

Nuclearity of Hankel operators for ultradifferentiable control systems *

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Abstract

Nuclearity of the Hankel operator is a known sufficient condition for convergence of Lyapunov-balanced truncations. We show how a previous result on nuclearity of Hankel operators of systems with an analytic semigroup can be extended to systems with a semigroup of class D^p with $p \geq 1$ (the case $p = 1$ being the analytic case). For semigroups that are generated by a Dunford-Schwartz spectral operator we prove that being of class D^p is equivalent to being (Gevrey) ultradifferentiable of order p . We illustrate how for certain partial differential equations our results lead to an easy way of showing nuclearity of the Hankel operator for a wide range of control and observation operators by considering several examples of damped beams.

1 Introduction

For the practical implementation of a controller for a system described by partial differential equations, approximation of such a system by a system described by ordinary differential equations is usually essential. For control purposes the proper notion of convergence of such approximations is convergence of the transfer functions in the H^∞ norm (or in the unstable case: in the gap metric). For such a sequence of approximations to exist it is necessary that the Hankel operator of the original system is compact and for convergence of several popular approximation methods such as Lyapunov balanced truncations it is sufficient that the Hankel operator of the original system is nuclear, see [4, 7] (we assume here that the input and output spaces are finite-dimensional, which in this context is a natural assumption). In this article we will only treat the sufficient nuclearity condition. Nuclearity of the Hankel operator is not easily directly obtained for specific partial differential equations. Therefore, in [4] two sufficient conditions for nuclearity were given that are relatively easily verifiable. These

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conditions pertain to the abstract system equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0 \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

and are, respectively,

CS1 A generates an exponentially stable C_0 semigroup $T(t)$ on a Hilbert space \mathcal{X} , $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with \mathcal{U} and \mathcal{Y} finite dimensional Hilbert spaces.

CS2 A generates an exponentially stable analytic C_0 semigroup $T(t)$ on a Hilbert space \mathcal{X} , $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)$, $C \in \mathcal{L}(\mathcal{X}_\alpha, \mathcal{Y})$ with \mathcal{U} and \mathcal{Y} finite dimensional Hilbert spaces and $\alpha - \beta < 1$.

In this last statement \mathcal{X}_α and \mathcal{X}_β are the usual interpolation spaces associated to the generator of a C_0 semigroup (see e.g. [12, Section 3.6]). This second set of conditions, when compared to the first, shows that unboundedness of the control and observation operators can to some extent be compensated for by analyticity of the semigroup. In this article we first point out that the proof of the above CS2 result from [4] actually allows for a much stronger conclusion than is drawn in that article. The crucial assumption in the proof is that $\|CT(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})}$ and $\|B^*T^*(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{U})}$ are bounded from above by some $L^2(0, \infty)$ function. This is certainly implied by the assumptions given in CS2, but there are many control systems described by partial differential equations for which these L^2 bounds are true, but which fail to satisfy CS2. We give some examples in Section 3. We note that a similar conservativity applies to [12, Theorem 5.7.3], where under the assumptions CS2 minus the finite-dimensionality of \mathcal{U} and \mathcal{Y} and the stability assumption, well-posedness of the control system is shown. Also here it is actually the above L^2 bounds that are used in the proof. So also here the proof actually gives a much stronger result than the theorem states.

As mentioned, CS2 is just a sufficient condition for the L^2 bounds to hold. There is another interesting class of sufficient conditions for the L^2 bounds to hold, namely: replace in CS2 ‘analytic C_0 semigroup’ by ‘ C_0 semigroup of class D^p ’ [3] and $\alpha - \beta < 1$ by $p(\alpha - \beta) < 1$. Condition CS2 now becomes the special case $p = 1$ and condition CS1 can (very roughly) be seen as the case $p \rightarrow \infty$. This result thus more or less bridges the two conditions CS1 and CS2. We show that in case the generator is a Dunford-Schwartz spectral operator, the semigroup is of class D^p if and only if the semigroup is (Gevrey) ultradifferentiable of order p .

In Section 3 we give some examples of control systems described by partial differential equations that give rise to ultradifferentiable semigroups. The typical situation in which they appear seems to be in elastic systems with relatively weak damping (with strong damping one typically obtains an analytic semigroup and with very weak damping one typically obtains a group). We use the results from Section 2 to list choices of sensors and actuators that lead

to a nuclear Hankel operator for these PDEs and therefore to controlled PDEs that can be approximated in the H^∞ norm by ODEs using one of the standard methods such a Lyapunov balanced truncation.

2 Theory

We assume in this section that \mathcal{X} , \mathcal{U} and \mathcal{Y} are Hilbert spaces. All semigroups mentioned are assumed to be strongly continuous. We refer to [12] for the general theory of well-posed linear systems and to [4] for a discussion of nuclearity of the Hankel operator of a well-posed linear system.

