Infinite-dimensional negative imaginary systems

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Abstract

We consider second order infinite-dimensional systems with force control and collocated position measurement interconnected with finite-dimensional controllers of the same type. We show that under assumptions that generalize those in the finite-dimensional case (the theory of negative imaginary systems), asymptotic stability of the closed-loop system can be concluded, but that the closed-loop system may not be exponentially stable nor input-output stable.

1 Introduction

Flexible systems with force control and collocated velocity measurement lead to positive real transfer functions. This class of systems is very well studied. The case of collocated position measurement has received far less attention. An important recent contribution to the latter situation was the introduction of the concept of a negative imaginary transfer function [7, 8, 10]. Many flexible systems (such as beams, strings and plates) are best described by partial differential equations and are therefore infinite-dimensional systems. These are not covered by the current theory on negative imaginary functions. In this article we investigate in how far the existing theory on negative imaginary functions generalizes to infinite-dimensional systems. The for applications most relevant case is that of second order systems with force control and collocated position measurement interconnected with finite-dimensional controllers of the same form, and we restrict ourselves to this case. We make extensive use of existing theory of second order infinite-dimensional systems, see e.g. [6] for an extensive discussion and bibliography of such systems.

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The paradigmatic example in this article is the following wave equation with boundary force control and collocated position measurement (see e.g. [12, Chapter 9] and see [4] for the similar situation involving a beam equation):

$$w_{tt}(x,t) = w_{xx}(x,t), \quad w_x(0,t) = u(t), \quad w(1,t) = 0, \quad y(t) = -w(0,t).$$

Here $w(x,t)$ denotes the displacement at position $x \in [0,1]$ and time $t \geq 0$. This can be written in an abstract second order operator-theoretic form as (the details are in Section 4):

$$\ddot{w}_p(t) + K_p w_p(t) = G_p u(t), \quad y(t) = G_p^* w_p(t), \tag{1}$$

where the state space is infinite-dimensional. Systems of this form, but with a finite-dimensional state space, are shown in the theory of negative imaginary systems to be stabilized by a finite-dimensional controller of the form

$$\ddot{w}_c(t) + D_c \dot{w}_c(t) + K_c w_c(t) = G_c y(t), \quad u(t) = G_c^* w_c(t),$$

under some assumptions on the controller parameters [14, Theorem 1], [15]. Stabilization means here that the closed-loop A-matrix which appears in the first order form of the closed-loop system is Hurwitz. We show that –under essentially the same conditions as in the finite-dimensional case– in the infinite-dimensional case the closed-loop system is asymptotically stable, but not necessarily exponentially stable (the eigenvalues are in the left half-plane, but can converge to the imaginary axis). If we add an input disturbance to the plant (i.e. replace u in (1) by $u + d$) and consider the transfer function from this input disturbance to the output y of the plant, then contrary to the finite-dimensional case, this transfer function is generally not stable.

2 Abstract second order systems

We review the set-up of abstract second order infinite-dimensional systems in the special case of bounded damping operators (a more general case can be found in e.g. [6]). How the above wave equation fits into this framework is explained in Section 4.

The second order equation that we consider is

$$\ddot{w}(t) + D\dot{w}(t) + Kw(t) = Gu(t), \quad y(t) = Hw(t). \tag{2}$$

The stiffness operator K is assumed to be a densely defined nonnegative self-adjoint operator on the Hilbert space $\mathscr{H}$ with a bounded inverse. We denote the domain of K by $\mathscr{K}_1$ and that of its (nonnegative self-adjoint) square root by $\mathscr{K}_{1/2}$, we equip these spaces with their graph norm, thus making them into Hilbert spaces. The space $\mathscr{K}_{-1/2}$ is defined as the completion of $\mathscr{H}$ under the norm $\|x\|_{-1/2} := \|(I + K^{1/2})^{-1}x\|_{\mathscr{H}}$, which makes it into a Hilbert space. The operator K extends to a bounded operator from $\mathscr{K}_{1/2}$ to $\mathscr{K}_{-1/2}$.

We assume that the damping operator D is a bounded operator on $\mathscr{H}$. The second order control operator G is assumed to be a bounded operator from the Hilbert space $\mathscr{U}$ (the input space) to $\mathscr{K}_{-1/2}$. The second order observation operator H is assumed to be a bounded operator from $\mathscr{K}_{1/2}$ to the Hilbert space $\mathscr{Y}$ (the output space).

