LQG Balancing for Continuous-Time Infinite-Dimensional Systems

Mark R. Opmeer

SIAM J. Control Optim. 46 (2007), no. 5, 1831–1848

Abstract

In this paper we study the existence of linear quadratic Gaussian (LQG)-balanced realizations for continuous-time infinite-dimensional systems. LQG-balanced realizations are those for which the optimal cost operator for the system and its dual system are equal (and diagonal). The class of systems we consider is that of distributional resolvent linear systems which includes well-posed linear systems as a subclass. We prove the existence of LQG-balanced realizations under a finite cost condition for both the system and its dual system. We also show that an LQG-balanced realization of a well-posed transfer function is well-posed. We further show that approximately controllable and observable LQG-balanced realizations are unique up to a unitary state-space transformation. Finally, we show that the spectrum of the product of the optimal cost operator of a system and its dual system is independent of the particular realization. Our method of proof shows the connections with coprime factorizations, Lyapunov-balanced realizations, and discrete-time systems. The main reason for studying LQG-balanced realizations is that truncated LQG-balanced realizations provide a good approximation of the original system. We show that, under certain conditions, this is also true in the infinite-dimensional case by proving an error bound in the gap-metric.

1 Introduction

Simple models are normally preferred over complex ones in control systems design. Sometimes it is obvious how to construct a simple model for a physical system, but sometimes it is not obvious what the characteristics essential to the controller design of a physical system are. One way of obtaining a simple model in the latter case is to first obtain a sophisticated model that takes every aspect of possible interest into account and then perform model reduction on this sophisticated model. A simple model reduction procedure was introduced by Moore [9] and is now a textbook subject (see, e.g., Zhou and Doyle [24, Chapter 7]). The method proposed by Moore consists of truncating a balanced realization. A balanced realization (also called Lyapunov- or internally balanced) is a realization for which the controllability and observability gramians are equal and diagonal. Lyapunov-balanced realizations are popular because they are relatively easy to compute and there exists an error bound in the H-infinity norm on the basis of which one can show that compensators based on the reduced order model have a certain performance when applied to the full order system. The Lyapunov-balanced realization method is applicable only to stable systems. Alternatively for unstable systems one can use truncations of a linear quadratic Gaussian (LQG)-balanced realization, which for rational transfer functions always exists. An LQG-balanced realization is a realization for which the optimal cost operator for the system and its dual system (with respect to the standard quadratic cost functional) are equal and diagonal. This method was proposed by Verriest [20], [21] and further developed by Jonckheere and Silverman [7]. For an alternative treatment see Mustafa and Glover [10], and for the discrete-time case see Hoffmann, Prätzel-Wolters, and Zerz [6]. The computation of an LQG-balanced realization can also be performed reasonably efficiently, and there exists an error bound in the gap-metric which provides advantages similar to those of the H-infinity error bound for the truncated Lyapunov-balanced realization.

In the case that the system is infinite-dimensional, the model/controller approximation becomes essential. One would like to use the methods of balanced truncation and LQG-balanced truncation in this case, too.

The existence of Lyapunov-balanced and LQG-balanced realizations for irrational transfer functions is nontrivial. A necessary and sufficient condition for the existence of Lyapunov-balanced realizations in discrete time was given by Young [23], [22] (see [19, section 9.5] for the continuous-time case). A necessary and sufficient condition for the existence of LQG-balanced realizations for discrete-time systems was given in [16]. The first of the main results of the present article shows the analogous result for the continuous-time case.

As in the finite-dimensional case it is essential for controller design to have convergence in the H-infinity norm (for Lyapunov-balanced realizations) or the gap-metric (for LQG-balanced realizations); see [2]. Additional assumptions need to be made to ensure this. Under appropriate additional assumptions, a priori error bounds ensuring such convergence were given in [5] for continuous-time Lyapunov-balanced realizations and in [1] for discrete-time Lyapunov-balanced realizations. The second of the main aims of the present article is to provide a priori error bounds in the gap-metric for LQG-balanced realizations in both discrete and continuous time.