2.1 L^2 estimates

The following theorem is (part of) what is actually proven in the proof of [12, Theorem 5.7.3].

Theorem 1 *Let T be a C_0 semigroup with generator A , let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$, $C \in \mathcal{L}(\mathcal{X}_1, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Assume that there exist $L^2_{\text{loc}}([0, \infty))$ functions (i.e. locally square integrable functions) b and c such that*

$$\begin{aligned} \|T(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X})} &\leq b(t), \quad t > 0 \\ \|CT(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} &\leq c(t), \quad t > 0. \end{aligned}$$

Then (A, B, C, D) generates a well-posed linear system on the state space \mathcal{X} .

Proof As mentioned, the proof follows [12, Theorem 5.7.3] almost exactly. We reproduce the main arguments to show that indeed only the assumptions of the present theorem are needed.

The second assumed inequality obviously shows that C is an (finite-time) admissible observation operator for T on the state space \mathcal{X} (i.e. that the output map is well-posed): for every $\tau > 0$ and $x \in \mathcal{X}_1$ we have

$$\int_0^\tau \|CT(t)x\|_{\mathcal{Y}}^2 dt \leq \int_0^\tau |c(t)|^2 dt \|x\|_{\mathcal{X}}^2.$$

By duality, the first inequality then shows that B is an admissible control operator for T , i.e. that the input map is well-posed.

We get the following bound for the impulse response

$$\|CT(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{Y})} \leq b(t/2)c(t/2), \quad t > 0$$

by using

$$\|CT(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{Y})} = \|CT(t/2)T(t/2)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{Y})} \leq \|CT(t/2)\|_{\mathcal{L}(\mathcal{U}, \mathcal{X})} \|T(t/2)B\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})}.$$

By the Cauchy-Schwarz inequality, the impulse response is in $L^1_{\text{loc}}([0, \infty), \mathcal{L}(\mathcal{U}, \mathcal{Y}))$. It follows from this that the input-output map is well-posed. That the Hankel

operator is the product of the input map and the output map follows from the equality

$$\int_{-\infty}^0 CT(t-s)Bu(s) ds = CT(t) \int_{-\infty}^0 T(-s)Bu(s) ds,$$

where taking $CT(t)$ out of the integral is permitted since $CT(t) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ by assumption and the integral converges absolutely for almost all $t > 0$. \square

Similarly, (a part of) the proof of [4, Theorem 6] actually shows the following.

Theorem 2 *Let T be a C_0 semigroup with generator A , let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$, $C \in \mathcal{L}(\mathcal{X}_1, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Assume \mathcal{U} and \mathcal{Y} are finite-dimensional and that there exist $L^2(0, \infty)$ functions b and c such that*

$$\begin{aligned} \|T(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X})} &\leq b(t), \quad t > 0 \\ \|CT(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} &\leq c(t), \quad t > 0. \end{aligned}$$

Then the well-posed linear system generated by (A, B, C, D) has a nuclear Hankel operator.

Proof As mentioned, the proof follows [4, Theorem 6] almost exactly. We reproduce the main arguments to show that indeed only the assumptions of the present theorem are needed.

Let $\{y_1, \dots, y_p\}$ be an orthonormal basis for \mathcal{Y} and define for $i \in \{1, \dots, p\}$

$$(\mathcal{C}_i x)(t) = \langle CT(t)x, y_i \rangle_{\mathcal{Y}}, \quad x \in \mathcal{X}, \quad t > 0.$$

Then

$$|(\mathcal{C}_i x)(t)| \leq \|CT(t)x\|_{\mathcal{Y}} \leq \|CT(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \|x\|_{\mathcal{X}} \leq c(t) \|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}, \quad t > 0.$$

Now an application of [4, Theorem 5] (which is a reformulation of [14, Theorem 6.12]) shows that $\mathcal{C}_i : \mathcal{X} \rightarrow L^2(0, \infty)$ is Hilbert-Schmidt. Consequently for any orthonormal basis $\{x_j\}_{j \geq 1}$ of \mathcal{X} we have

$$\sum_{j \geq 1} \|\mathcal{C}_i x_j\|_{L^2(0, \infty)}^2 < \infty.$$

It follows that

$$\sum_{j \geq 1} \|CT(\cdot)x_j\|_{L^2(0, \infty, \mathcal{Y})}^2 = \sum_{i=1}^p \sum_{j \geq 1} \|\mathcal{C}_i x_j\|_{L^2(0, \infty)}^2 < \infty,$$

and so $CT(\cdot) : \mathcal{X} \rightarrow L^2(0, \infty, \mathcal{Y})$ is Hilbert-Schmidt. Similarly, $B^*T(\cdot)^* : \mathcal{X} \rightarrow L^2(0, \infty, \mathcal{U})$ is Hilbert-Schmidt, from which it follows that its adjoint, the input map $\mathcal{B} : L^2(-\infty, 0; \mathcal{U}) \rightarrow \mathcal{X}$ given by $\mathcal{B}u = \int_{-\infty}^0 T(-t)Bu(t) dt$, is Hilbert-Schmidt. The Hankel operator is then seen to be nuclear since it equals $\mathcal{C}\mathcal{B}$, the product of two Hilbert-Schmidt operators. \square