The first order form of the second order equation (2) is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

with state space $\mathscr{X} := \mathscr{K}_{1/2} \times \mathscr{H}$ (which in physical situations will be the finite energy space) and

$$A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad C = \begin{bmatrix} H & 0 \end{bmatrix}.$$

The position w is the first component of the state and the velocity $\dot{w}$ is the second component of the state. The state operator A has domain $D(A) = \mathscr{K}_1 \times \mathscr{K}_{1/2}$ and (by the Lumer-Philips theorem, see e.g. [1, Theorem 3.4.5], [9, Proposition 2.25], [11, Theorem 3.4.8], [13, Theorem 3.8.4]) generates a strongly continuous contraction semigroup on the state space. The first order control operator B is a bounded operator from $\mathscr{U}$ to $\mathscr{H} \times \mathscr{K}_{-1/2}$ (but not necessarily to the state space) and the first order observation operator C is a bounded operator from the state space to $\mathscr{Y}$.

We note for future reference that

$$A^{-1} = \begin{bmatrix} -K^{-1}D & -K^{-1} \\ I & 0 \end{bmatrix}, \tag{3}$$

which is a bounded operator on the state space.

3 Asymptotic stability of the closed-loop system

For the plant

$$\ddot{w}_p(t) + K_p w_p(t) = G_p u(t), \quad y(t) = G_p^* w_p(t),$$

we make the assumptions of Section 2 (with $D = 0$) and denote the corresponding spaces with a superscript p. We further assume that the input and output spaces are finite-dimensional. For the controller

$$\ddot{w}_c(t) + D_c \dot{w}_c(t) + K_c w_c(t) = G_c u_c(t), \quad y_c(t) = G_c^* w_c(t),$$

we assume that D_c and K_c are positive self-adjoint operators on the finite-dimensional space $\mathscr{H}^c$. Naturally, the input space of the controller equals the output space of the plant and the output space of the controller equals the input space of the plant. This makes the controller a very special case of the abstract second order systems of Section 2. The interconnected system (with $u = y_c + d$ and $u_c = y$) then has the second order representation

$$\ddot{w}(t) + D\dot{w}(t) + Kw(t) = Gd(t), \quad y(t) = Hw(t),$$

with $w = \begin{bmatrix} w_p \\ w_c \end{bmatrix}$ and

$$D = \begin{bmatrix} 0 & 0 \\ 0 & D_c \end{bmatrix}, \quad K = \begin{bmatrix} K_p & -G_p G_c^* \\ -G_c G_p^* & K_c \end{bmatrix}, \quad G = \begin{bmatrix} G_p \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} G_p^* & 0 \end{bmatrix},$$

where D is a bounded operator on $\mathscr{H}^p \times \mathscr{H}^c$, K is an operator on $\mathscr{H}^p \times \mathscr{H}^c$ with domain

$$D(K) = \left\{ \begin{bmatrix} z_p \\ z_h \end{bmatrix} \in \mathscr{K}_{1/2}^p \times \mathscr{H}^c : K_p z_p - G_p G_c^* z_h \in \mathscr{H}^p \right\},$$

G is a bounded operator from $\mathscr{U}$ to $\mathscr{K}_{-1/2}^p$ and H is a bounded operator from $\mathscr{K}_{1/2}^p \times \mathscr{H}^c$ to $\mathscr{Y}$.

For now we disregard the control and observation operators of this closed-loop system and concentrate on the state operators (we return to the control and observation operators in the example in Section 4). For the closed-loop system to fit into the abstract second order framework of Section 2, we need K to be nonnegative self-adjoint with a bounded inverse on $\mathscr{H}^p \times \mathscr{H}^c$. The following lemma provides a necessary and sufficient condition for this. It involves the DC loop gain condition known in the theory of negative imaginary systems [8, 10, 14, 15]. The transfer function of a system is denoted by G with the relevant subscript (p for the plant, c for the controller).