The class of continuous-time systems we consider is very general: it includes virtually all causal time-invariant linear systems studied in the literature. Details on this class of systems are given in section 4.

The proofs of our results are based on the discrete-time case [16] supplemented by recent results on coprime factorizations [3] and on the Cayley transform and the linear quadratic regulator (LQR) problem [15], [12].

2 LQG-balanced realizations: The finite-dimensional case

In this section we review some of the results on finite-dimensional LQG-balanced realizations. We consider systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t), \tag{1}$$

where A, B, C, D are matrices of compatible dimensions. We consider the linear quadratic regulator (LQR) problem for the cost functional

$$J(x_0, u) := \int_0^\infty \|u(t)\|^2 + \|y(t)\|^2 \, dt,$$

where y is given in terms of x_0 and u by (1). The LQR problem consists of finding for a given x_0 that u for which $J(x_0, u)$ is minimal. As is well known, this problem has a unique solution when the system is minimal: the optimal input u^{opt} is given by the state feedback $u^{\mathrm{opt}}(t) = -(I + D^*D)^{-1}(D^*C + B^*Q)x(t)$, where Q is the unique nonnegative solution of the Riccati equation

$$A^*Q + QA + C^*C = (C^*D + QB)(I + D^*D)^{-1}(D^*C + B^*Q),$$

and the optimal cost is given by $J(x_0, u^{\mathrm{opt}}) = \langle x_0, Qx_0 \rangle$. By duality the "optimal filter cost" is given by $\langle x_0, Px_0 \rangle$, where P is the unique nonnegative solution of the Riccati equation

$$PA^* - AP + BB^* = (BD^* + PC^*)(I + DD^*)^{-1}(DB^* + CP).$$

The quantity $\langle x_0, Px_0 \rangle$ can be interpreted as a measure of the difficulty of reconstructing the initial state x_0 from noisy measurements. The eigenvalues of the product PQ are similarity invariants; their square roots are called the LQG-characteristic values of the system. These invariants can be interpreted as a measure of how important the subspace generated by the eigenvector is for the compensator design. This can be seen from the LQG-balanced realization. An LQG-balanced realization is a realization such that $P = Q = \Lambda$, where Λ is the diagonal matrix containing the LQG-characteristic values. Let λ_i be the square root of an eigenvalue of PQ with eigenvector x_i of length one. Then, in the LQG-balanced realization, the optimal cost with initial condition x_i is λ_i and the difficulty of reconstructing this initial state from noisy measurements is also λ_i. The idea behind LQG-balanced truncation is to restrict the system to the subspace generated by the eigenvectors corresponding to the largest eigenvalues. Since this subspace is most important for compensator design, the system obtained by LQG-balanced truncation seems to be a reasonable approximation. As mentioned in the introduction there is also an error bound which justifies the above heuristics. Let δ_g denote the gap-metric (see Zhou and Doyle [24, Chapter 7]), Σ the original (n-dimensional) system, and Σ_k the k-dimensional LQG-balanced truncation. Then

$$\delta_g(\Sigma, \Sigma_k) \leq 2 \sum_{i=k+1}^{n} \frac{\lambda_i}{\sqrt{1 + \lambda_i^2}};$$

see Mustafa and Glover [10, section 8.4.5].

3 Discrete-time systems

In this section we review the results in [16] on discrete-time infinite-dimensional LQG-balanced realizations and give some extensions. The discrete-time case is a key ingredient for the proof in continuous time.