2.2 Semigroups of class D^p

We review a concept from Crandall and Pazy [3] that generalizes the concept of an analytic semigroup and will allow us to formulate a generalization of the results [12, Theorem 5.7.3], [4, Theorem 6] that is easier to apply than the above theorems involving L^2 estimates.

Definition 3 *Let $p \in [1, \infty)$. A C^∞ semigroup which satisfies*

$$\limsup_{t \rightarrow 0^+} t^p \|AT(t)\|_{\mathcal{L}(\mathcal{X})} < \infty$$

is called a semigroup of class D^p .

Note that a semigroup is analytic if and only if it is a D^1 semigroup. Also note that a semigroup of class D^p when restricted or extended to one of its interpolation spaces \mathcal{X}_γ is again of class D^p .

In Lemmas 4 and 5 we present two alternative characterizations of semigroups of class D^p . These are used in Theorems 6 and 7, which provide the desired generalizations of the analytic well-posedness and nuclearity results.

Lemma 4 *A C^∞ semigroup is of class D^p if and only if for every ω larger than the growth-bound of the semigroup there exist a constant $M > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}_1)} \leq M(1 + t^{-p})e^{\omega t} \text{ for } t > 0. \quad (2)$$

Proof Assume the condition mentioned in the lemma. Using that the \mathcal{X}_1 norm is equivalent to the graph norm of $D(A)$ we have

$$\|AT(t)\|_{\mathcal{L}(\mathcal{X})} \leq C\|T(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}_1)} \leq CM(1 + t^{-p})e^{\omega t} \text{ for } t > 0.$$

Then obviously

$$\limsup_{t \rightarrow 0^+} t^p \|AT(t)\|_{\mathcal{L}(\mathcal{X})} \leq CM < \infty,$$

so that the semigroup is of class D^p .

Conversely, assume that the semigroup is of class D^p . Define

$$N := 2 \limsup_{t \rightarrow 0^+} t^p \|AT(t)\|_{\mathcal{L}(\mathcal{X})}.$$

Then there exists an $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$ we have

$$t^p \|AT(t)\|_{\mathcal{L}(\mathcal{X})} \leq N.$$

Since we obviously also have

$$t^p \|T(t)\|_{\mathcal{L}(\mathcal{X})} \leq N$$

for $t > 0$ small enough, it follows that for a certain C

$$t^p \|T(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}_1)} \leq Ct^p (\|T(t)\|_{\mathcal{L}(\mathcal{X})} + \|AT(t)\|_{\mathcal{L}(\mathcal{X})}) \leq CN,$$

(for all small enough $t > 0$), where we have used that the \mathcal{X}_1 norm is equivalent to the graph norm induced by the operator A . This shows that (2) holds for $t > 0$ small enough.

We now show that for each $\delta > 0$ and ω larger than the growth bound of T , there exists a $M > 0$ such that

$$\|AT(t)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{\omega t}. \quad (3)$$

This, together with the equivalence of the \mathcal{X}_1 norm and the graph norm induced by A gives the desired (2) for $t \geq \delta$. To prove (3) we argue as follows. We first note that by the differentiability of the semigroup $AT(\delta)$ is a bounded operator on \mathcal{X} . We then use the semigroup property to obtain (for $t \geq \delta$)

$$\|AT(t)\|_{\mathcal{L}(\mathcal{X})} \leq \|AT(\delta)\|_{\mathcal{L}(\mathcal{X})} \|T(t-\delta)\|_{\mathcal{L}(\mathcal{X})} \leq Me^{\omega t},$$

where ω is any number larger than the growth-bound of T .

Combining the result for t small enough with the result for $t \geq \delta$ gives the desired estimate. \square

Lemma 5 *A C^∞ semigroup is of class D^p if and only if for all $\alpha > 0$ and all ω larger than the growth-bound of the semigroup, there exists a constant $M > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}_\alpha)} \leq M(1 + t^{-\alpha p})e^{\omega t} \text{ for } t > 0. \quad (4)$$

Proof That the given condition implies that the semigroup is of class D^p trivially follows from Lemma 4. We first show that condition (4) for $\alpha = 1$ (which by Lemma 4 is equivalent to the semigroup being of class D^p) implies that it holds for all $\alpha \in \mathbb{N}$.

