$\sigma_n \leq MR^n,$$

for some $M > 0$ and $R \in (0, 1)$.

Proof: The proof is based on [13, Exercise 44].

Define

$$h_k := CA^{k-1}B.$$

We first note that by the assumption $r(A) < 1$ we have that there exist $\tilde{M} > 0$ and $r \in (0, 1)$ such that

$$\|A^j\| \leq \tilde{M} r^j.$$

It follows that

$$\|h_j\| = \|CA^{j-1}B\| \leq \|C\| \; \tilde{M} r^{j-1} \; \|B\| \leq N r^{j-1},$$

with

$$N := \|C\| \; \tilde{M} \; \|B\|.$$

We conclude that

$$\sum_{j=k+1}^{\infty} \|h_j\| \leq \sum_{j=k+1}^{\infty} N r^{j-1} = r^k \frac{N}{1 - r}. \tag{3}$$

Now, for $k \in \mathbb{N}$ define the Hankel operator H^k through

$$H^k_{ij} = \begin{cases} h_{i+j} & i + j \leq k, \\ 0 & i + j > k. \end{cases}$$

With $m := \min\{\dim \mathscr{U}, \dim \mathscr{Y}\}$ we have rank $H^k \leq km$. We conclude that

$$\sigma_{km+1} \leq \|H - H^k\|.$$

The operator $H - H^k$ is also a Hankel operator. The $\ell^1(\mathbb{N})$ norm of its impulse response is

$$\sum_{j=k+1}^{\infty} \|h_j\|.$$

Since the operator norm of a Hankel operator is smaller than or equal to the $\ell^1(\mathbb{N})$ norm of the impulse response, we conclude that for all $k \in \mathbb{N}$

$$\sigma_{km+1} \leq \sum_{j=k+1}^{\infty} \|h_j\|.$$

Using (3) we then have

$$\sigma_{km+1} \leq r^k \frac{N}{1 - r}. \tag{4}$$

Define

$$M := \frac{N}{1 - r}, \quad R := r^{1/(m+1)}.$$

Then it follows from (4) that for all $k \in \mathbb{N}$

$$\sigma_{km+1} \leq MR^{km+m}.$$

By using that the singular values form a non-increasing sequence we then obtain for all $n \in \mathbb{N}$

$$\sigma_n \leq MR^n,$$

as desired. ■

The continuous-time case follows.

Theorem 8: Assume that $A \in \mathcal{L}(\mathscr{X})$, with $\mathscr{X}$ a Hilbert space, generates an exponentially stable strongly continuous semigroup and that $B \in \mathcal{L}(\mathscr{U}, \mathscr{X})$ and $C \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ with at least one of $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Then the Hankel singular values satisfy

$$\sigma_n \leq MR^n,$$

for some $M > 0$ and $R \in (0, 1)$.

Proof: Use the Cayley transform to translate the system to discrete-time. Since A is bounded and generates an exponentially stable strongly continuous semigroup, its Cayley transform has spectral radius strictly smaller than 1. The discrete-time Hankel operator associated to the Cayley transform is unitarily equivalent to the continuous-time Hankel operator. In particular, the continuous-time system and its Cayley transform have the same Hankel singular values. The result then follows from Theorem 7. ■

Remark 9: Note that if A is unbounded, then its Cayley transform has -1 in its spectrum so that the spectral radius of the Cayley transform in that case is not strictly smaller than one. This is the reason that the above proof fails for the case where A is unbounded. In fact, as the following example shows, the conclusion of Theorem 8 is false if the assumption that A is bounded is dropped.

Example 10: Consider the following modification of Example 3:

$$\begin{aligned} &\frac{\partial w}{\partial t} = -\frac{\partial w}{\partial \xi} - \varepsilon w, \quad t > 0, \quad \xi \in (0, 1), \\ &w_t(t, 0) + w(t, 0) = u(t), \\ &z_t(t) + z(t) = w(t, 1), \\ &y(t) = z(t). \end{aligned}$$

This can be written in the standard state space form by choosing the spaces

$$\begin{aligned} \mathscr{X} &= L^2(0, 1) \times \mathbb{C}^2, \\ \mathscr{U} &= \mathscr{Y} = \mathbb{C}, \end{aligned}$$

and the operators

$$A = \begin{bmatrix} -\frac{\partial}{\partial \xi} - \varepsilon & 0 & 0 \\ 0 & -1 & 0 \\ T_1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$
$$D(A) = \left\{ [z_1; z_2; z_3] \in W^{1,2}(0,1) \times \mathbb{C}^2 : z_1(0) = z_2 \right\},$$

where $T_1 x = x(1)$ with $D(T_1) = W^{1,2}(0,1)$.