Let $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ be separable Hilbert spaces and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}, \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \right).$$

Such a block operator will be called a *discrete-time system*. We will also denote such a block operator using the notation $[A, B; C, D]$ (we denote a block row of operators by $[X, Y]$ and a block column by $[X; Y]$). We denote the set of nonnegative integers by $\mathbb{Z}^+$. For a given initial state $x_0 \in \mathcal{X}$ and input $u : \mathbb{Z}^+ \to \mathcal{U}$ define the state $x : \mathbb{Z}^+ \to \mathcal{X}$ and output $y : \mathbb{Z}^+ \to \mathcal{Y}$ by

$$x_{n+1} = Ax_n + Bu_n, \quad x_0 = x^0, \quad y_n = Cx_n + Du_n. \tag{2}$$

A sequence $h : \mathbb{Z}^+ \to \mathcal{H}$ is called Z-transformable if the power series

$$\sum_{i=0}^{\infty} h_i z^i$$

has a positive radius of convergence. The Z-transform of a Z-transformable sequence h is defined to be the sum of this series and is denoted by $\hat{h}$. For operators A, B, C, D as above define the *transfer function* $G : \mathbb{D}_r \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ by

$$G(z) = D + \sum_{i=0}^{\infty} CA^i B z^{i+1},$$

where $\mathbb{D}_r$ is defined to be the largest disc centered at the origin for which the above sum converges (note that it definitely converges on the disc centered at the origin with radius $1/r(A)$, where $r(A)$ is the spectral radius of the operator A). If the input sequence u is Z-transformable, then the output sequence y is also Z-transformable, and if $x^0 = 0$, then the Z-transform of the output is given by

$$\hat{y}(z) = G(z)\hat{u}(z)$$

on some neighborhood of the origin. The function $D + Cz(I - zA)^{-1}B$ is called the *characteristic function* of the discrete-time system. Note that the transfer function and the characteristic function are equal on some neighborhood of the origin but may not be identically equal. A discrete-time system is called a *realization* of the function G if $G(z) = D + Cz(I - zA)^{-1}B$ on some neighborhood of the origin. Any $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function that is holomorphic at the origin can be realized as the transfer function of some discrete-time system. This discrete-time system is far from unique.

3.1 Lyapunov-balanced realizations in discrete time

Although we are studying LQG-balanced realizations, we do this by relating them to Lyapunov-balanced realizations. In this subsection we review some results on Lyapunov-balanced realizations that are needed in what follows. To define what we exactly mean by a Lyapunov-balanced realization we first have to define the input and output maps and the gramians of a discrete-time system.

The input map of a discrete-time system is defined for finitely nonzero $u : \mathbb{Z}^- \to \mathcal{U}$ by (here $\mathbb{Z}^-$ is the set of negative integers)

$$\mathcal{B}u := \sum_{i=0}^{\infty} A^i B u_{-i-1}.$$

A discrete-time system is called *approximately controllable* if the range of $\mathcal{B}$ is dense in $\mathcal{X}$, and it is called *input stable* if $\mathcal{B}$ extends to a bounded operator from $l^2(\mathbb{Z}^-;\mathcal{U})$ to $\mathcal{X}$. For an input stable discrete-time system we define the *controllability gramian* $L_B \in \mathcal{L}(\mathcal{X})$ by $L_B := \mathcal{B}\mathcal{B}^*$.

The output map of a discrete-time system is defined for $x \in \mathcal{X}$ by

$$(\mathcal{C}x)_k := CA^k x, \quad k \in \mathbb{Z}^+.$$

A discrete-time system is called *approximately observable* if $\mathcal{C}$ is one-to-one, and it is called *output stable* if $\mathcal{C}$ is a bounded operator from $\mathcal{X}$ to $l^2(\mathbb{Z}^+;\mathcal{Y})$. For an output stable discrete-time system we define the *observability gramian* $L_C \in \mathcal{L}(\mathcal{X})$ by $L_C := \mathcal{C}^*\mathcal{C}$. A discrete-time system is called *minimal* if it is both approximately controllable and approximately observable.