We have for $n \in \mathbb{N}$ and $\sigma \in \rho(A)$

$$(\sigma - A)^n T(t) = \left((\sigma - A) T\left(\frac{t}{n}\right) \right)^n,$$

so

$$\begin{aligned} \|(\sigma - A)^n T(t)\|_{\mathcal{L}(\mathcal{X})} &\leq \left\| (\sigma - A) T\left(\frac{t}{n}\right) \right\|_{\mathcal{L}(\mathcal{X})}^n \\ &\leq \left(M \left(1 + \left(\frac{t}{n}\right)^{-p} \right) e^{\omega t/n} \right)^n \\ &= M^n n^p (n^{-p} + t^{-p})^n e^{\omega t}. \end{aligned}$$

The result then follows using the elementary fact that $(n^{-p} + t^{-p})^n \leq C(1 + t^{-np})$ for some C independent of t .

The case for general $\alpha > 0$ goes as follows. The key ingredient is the second inequality from [12, Lemma 3.9.8]. This gives the existence of a $C > 0$ such that for all $\alpha \in (0, 1)$, $x \in \mathcal{X}$

$$\|T(t)x\|_{\mathcal{X}_\alpha} \leq C \|T(t)x\|_{\mathcal{X}}^{1-\alpha} \|T(t)x\|_{\mathcal{X}_1}^\alpha.$$

Using (4) for $\alpha = 0, 1$ (without loss of generality with the same M and ω) we obtain

$$\begin{aligned} \|T(t)x\|_{\mathcal{X}_\alpha} &\leq C (Me^{\omega t}\|x\|_{\mathcal{X}})^{1-\alpha} (M(1+t^{-p})e^{\omega t}\|x\|_{\mathcal{X}})^\alpha \\ &= CM(1+t^{-p})^\alpha e^{\omega t}\|x\|_{\mathcal{X}} \end{aligned}$$

The result then follows using the elementary fact that $(1+t^{-p})^\alpha \leq C(1+t^{-\alpha p})$ for some C independent of α .

The general case follows by writing $\alpha = \alpha' + n$ with $\alpha' \in [0, 1)$ and $n \in \mathbb{N}_0$ and applying the above with $(\sigma - A)^n T(t)x$ instead of with $T(t)x$, using the result for integers values of α and $(1+t^{-c})(1+t^{-d}) \leq 3(1+t^{-c-d})$. \square

The concept of a semigroup of class D^p combined with Theorem 1 gives the following.

Theorem 6 *Let T be a semigroup of class D^p with generator A , let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)$, $C \in \mathcal{L}(\mathcal{X}_\alpha, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Assume that $p(\alpha - \beta) < 1$. Then (A, B, C, D) generates a well-posed linear system on the state space \mathcal{X}_γ for all $\gamma \in (\alpha - \frac{1}{2p}, \beta + \frac{1}{2p})$.*

Proof The proof is almost identical to the special case $p = 1$ from [12, Theorem 5.7.3].

We first note that we may assume that $\beta \leq \alpha$. The case $\beta > \alpha$ is reduced to the case $\alpha = \beta$ as follows. If $\beta > \alpha$ the assumptions of the theorem hold with β and α both replaced by δ for any $\delta \in [\alpha, \beta]$. We may then apply the theorem for the case $\beta = \alpha = \delta$ to obtain that the system is well-posed on \mathcal{X}_γ for $\gamma \in (\delta - \frac{1}{2p}, \delta + \frac{1}{2p})$. Since δ is allowed to be any number in $[\alpha, \beta]$ we obtain well-posedness on \mathcal{X}_γ for $\gamma \in (\alpha - \frac{1}{2p}, \beta + \frac{1}{2p})$, as desired.

We check the sufficient conditions from Theorem 1. We have if $\beta \leq \gamma$

$$\begin{aligned} \|T(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\gamma)} &\leq \|T(t)\|_{\mathcal{L}(\mathcal{X}_\beta, \mathcal{X}_\gamma)} \|B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)} = \|T(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}_{\gamma-\beta})} \|B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)} \\ &\leq M(1+t^{-(\gamma-\beta)p})e^{\omega t} \|B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)}. \end{aligned}$$

The upper-bound belongs to $L^2_{\text{loc}}(0, \infty)$ if and only if $(\gamma - \beta)p < 1/2$. If $\beta > \gamma$, then we have

$$\|T(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\gamma)} \leq \|T(t)B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)} \leq \|T(t)\|_{\mathcal{L}(\mathcal{X})} \|B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)} \leq Me^{\omega t} \|B\|_{\mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)},$$

and this upper-bound is always in $L^2_{\text{loc}}(0, \infty)$.