Lemma 1. *The above operator K is nonnegative self-adjoint with a bounded inverse on $\mathscr{H}^p \times \mathscr{H}^c$ if and only if the spectral radius of the DC loop gain $\mathsf{G}_p(0)\mathsf{G}_c(0)$ is strictly smaller than 1.*

Proof. By Schur complements, K being nonnegative self-adjoint with a bounded inverse is equivalent to the same property holding for

$$K_c - G_c G_p^* K_p^{-1} G_p G_c^*.$$

We recognize $G_p^* K_p^{-1} G_p$ as the transfer function of the plant evaluated in zero, $\mathsf{G}_p(0)$, so that the condition is equivalent to (using finite-dimensionality of $\mathscr{H}^c$):

$$K_c - G_c \mathsf{G}_p(0) G_c^* > 0.$$

This is equivalent to
$$K_c^{-1/2} G_c \mathsf{G}_p(0) G_c^* K_c^{-1/2} < I,$$

which in turn is equivalent to the spectral radius condition

$$r(K_c^{-1/2} G_c \mathsf{G}_p(0) G_c^* K_c^{-1/2}) < 1.$$

The ordering of operators is irrelevant for the spectral radius of a product (so long as the involved products make sense), so that the above condition is equivalent to
$$r(\mathsf{G}_p(0) G_c K_c^{-1} G_c^*) < 1.$$

We recognize that $G_c K_c^{-1} G_c^* = \mathsf{G}_c(0)$ and the result follows. $\qquad\square$

By the theory of Section 2, under the assumption of Lemma 1, the closed-loop state operator A generates a strongly continuous contraction semigroup on the state space $\mathscr{K}_{1/2} \times \mathscr{H}$. We note that this state space is generally not equal to the product (up to re-ordering) of the state spaces of the plant and the controller, i.e. it is not generally equal to $\mathscr{K}_{1/2}^p \times \mathscr{K}_{1/2}^c \times \mathscr{H}^p \times \mathscr{H}^c$.

The following is our main theorem. It uses the notion of approximate observability which can be found in e.g. [3, Section 4.1], [11, Section 9.4], [13, Chapter 6].

Theorem 2. *If the following hold:*

1. *the standing assumptions on the plant and the controller mentioned at the beginning of this section, namely*

   - K_p *is a densely defined nonnegative self-adjoint operator on the Hilbert space $\mathscr{H}^p$ with a bounded inverse,*

   - G_p *is a bounded operator from the finite-dimensional space $\mathscr{U}$ to $\mathscr{K}_{-1/2}^p$,*

   - K_c *is a bounded nonnegative self-adjoint operator on the finite-dimensional space $\mathscr{H}^c$ with a bounded inverse,*

   - D_c *is a bounded nonnegative self-adjoint operator on the finite-dimensional space $\mathscr{H}^c$ with a bounded inverse,*

   - G_c *is a bounded operator from the finite-dimensional space $\mathscr{Y}$ to $\mathscr{H}^c$,*

2. *the DC loop gain condition $r(\mathsf{G}_p(0)\mathsf{G}_c(0)) < 1$,*

3. *the plant is approximately observable,*

4. *G_c is injective,*

5. *K_p^{-1} is compact,*

then the strongly continuous semigroup generated by the closed-loop A is asymptotically stable.

Proof. Since the closed-loop state operator A generates a strongly continuous contraction semigroup, its spectrum is contained in the closed left half-plane. We now show that in fact A has no eigenvalues on the imaginary axis. We note that this doesn't use assumption 5.

It follows from (3) applied to the closed-loop system that zero is not an eigenvalue of A. So suppose that $A\phi = i\omega\phi$ with $\omega \in \mathbb{R}$, $\omega \neq 0$, and $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in D(A)$. Then
$$(-\omega^2 + i\omega D + K)\phi_1 = 0, \quad \phi_2 = i\omega\phi_1.$$
Consider the inner-product on $D(K^{1/2}) \times \mathscr{H}$ given by
$$\langle x, z \rangle_K := \langle x_2, z_2 \rangle_{\mathscr{H}} + \langle K^{1/2}x_1, K^{1/2}z_1 \rangle_{\mathscr{H}},$$

which is equivalent to the inner-product obtained from the graph norm. Then on the one hand