The Hankel operator $\mathcal{H}$ of a discrete-time system is defined for finitely nonzero $u : \mathbb{Z}^- \to \mathcal{U}$ by

$$(\mathcal{H}u)_k = \sum_{i=0}^{\infty} CA^i B u_{k-i-1}, \quad k \in \mathbb{Z}^+.$$

Note that $\mathcal{H} = \mathcal{C}\mathcal{B}$ and that $\mathcal{H}$ depends only on the transfer function of the system.

Definition 3.1. *A discrete-time system is called* Lyapunov-balanced *if it is input and output stable and $L_B = L_C$, and it is called* compact Lyapunov-balanced *if, in addition, $L_B = L_C$ is compact.*

Young [23], [22] proved that every holomorphic uniformly bounded function on the unit disc (i.e., every element of $\mathbf{H}^\infty(\mathbb{D}, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$) has a minimal Lyapunov-balanced realization. He also noted that if the Hankel operator that has this given function as symbol is compact then there exists a minimal compact Lyapunov-balanced realization. A simplification of the proof of Young can be found in Peller [17, section 11.2] and an alternative proof can be found in Staffans [19, section 9.5]. Young [23], [22] has also shown that minimal Lyapunov-balanced realizations are unique up to a unitary transformation in the state space. Let $[A, B; C, D]$ be a compact Lyapunov-balanced realization and denote by $P : \mathcal{X} \to \mathcal{X}$ the projection onto the subspace spanned by the eigenvectors of $L_B = L_C$ corresponding to the largest n eigenvalues (eigenvalues are counted with multiplicity and it is assumed here that the $n+1$st eigenvalue is different from the nth eigenvalue). Then $[PAP, PB; CP, D]$ is called the n-dimensional *truncated Lyapunov-balanced realization*. Note that the n-dimensional truncated Lyapunov-balanced realization may not be defined for every $n \in \mathbb{Z}^+$ due to repeated eigenvalues. Since eigenvalues of compact operators have finite multiplicity it is defined for infinitely many values of n. When we mention n-dimensional truncated Lyapunov-balanced realizations in what follows we will implictely assume that n is such that this notion is well defined. A truncated Lyapunov-balanced realization is not unique, since the Lyapunov-balanced realization is not. However, since two minimal Lyapunov-balanced realizations of the same transfer function are unitarily equivalent, so are all n-dimensional truncated balanced realizations. Consequently the transfer function of an n-dimensional truncated balanced realization G^n is well defined. The whole idea of Lyapunov-balanced realizations is that G^n is a good approximation of G. That this is indeed the case under certain conditions was proven by Bonnet [1]. The result of Bonnet is the discrete-time version of the continuous-time result in [5]. We summarize the results of Young and Bonnet in the following theorem. We note that the singular values of a compact operator T are the square roots of the eigenvalues of T^*T and that an operator is called *nuclear* if it is compact and its singular values form a summable sequence. The *Hankel singular values* of a system are the singular values of its Hankel operator.

Theorem 3.2. 1. *Every function in $\mathbf{H}^\infty(\mathbb{D}, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$ has a minimal Lyapunov-balanced realization. If the Hankel operator of the function is compact, then it has a minimal compact Lyapunov-balanced realization.*

2. *If, in addition, the Hankel operator of the function is nuclear and the input and output spaces are finite-dimensional, then*

$$\|G - G^n\|_\infty \leq 2 \sum_{i=n+1}^{\infty} \sigma_i,$$

where G^n is the transfer function of a truncated compact Lyapunov-balanced realization of G and the σ_i are the Hankel singular values.

Part 2 of the above theorem was proven in [1], following the continuous-time version in [5], only for the case where the Hankel singular values are distinct. As indicated in [5] the generalization to the case of possibly repeating Hankel singular values is not difficult, except notationally. Details may be found in [14, Chapter 10].