The transfer function of this system is

$$G(s) = \frac{1}{(s+1)^2} \mathrm{e}^{-(s+\varepsilon)}.$$

This system satisfies all the assumptions of Theorem 8 except that in this case A is unbounded. Using the explicit description of the transfer function, it follows from [5] that

$$n^2 \sigma_n \to C \neq 0.$$

In particular, the Hankel singular values of this system do not decay at an exponential rate.

V. Numerical methods

As mentioned, balanced truncations of irrational functions can usually not be calculated explicitly. The standard procedure to obtain a reduced order model is to use a numerical discretization method (such as finite elements) for the underlying partial differential equation and then to calculate a balanced truncation of that system using numerical linear algebra. The first question that should be asked when using this method is whether the so obtained reduced order model is close to the exact balanced truncation of the underlying partial differential equation. The following result taken from [9] shows that –under very reasonable assumptions (which however exclude our Example 1)– this is (asymptotically) true.

Theorem 11: Assume that $\mathscr{U}$ and $\mathscr{Y}$ are finite-dimensional and that the impulse responses $(h_m)_{m\in\mathbb{N}}$ of the numerical discretization converge in $L^1(0,\infty;\mathcal{L}(\mathscr{U},\mathscr{Y}))$ to the impulse response h of the full order system. Denote the reduced order transfer functions obtained from the numerical discretization by balanced truncation by $(G_n^m)_{m\in\mathbb{N}}$ and the reduced order transfer function obtained by balanced truncation of the full order system by G_n. Then there exists a subsequence of $(G_n^m)_{m\in\mathbb{N}}$ that converges in H^∞ to G_n. In the case where the Hankel singular values of the full order system are distinct, convergence of the full sequence $(G_n^m)_{m\in\mathbb{N}}$ to G_n in H^∞ can be concluded.

The second question is how well the initial numerical discretization approximates the full order system. We give some results for our first two examples.

A. Interior control

In this case the standard estimates for finite element approximations give the following error-bound:

$$\|G - G_N\|_\infty \leq \frac{C}{N^2},$$

where G_N is the transfer function of the piecewise linear finite element approximation. Comparing this to the lower-bound (2), we see that piecewise linear finite elements is asymptotically optimal for this example (i.e. the power 2 is optimal). For slightly different norms this is known [18].

B. Boundary control

In the case of boundary control and observation, the standard estimates for finite element approximations give the following error-bound:

$$\|G - G_N\|_\infty \leq \frac{C}{N},$$

where G_N is the transfer function of the piecewise linear finite element approximation. The loss from N^2 to N compared to the interior control case is due to unboundedness of the control and observation operators in the boundary control case.

Numerical calculations indicate that this bound is sharp: the error very much seems to behave like $\frac{C}{N}$ (see figure 1). As we saw in Section IV-B, the balanced truncation error decays at a rate faster than any polynomial rate. Yet the piecewise linear finite element approximation decays only at a linear rate. So in contrast to the interior control case, in the boundary control case, piecewise linear finite elements is very far from asymptotically optimal.

An alternative numerical method based on the eigenvector expansion of the differential operator A leads to modal approximation and was already discussed in Section IV-B. This also gives a decay rate of $\frac{C}{N}$ and is therefore also very far from asymptotically optimal.

The Chebyshev collocation method performs asymptotically better for this example: numerical calculations indicate that the decay rate for that method is $\frac{C}{N^2}$. This is however still very far from asymptotically optimal.

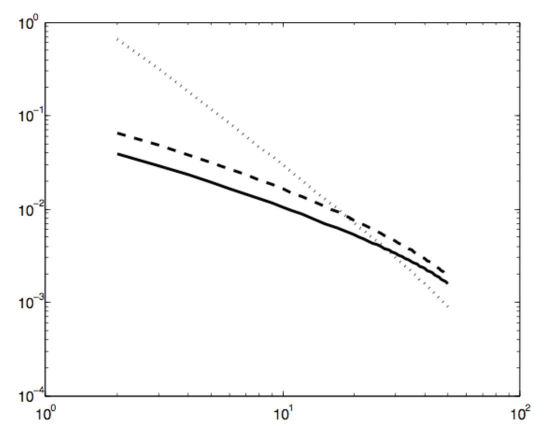

Fig. 1. Numerical approximation of Example 2 by finite elements (-), modal truncation (- -) and Chebyshev collocation (:). Dimension of reduced order system on horizontal axis, H^∞ error on vertical axis.

VI. Conclusion

We have considered model reduction of systems described by partial differential equations from a functional analytic and numerical analysis perspective.

The functional analytic perspective provided a lower-bound on the H^∞ error and -by analyzing balanced truncation- an upper-bound on the achievable H^∞ error as well. These bounds are both in terms of the Hankel singular values of the full order system. By considering specific system classes, these Hankel singular values can be estimated, so providing decay rates in terms of the McMillan degree of the reduced order system.

The numerical analysis of the underlying partial differential equations shows that some commonly used numerical discretizations are asymptotically optimal for an interior control example, but are very far from asymptotically optimal for a boundary control example.

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