$$\langle A\phi, \phi\rangle_K = i\omega\langle\phi, \phi\rangle_K,$$

so that $\mathrm{Re}\langle A\phi, \phi\rangle_K = 0$, and on the other hand

$$\langle A\phi, \phi\rangle_K = -\langle K\phi_1 + D\phi_2, \phi_2\rangle_{\mathscr{H}} + \langle\phi_2, K\phi_1\rangle_{\mathscr{H}} = -\langle D\phi_2, \phi_2\rangle_{\mathscr{H}} - 2i\,\mathrm{Im}\langle K\phi_1, \phi_2\rangle_{\mathscr{H}}, \tag{4}$$

so that $\mathrm{Re}\langle A\phi, \phi\rangle_K = -\langle D\phi_2, \phi_2\rangle_{\mathscr{H}}$. It follows that $\langle D\phi_2, \phi_2\rangle_{\mathscr{H}} = 0$. Since $D = \begin{bmatrix} 0 & 0 \\ 0 & D_c \end{bmatrix}$, it follows that $\langle D_c\phi_2^c, \phi_2^c\rangle = 0$, where $\phi_2 = \begin{bmatrix} \phi_2^p \\ \phi_2^c \end{bmatrix}$. Since D_c is assumed to be positive, it follows that $\phi_2^c = 0$. It then follows from $\phi_2 = i\omega\phi_1$ that $\phi_1^c = 0$. The equation $(-\omega^2 + i\omega D + K)\phi_1 = 0$ then becomes

$$0 = \begin{bmatrix} -\omega^2 + K_p & G_p G_c^* \\ G_c G_p^* & -\omega^2 + i\omega D_c + K_c \end{bmatrix} \begin{bmatrix} \phi_1^p \\ 0 \end{bmatrix}.$$

It follows that

$$G_c G_p^* \phi_1^p = 0, \quad K_p \phi_1^p = \omega^2 \phi_1^p.$$

Since G_c is assumed to be injective, it follows that

$$G_p^* \phi_1^p = 0, \quad K_p \phi_1^p = \omega^2 \phi_1^p,$$

we also have

$$\phi_2^p = i\omega\phi_1^p,$$

so that ϕ^p is unobservable for the plant. It follows that $\phi^p = 0$ because the plant is assumed to be approximately observable. We conclude that $\phi = 0$, which is a contradiction. We conclude that A has no eigenvalues on the imaginary axis.

We note that in the finite-dimensional case, the above eigenvalue result implies stability. In the infinite-dimensional case, it does not [3, Example 5.1.4], [9, Example 3.6], [1, Example 5.1.10], more information is needed. The eigenvalue result together with A^{-1} being compact does provide enough information to conclude asymptotic stability [9, Theorem 3.26]. We now show that A^{-1} is indeed compact. This does use assumption 5.

By Schur complements and the fact that $G_p^* K_p^{-1} G_p = \mathsf{G}(0)$ we have

$$K^{-1} = \begin{bmatrix} K_p^{-1} + K_p^{-1} G_p G_c^* (K_c - G_c \mathsf{G}_p(0) G_c^*)^{-1} G_c G_p^* K_p^{-1} & -K_p^{-1} G_p G_c^* (K_c - G_c \mathsf{G}_p(0) G_c^*)^{-1} \\ -(K_c - G_c \mathsf{G}_p(0) G_c^*)^{-1} G_c G_p^* K_p^{-1} & (K_c - G_c \mathsf{G}_p(0) G_c^*)^{-1} \end{bmatrix}.$$

The inverses here are well-defined by the DC loop gain condition (see the proof of Lemma 1). The fact that the state space of the controller is finite-dimensional implies that all operators except possibly the operator in the upper left corner are compact. Compactness of that operator however follows from the fact that K_p^{-1} is compact. Using (3) for the closed-loop system it now follows that A^{-1} is compact on the state space (the fact that the damping operator D is bounded is important here). □

Remark 3. Note that the DC loop gain condition $r(\mathsf{G}_p(0)\mathsf{G}_c(0)) < 1$ looks different from the one in previous articles on negative imaginary systems, which have the condition that the largest eigenvalue of $\mathsf{G}_p(0)\mathsf{G}_c(0)$ must be strictly smaller than one. However, since $\mathsf{G}_p(0)$ and $\mathsf{G}_c(0)$ are nonnegative self-adjoint, the eigenvalues of their product are nonnegative, so that the two conditions –in the particular case of second order systems considered here– are in fact equivalent.