Similarly we have if $\gamma \leq \alpha$

$$\|CT(t)\|_{\mathcal{L}(\mathcal{X}_\gamma, \mathcal{Y})} \leq M(1+t^{-(\alpha-\gamma)p})e^{\omega t} \|C\|_{\mathcal{L}(\mathcal{X}_\alpha, \mathcal{Y})}$$

and the upper-bound is in $L^2_{\text{loc}}(0, \infty)$ if and only if $(\alpha - \gamma)p < 1/2$. The case $\alpha < \gamma$ is dealt with as above. So both conditions on the respective upper-bounds hold if and only $\gamma \in (\alpha - \frac{1}{2p}, \beta + \frac{1}{2p})$. The condition $p(\alpha - \beta) < 1$ ensures that this is a nonempty interval. \square

Combining the concept of a semigroup of class D^p with Theorem 2 gives the following.

Theorem 7 *Let T be an exponentially stable semigroup of class D^p with generator A , let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_\beta)$, $C \in \mathcal{L}(\mathcal{X}_\alpha, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Assume that $p(\alpha - \beta) < 1$ and that \mathcal{U} and \mathcal{Y} are finite-dimensional. Then the Hankel operator of the well-posed linear system generated by (A, B, C, D) is nuclear.*

Proof The proof is almost identical to that of Theorem 6. Now the exponential stability is used to deduce that the upper-bounds are actually in $L^2(0, \infty)$ and not only in $L^2_{\text{loc}}(0, \infty)$. So the sufficient conditions for nuclearity from Theorem 2 hold. \square

2.3 Ultradifferentiable semigroups

Following [13, Chapter 5], we define (Gevrey) ultradifferentiable semigroups (see also [10]).

Definition 8 *Let $p \in [1, \infty)$. A C^∞ semigroup T is called ultradifferentiable of order p if for all $\tau, \theta > 0$ there exists a constant C such that for all $n \in \mathbb{N}_0$ and $t \in (0, \tau]$*

$$\|T^{(n)}(t)\|_{\mathcal{L}(\mathcal{X})} \leq C\theta^n(n!)^p. \quad (5)$$

Note that a semigroup is analytic if and only if it is ultradifferentiable of order one.

Theorem 9 *Let $p \in [1, \infty)$ and assume that A is the generator of a C_0 semigroup and that A is similar to a normal operator. The following are equivalent.*

1. *A generates an ultradifferentiable semigroup of order p .*
2. *A generates a semigroup of type D^p .*
3. *There exist $b > 0$, $a \in \mathbb{R}$ such that*

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq a - b|\operatorname{Im}\lambda|^{1/p}\}. \quad (6)$$

Proof That 2. implies 3. was shown in [3, Theorem 2.1 and 2.2] and that 1. implies 3. was shown in [13, Theorem 5.6] in both cases even without the assumption that A is similar to a normal operator. The equivalence of 1. and 3. was shown by Markin [10, Theorem 5.1]. This leaves to show 3. implies 2., which we will do now.

It is easy to see that for all $b > 0$, $a \in \mathbb{R}$, there exist $\beta > 0$, $\alpha \in \mathbb{R}$ such that

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq \alpha - \beta|\lambda|^{1/p}\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq a - b|\operatorname{Im}\lambda|^{1/p}\}.$$

It follows that condition 3. is equivalent to: there exist $b > 0$, $a \in \mathbb{R}$ such that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq a - b|\lambda|^{1/p}\} \quad (7)$$

(the other part of this equivalence being completely trivial).

With E_A the spectral measure of A and ϕ a continuous function on $\sigma(A)$ we have

$$\|\phi(A)x\|^2 = \int_{\sigma(A)} |\phi(s)|^2 d\langle E_A x, x \rangle.$$

Applying this with $\phi_1(s) = se^{st}$ and $\phi_2(s) = |s|e^{t(a-b|s|^{1/p})}$ respectively we have

$$\|Ae^{At}x\|^2 = \int_{\sigma(A)} |\phi_1(s)|^2 d\langle E_A x, x \rangle,$$

and

$$\| |A|e^{t(a-b|A|^{1/p})}x\|^2 = \int_{\sigma(A)} |\phi_2(s)|^2 d\langle E_A x, x \rangle.$$