3.2 LQG-balanced realizations in discrete time

In this subsection we summarize the results obtained in [16] on LQG-balanced realizations in discrete time. We also extend these by obtaining an error bound on truncated compact LQG-balanced realizations.

To exactly define the concept of an LQG-balanced discrete-time system we first consider the LQR problem. This problem is as follows: for given $x^0 \in \mathcal{X}$ find an input u that minimizes

$$J(x^0, u) := \sum_{n=0}^{\infty} \|u_n\|^2 + \|y_n\|^2, \tag{3}$$

where y is given in terms of x^0 and u by (2). We introduce the following concept: a discrete-time system satisfies the *finite cost condition* if for every $x^0 \in \mathcal{X}$ there exists a $u \in l^2(\mathbb{Z}^+;\mathcal{U})$ such that the corresponding output $y \in l^2(\mathbb{Z}^+;\mathcal{Y})$. It is well known (see, e.g., [4]) that if the finite cost condition is satisfied, then for every $x^0 \in \mathcal{X}$ there exists a unique $u^{\mathrm{opt}} \in l^2(\mathbb{Z}^+;\mathcal{U})$ that minimizes the cost function (3) and there exists a bounded nonnegative operator Q such that the minimal cost is given by $\langle Qx^0, x^0\rangle$. This operator Q is called the *optimal cost operator*. Similarly, if for the *dual system*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$$

the finite cost condition is satisfied, then there exists an optimal cost operator P for this dual system. This operator P is called the *dual optimal cost operator* of the original system.

Definition 3.3. *A discrete-time system is called* LQG-balanced *if it and its dual both satisfy the finite cost condition and $P = Q$, and it is called* compact LQG-balanced *if, in addition, $P = Q$ is compact.*

Below we state not only the main results obtained in [16], but also some main steps in the proof. These intermediate results are also necessary to obtain the continuous-time analogues.

The optimal cost operator satisfies the following Riccati equation:

$$A^*QA - Q + C^*C = (C^*D + A^*QB)(I + D^*D + B^*QB)^{-1}(B^*QA + D^*C).$$

The optimal input u^{opt} can be given by a state feedback. To explain this we consider the concept of an admissible state feedback pair.

Definition 3.4. *An admissible state feedback pair for a discrete-time system is a pair $[K, F] \in \mathcal{L}([\mathcal{X},\mathcal{U}],\mathcal{U})$ such that $I - F$ is boundedly invertible. The closed-loop system is given by*

$$A^{\mathrm{cl}} := A + B(I-F)^{-1}K, \quad B^{\mathrm{cl}} := B(I-F)^{-1},$$

$$C^{\mathrm{cl}} := \begin{bmatrix} (I-F)^{-1}K \\ C + D(I-F)^{-1}K \end{bmatrix}, \quad D^{\mathrm{cl}} := \begin{bmatrix} (I-F)^{-1} \\ D(I-F)^{-1} \end{bmatrix}.$$

This closed-loop system is obtained by adding the equation $u_n = Kx_n + Fu_n + r_n$ to (2), considering $[u; y]$ as the new output and r as the new input, and solving. The state feedback pair

$$K := -(I+D^*D+B^*QB)^{-1/2}(D^*C+B^*QA), \quad F := I-(I+D^*D+B^*QB)^{-1/2}$$

is admissible and with zero input and initial condition x^0 the output of the closed-loop system is exactly $[u^{\mathrm{opt}}; y^{\mathrm{opt}}]$, the optimal input and output for the system $[A, B; C, D]$. The closed-loop system with this specific admissible state feedback pair will be called the *optimal closed-loop system* corresponding to the system $[A, B; C, D]$.

We give some properties of the optimal closed-loop system that were proven in [16] and some that follow from results in [3]. To formulate these we first recall the concept of a (normalized) right coprime factor.