The example in the next section shows that the closed-loop semigroup need not be exponentially stable under the conditions of Theorem 2 and that the closed-loop system need not be input-output stable under the conditions of Theorem 2 either.

4 An example

We return to the example of a wave equation with force control and position measurement introduced in the introduction:

$$w_{tt}(x,t) = w_{xx}(x,t), \quad w_x(0,t) = u(t), \quad w(1,t) = 0, \quad y(t) = -w(0,t). \quad (5)$$

This can be written in abstract second order operator-theoretic form as:

$$\ddot{w}_p(t) + K_p w_p(t) = G_p u(t), \quad y(t) = G_p^* w_p(t),$$

where $\mathscr{H}^p := L^2(0,1)$ and (with $W^{2,2}$ the standard Sobolev space)

$$K_p := -\frac{\partial^2}{\partial x^2}, \quad \mathscr{K}^p = D(K_p) := \{z \in W^{2,2}(0,1) : z'(0) = 0, \quad z(1) = 0\},$$

and

$$G_p^* z := -z(0), \quad \mathscr{K}_{1/2}^p = D(G_p^*) = \{z \in W^{1,2}(0,1) : z(1) = 0\}.$$

That the operator K_p^{-1} is compact on $L^2(0,1)$ can be shown similarly as in [3, Example A4.26] (or more abstractly by invoking Sobolev embedding). This system is approximately observable since putting $y = 0$ and $u = 0$ gives the boundary value problem

$$w_{tt}(x,t) = w_{xx}(x,t), \quad w_x(0,t) = 0, \quad w(1,t) = 0, \quad w(0,t) = 0,$$

which has only the zero solution (e.g by Laplace transforming in t and then considering it as an initial value problem in x with zero initial conditions at $x = 0$). The open-loop transfer function can be calculated (as in [3, Section 4.3] or [2]) to be

$$\mathsf{G}_p(s) = \frac{\sinh s}{s \cosh s}.$$

It follows from Theorem 2 that with the one-dimensional controller

$$\ddot{w}_c(t) + D_c \dot{w}_c(t) + K_c w_c(t) = y(t), \quad u(t) = w_c(t),$$

7

with $D_c > 0$ and $K_c > 1$ (to satisfy the DC loop gain condition), the closed-loop system is asymptotically stable.

The closed-loop transfer function can be calculated to be

$$\mathsf{G}(s) = \frac{\sinh s}{s \cosh s - \frac{\sinh s}{s^2 + D_c s + K_c}}.$$

Consider the points $s_n := \frac{\pi}{2}i + n\pi i$ with $n \in \mathbb{Z}$ (which are the open-loop poles). We have $\cosh s_n = 0$ and $\sinh s_n = \pm 1$ so that $\mathsf{G}(s_n) = -(s_n^2 + D_c s_n + K_c)$. It follows that $|\mathsf{G}(s_n)| \to \infty$ as $n \to \infty$ so that $\mathsf{G} \notin L^\infty(i\mathbb{R})$ and hence the closed-loop system is not input-output stable. The eigenvalues λ_n of the closed-loop state operator coincide with the poles of the closed-loop transfer function, i.e. they are the solutions of the equation

$$s \cosh s - \frac{\sinh s}{s^2 + D_c s + K_c} = 0.$$

This equation can be re-written as

$$-\mathrm{e}^{2s} = \frac{s^3 + D_c s^2 + K_c s + 1}{s^3 + D_c s^2 + K_c s - 1}.$$

By keeping only the asymptotically largest term on the right hand-side, we obtain the asymptotic equation $-\mathrm{e}^{2s} = 1$, which has as solutions the eigenvalues of the open-loop system $s_n := \frac{\pi}{2}i + n\pi i$. It can be easily shown (e.g. using the method described in [5] or in a more elementary way by an application of Rouché's theorem) that $\lambda_n - s_n \to 0$, so that $\operatorname{Re} \lambda_n \to 0$ as $n \to \infty$. Hence the closed-loop system is not exponentially stable.

5   Conclusion

We have shown that –at least for second order systems with force control and position measurement and for finite-dimensional controllers of the same type– negative imaginary stability theory carries over to the infinite-dimensional case if stability is understood as asymptotic stability, but not if it is understood as exponential stability or as input-output stability.

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