We note that on $\sigma(A)$ we have, due to the inclusion (7), $|\phi_1(s)| \leq |\phi_2(s)|$ for all $t \geq 0$ and $s \in \sigma(A)$. It follows that for all $t \geq 0$ and all $x \in \mathcal{X}$

$$\|Ae^{At}x\|^2 \leq \| |A|e^{t(a-b|A|^{1/p})}x\|^2. \quad (8)$$

Define $G := -|A|^{1/p}$. Then G is a self-adjoint operator that is bounded from above, so it generates an analytic semigroup. The equation (8) can be rewritten as

$$\|Ae^{At}x\| \leq \|G^p e^{t(a+bG)}x\|.$$

Since an analytic semigroup is of type D^1 we have, e.g. using Theorem 5, that

$$\limsup_{t \rightarrow 0^+} t^p \|G^p e^{t(a+bG)}\| < \infty.$$

It follows that

$$\limsup_{t \rightarrow 0^+} t^p \|Ae^{At}\| < \infty,$$

which shows that A generates a semigroup of class D^p . \square

Remark 10 In [3, Theorem 2.3] it is shown that condition 3. of Theorem 9, with instead of the assumption that A is similar to a normal operator a growth condition on the resolvent, implies that the semigroup generated by A is of type D^{2p-1} . This is a weaker conclusion than that in Theorem 9 (except in the analytic case $p = 1$).

The following lemmas show that generating a semigroup of type D^p or an ultra-differentiable semigroup is preserved under bounded commuting perturbation.

Lemma 11 Assume that A generates a semigroup of type D^p and that P is bounded and commutes with A , then $A + P$ generates a semigroup of type D^p .

Proof Writing

$$t^p(A + P)e^{(A+P)t} = Ae^{At}e^{Pt} + t^pPe^{At}e^{Pt}$$

and taking norms gives

$$t^p \|(A + P)e^{(A+P)t}\| \leq t^p \|Ae^{At}\| \|e^{Pt}\| + t^p \|Pe^{At}e^{Pt}\|.$$

The first term on the right-hand side is bounded as $t \rightarrow 0^+$ since A generates a semigroup of type D^p and the second term on the right-hand side converges to zero as $t \rightarrow 0^+$. It follows that the left-hand side stays bounded as $t \rightarrow 0^+$, so $A + P$ generates a semigroup of type D^p . \square

Lemma 12 *Assume that A generates an ultradifferentiable semigroup of order p and that P is bounded and commutes with A . Then $A + P$ generates an ultradifferentiable semigroup of order p .*

Proof Assume that f and g are commuting Banach-space valued ultradifferentiable functions of order p_1 and p_2 , respectively. Let p denote the maximum of p_1 and p_2 . Use Leibniz formula to write

$$(fg)^{(n)}(t) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(t) g^{(n-k)}(t)$$

and take norms to obtain

$$\|(fg)^{(n)}(t)\| \leq \sum_{k=0}^n \binom{n}{k} \|f^{(k)}(t)\| \|g^{(n-k)}(t)\|.$$

Now use that f and g are ultradifferentiable to obtain, for fixed $\theta, \tau > 0$, a constant C such that

$$\|(fg)^{(n)}(t)\| \leq C\theta^n (n!)^p \sum_{k=0}^n \binom{n}{k} \leq C(2\theta)^n (n!)^p.$$

It follows that fg is ultradifferentiable of order p .

Apply the above with f the semigroup generated by A and g the semigroup generated by P to obtain the desired conclusion. \square

We recall the concept of a (Dunford-Schwartz) spectral operator [5], [6]. On a Hilbert space this is an operator that can be written as $S + N$, with S similar to a normal operator, N a quasi-nilpotent operator (i.e. an operator whose spectrum consists only of zero) and with S and N commuting. It is known that the spectrum of a spectral operator and that of its scalar part S are equal.

Theorem 9 and Lemmas 11 and 12 immediately give the following.

Theorem 13 *Let $p \in [1, \infty)$ and assume that A is the generator of a C_0 semigroup and that A is a (Dunford-Schwartz) spectral operator. The following are equivalent.*

1. A generates an ultradifferentiable semigroup of order p .

2. A generates a semigroup of type D^p .

3. There exist $b > 0$, $a \in \mathbb{R}$ such that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq a - b|\operatorname{Im}\lambda|^{1/p}\}. \quad (9)$$

Proof As mentioned at the start of the proof of Theorem 9, 2. implies 3. and 1. implies 3. hold in this case as well. So we only have to show that 3. implies 1. and 2.

So assume that A satisfies 3. and let A_S denote the scalar part of A . Then the corresponding 3. holds for A_S as well and A_S generates a C_0 semigroup since it is a bounded perturbation of A , so that by Theorem 9 A_S generates an ultradifferentiable semigroup of order p and a semigroup of type D^p . Since $A = A_S + N$ where N is bounded and commutes with A_S it follows from Lemmas 11 and 12 that A has the same generation properties. \square

3 Examples

We will work out one example in some detail and mention two other examples that can be worked out in a very similar way.