Definition 3.5. *A function $[M; N] \in \mathbf{H}^\infty(\mathbb{D}, \mathcal{L}(\mathcal{U}, [\mathcal{U}; \mathcal{Y}]))$ is called a right* factor *of a function G if M is invertible on some neighborhood of the origin and $G = NM^{-1}$ on this region. M and N as above are called right* coprime *if there exists $[\tilde{X}, \tilde{Y}] \in \mathbf{H}^\infty(\mathbb{D}, \mathcal{L}([\mathcal{U}, \mathcal{Y}], \mathcal{U}))$ such that $\tilde{X}M - \tilde{Y}N = I$ on the unit disc. $[M; N]$ as above is called* normalized *if $M^*M + N^*N = I$ almost everywhere on the unit circle.*

We note that normalized right coprime factors are unique up to a unitary transformation in $\mathcal{L}(\mathcal{U})$. The following theorem relates factorizations to the optimal closed-loop system.

Theorem 3.6. *If the system $[A, B; C, D]$ satisfies the finite cost condition, then the transfer function of its optimal closed-loop system is a normalized right factor of its transfer function. If, in addition, the dual system also satisfies the finite cost condition, then this factor is right coprime.*

Proof. The first part of the proof follows from Corollary 5.8 of [16]. The second part follows from Lemma 6.7 of [16] and the discrete-time version of [3, Corollary 7.2]. □

Given a realization $[\check{A}, \check{B}; [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2]]$ of a factor $[M; N]$, we can obtain a realization

$$A := \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1, \quad B := \check{B}\check{D}_1^{-1}, \quad C := \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1, \quad D := \check{D}_2\check{D}_1^{-1} \tag{4}$$

of NM^{-1}. This follows from [16, Lemma 5.7]. The next result shows how one can obtain an LQG-balanced realization from a Lyapunov-balanced realization of a normalized right coprime factor.

Theorem 3.7. *Suppose that G has a normalized right coprime factor $[M; N]$. Let $[\check{A}, \check{B}; [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2]]$ be a minimal Lyapunov-balanced realization of this normalized right coprime factor. Define $[A, B; C, D]$ by (4). Then $[A, B; C, D]$ and its dual both satisfy the finite cost condition; its optimal cost operator is L, and its dual optimal cost operator is $L(I - L^2)^{-1}$, where L is the (controllability and observability) gramian of the Lyapunov-balanced realization.*

Proof. That $[M; N]$ has a minimal Lyapunov-balanced realization follows from Theorem 3.2. The rest follows from the first lines of the proof of [16, Theorem 8.2]. □

We note that $I - L$ has a bounded inverse since the Hankel singular values of the optimal closed-loop system are all strictly smaller than one. This last fact follows from the coprimeness of the factorization as in [3, Corollary 7.2].

From the system $[A, B; C, D]$ in Theorem 3.7 we obtain the LQG-balanced realization $[SAS^{-1}, SB; CS^{-1}, D]$, where $S := (I - L^2)^{-1/4}$. The following theorem summarizes some properties of LQG-balanced realizations.

Theorem 3.8. 1. *Let $[A_i, B_i; C_i, D_i]$ with $i = 1, 2$ be discrete-time systems that satisfy the finite cost condition and whose duals also satisfy the finite cost condition. If these two systems have the same transfer function, then the nonzero elements of $\sigma(P_1Q_1)$ equal the nonzero elements of $\sigma(P_2Q_2)$.*

2. *If $[A, B; C, D]$ and its dual both satisfy the finite cost condition, then its transfer function has an LQG-balanced realization.*

3. *If $[A_i, B_i; C_i, D_i]$ with $i = 1, 2$ are two minimal LQG-balanced realizations of the same transfer function, then there exists a unitary $U \in \mathcal{L}(\mathcal{X})$ such that $[A_1, B_1; C_1, D_1] = [UA_2U^{-1}, UB_2; C_2U^{-1}, D_2]$.*

Proof. 1. This is [16, Lemma 7.2] up to an additional assumption that was made there. There it was assumed that the systems were approximately observable. The reason for this was that in [16, Lemma 6.9] this assumption was needed. It was shown in [3, Lemma 4.9] how this assumption can be removed from [16, Lemma 6.9] and this implies that it can also be removed from [16, Lemma 7.2].