In Luo and Guo [9] the following partial differential equation is studied:

$$\begin{aligned} w_{tt} + w_{\xi\xi\xi\xi} &= 0, & t > 0, \xi \in (0, 1), \\ w(0, t) = w_\xi(0, t) = w_{\xi\xi}(1, t) = w_{\xi\xi\xi}(1, t) - kw_{t\xi}(1, t) &= 0 & t > 0. \end{aligned}$$

This model was used there to describe shear force feedback control of a single-link flexible robot arm with a revolute joint.

In a subsequent paper by Guo, Wang and Yung [8] it is shown that for $k \in (0, 1) \cup (1, \infty)$ this system is described by an exponentially stable C_0 semigroup whose generator is a Dunford-Schwartz spectral operator (it has a Riesz basis of generalized eigenvectors) and that satisfies the spectrum location condition (9) with p any number larger than 2 (actually, more precise asymptotics of the eigenvalues are obtained). It is only concluded in [8] that the semigroup is differentiable, but Theorem 13 gives the stronger conclusion that the semigroup is of class D^p for any $p > 2$ (and that it is ultradifferentiable of order p).

The results obtained in [8] were also obtained by Shubov [11] by slightly different methods. Yet another approach to this problem was taken by Belinskiy and Lasiecka [1]. This last approach however does not show that the system is described by a Dunford-Schwartz spectral operator.

In the following the state is taken as $x(t) = [w(\cdot, t); w_t(\cdot, t)]$.

The interpolation spaces \mathcal{X}_η with $\eta \in (-1, 1)$, $\eta \neq \pm 1/4, \pm 3/4$ can be calculated to be the following (H^θ denotes the usual scale of L^2 type Sobolev spaces):

- If $\eta \geq 0$ the space $H^{2+2\eta}(0, 1) \times H^{2\eta}(0, 1)$ with boundary conditions

- If $\eta \geq 0$: $f_1(0) = f_1'(0) = 0$
- If $\eta > 1/4$: $f_1''(1) - kf_2(1) = f_2(0) = 0$
- If $\eta > 3/4$: $f_1'''(1) = f_2'(0) = 0$.
- If $\eta \leq 0$ the dual space of $H^{2+2\eta}(0, 1) \times H^{2\eta}(0, 1)$ with boundary conditions
 - If $\eta \geq 0$: $f_1(0) = f_1'(0) = 0$
 - If $\eta > 1/4$: $f_1''(1) = f_2(0) = 0$
 - If $\eta > 3/4$: $f_1'''(1) - kf_2'(1) = f_2'(0) = 0$.

We restrict ourselves here to the single-input single-output case, the multi-input multi-output case can be dealt with similarly. Some reasonable choices for the observation are given in Table 1. We note that in the integrals an $L^2(0, 1)$ weighting function can be inserted without changing the boundedness properties of the observation operator. We also note that boundary observation is included (this follows from the indicated range for ζ in the point observation cases). The second line in Table 1 for example indicates that for the output $y(t) = w(\zeta, t)$ with $\zeta \in (0, 1]$ the corresponding observation operator is in $\mathcal{L}(\mathcal{X}_\alpha, \mathbb{C})$ for any $\alpha < -3/4$.

In Table 2 we consider interior controls of the form:

$$w_{tt}(t, \xi) + w_{\xi\xi\xi\xi}(t, \xi) = b(\xi)u(t),$$

for several choices of b . For example the second line means that if we exert a force at the point $\xi = \zeta$ with $\zeta \in (0, 1)$ (i.e. ζ is not a boundary point), then the corresponding control operator is in $\mathcal{L}(\mathbb{C}, \mathcal{X}_\beta)$ for any $\beta < -1/4$.

In Table 3 we consider boundary control. The options of what to control are now limited by the boundary conditions of the uncontrolled system.

Remark 14 *We note that the given degree of unboundedness is easily calculated from the above description of interpolation spaces \mathcal{X}_η .*

For example δ_ζ is well-known to be an element of the dual space of $H^\theta(0, 1)$ for any $\theta > 1/2$. Using that the second component of \mathcal{X}_η is contained in the dual space of $H^{2\eta}(0, 1)$ it follows that $[0; \delta_\zeta] \in \mathcal{X}_\eta$ for $\eta < -1/4$. This gives the result mentioned on the second line of Table 2.

An example for the observation operators is the following. For the third line of Table 1 we need $w_\xi \in L^2(0, 1)$, so $w \in H^1(0, 1)$. From the description of the interpolation spaces \mathcal{X}_η we see that if we take $\eta = -1/2$ (then $2\eta + 2 = 1$), this is satisfied. So we can take $\alpha = -1/2$ for the observation operator corresponding to this observation. For point observation one can use that one loses $1/2 + \varepsilon$ (where ε can be any strictly positive number) in Sobolev order when compared to the corresponding distributed observation. Since a change in the parameter η in \mathcal{X}_η corresponds to double that change in Sobolev order (because of the appearance of $H^{2+2\eta}$ and $H^{2\eta}$) the difference in terms of degree of unboundedness is $1/4 + \varepsilon$. So from the above remark on the third line of Table 1, the degree of unboundedness as given on the fourth line of that table follows.

Table 1: Observation operators

Observation	Condition	α
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	-1
$w(\zeta, t)$	$\zeta \in (0, 1]$	$< -3/4$
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w_\xi(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	-1/2
$w_\xi(\zeta, t)$	$\zeta \in (0, 1]$	$< -1/4$
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w_{\xi\xi}(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	0
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w_t(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	0
$w_{\xi\xi}(\zeta, t)$	$\zeta \in [0, 1)$	$< 1/4$
$w_t(\zeta, t)$	$\zeta \in (0, 1]$	$< 1/4$
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w_{\xi\xi\xi}(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	1/2
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w_{t\xi}(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	1/2
$w_{\xi\xi\xi}(\zeta, t)$	$\zeta \in [0, 1)$	$< 3/4$
$w_{t\xi}(\zeta, t)$	$\zeta \in (0, 1]$	$< 3/4$
$\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} w_{t\xi\xi}(\xi, t) d\xi$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	1

Table 2: Interior control operators

$b(\xi)$	condition	β
δ'_ζ	$\zeta \in (0, 1)$	$< -3/4$
δ_ζ	$\zeta \in (0, 1)$	$< -1/4$
$1_{(\zeta-\varepsilon, \zeta+\varepsilon)}(\xi)$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	0
$1_{(\zeta-\varepsilon, \zeta+\varepsilon)}(\xi) \frac{1}{\varepsilon} e^{-\frac{\varepsilon^2}{\varepsilon^2 - \xi - \zeta ^2}}$	$(\zeta - \varepsilon, \zeta + \varepsilon) \subset (0, 1)$	1

Table 3: Boundary control operators

$u(t)$	β
$w_{\xi\xi}(t, 1)$	$< -3/4$
$w_t(t, 0)$	$< -3/4$
$w_{\xi\xi\xi}(t, 1) - kw_{t\xi}(t, 1)$	$< -1/4$
$w_{t\xi}(t, 0)$	$< -1/4$

From Theorem 7 and the remarks about this example made above we conclude that if we choose control operators and observation operators such that $2(\alpha - \beta) < 1$, then the system has a nuclear Hankel operator. So we could for example combine the observation $y(t) = w(\zeta, t)$ with $\zeta \in (0, 1]$ and any control operator mentioned in Table 2 or Table 3. Or we could combine the observation $y(t) = w_t(\zeta_1, t)$ with $\zeta_1 \in (0, 1]$ and the interior control operator associated to the indicator function $1_{(\zeta_2 - \varepsilon, \zeta_2 + \varepsilon)}(\xi)$ with $(\zeta_2 - \varepsilon, \zeta_2 + \varepsilon) \subset (0, 1)$.

It follows that with choices of control and observation operators with $2(\alpha - \beta) < 1$, Lyapunov balanced truncations converge in the H^∞ norm to the original system, a desirable aspect in reduced order controller design.

Remark 15 *The dual system of the above mentioned example is:*

$$\begin{aligned} w_{tt} + w_{\xi\xi\xi\xi} &= 0, & t > 0, \xi \in (0, 1), \\ w(0, t) = w_\xi(0, t) = w_{\xi\xi\xi}(1, t) = w_{\xi\xi}(1, t) + kw_t(1, t) &= 0 & t > 0. \end{aligned}$$

The situation here is entirely similar and the above translates to this case with minor changes. These equations could be used as a model for the same application as in Luo and Guo [9], but now obviously with a different feedback.

Remark 16 *Another situation where ultradifferentiable semigroups appear is in abstract equations like*

$$\ddot{w} + T^\alpha \dot{w} + Tw = 0,$$

with T a positive self-adjoint operator. For $\alpha \geq 1/2$ the evolution is described by an analytic semigroup, for $\alpha \in (0, 1/2)$ by an ultradifferentiable one of order $\frac{1}{2\alpha}$. See Taylor [13, Chapter 5], Chen and Triggiani [2]. These equations can be seen as a description of elastic systems with T the stiffness operator and T^α the damping operator. Similarly as in the above example, one can rather easily obtain well-posedness and nuclearity for certain choices of control and observation operators.

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