2. This is [16, Theorem 8.2].

3. This is [16, Lemma 8.3]. □

The square roots of the nonzero elements of $\sigma(PQ)$ are called the *LQG-characteri-stic values.* According to part 1 of Theorem 3.8 they do not depend on the realization but only on the transfer function.

We now introduce the metric in which LQG-balanced approximations converge under suitable assumptions. Assume both G_1 and G_2 have normalized right coprime factors $[M_1; N_1]$ and $[M_2; N_2]$, respectively. Define for $i = 1, 2$ the set $Z_i \subset H^2(\mathbb{D}, [U; Y])$ by $Z_i = \{(M_iv; N_iv) : v \in H^2(\mathbb{D}; U)\}$, and let P_i be the orthogonal projection from $H^2(\mathbb{D}, [U; Y])$ onto Z_i. Further define

$$\delta_g(G_1, G_2) = \|P_1 - P_2\|.$$

Note that this does not depend on the particular normalized right coprime factors chosen. The function δ_g is called the *gap-metric.* More information on the gap-metric can be found in [24, Chapter 17] and for nonrational functions in [25]. What is important for us is that

$$\delta_g(G_1, G_2) \leq \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} \right\|_\infty . \tag{5}$$

We define truncated LQG-balanced realizations similarly to truncated Lyapunov-balanced realizations. The following new result provides an a priori error bound in the gap-metric for truncated compact LQG-balanced realizations.

Theorem 3.9. *Suppose a discrete-time system satisfies the following assumptions:*

- *the finite cost condition is satisfied,*
- *the finite cost condition for the dual system is satisfied,*
- *the product PQ of the optimal cost operator and the dual optimal cost operator is nuclear, and*
- *the input and output spaces are finite-dimensional.*

Then the transfer function G has a compact LQG-balanced realization and

$$\delta_g(G, G^n) \le 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}, \tag{6}$$

where G^n is the transfer function of an n-dimensional truncated LQG-balanced realization of G.

Proof. From Theorem 3.6 it follows that the transfer function of the given system has a normalized right coprime factor. We show that the Hankel operator of this normalized right coprime factor is nuclear. It follows from [16, Lemmas 5.1 and 6.9] that $L_B L_C = (I + PQ)^{-1} PQ$, where L_B is the controllability gramian of the optimal closed-loop system and L_C is its observability gramian. Since the product PQ is assumed compact, this shows that the product $L_B L_C$ is compact. The eigenvalues are related by

$$\mu_i = \frac{\sigma_i}{\sqrt{1-\sigma_i^2}}, \quad \sigma_i = \frac{\mu_i}{\sqrt{1+\mu_i^2}}, \tag{7}$$

where the μ_i are the square roots of the eigenvalues of PQ and the σ_i are the square roots of the eigenvalues of $L_B L_C$. This shows that the square roots of the eigenvalues of $L_B L_C$ are summable. Denote the Hankel operator of the normalized right coprime factor by Γ. As in [16, Lemma 7.2] it follows that the spectrum of $\Gamma^*\Gamma$ equals the spectrum of $L_B L_C$ and the point spectrum of $\Gamma^*\Gamma$ equals the point spectrum of $L_B L_C$ (both with the possible exception of zero). This shows that $\Gamma^*\Gamma$ has only point spectrum (with the possible exception of zero) and that the square roots of the eigenvalues are summable. This shows that the Hankel operator is nuclear.

Denote the normalized eigenvectors of the gramian L of the Lyapunov-balanced realization of the normalized coprime factor by e_n. Since for the optimal control operators of the LQG-balanced realization we have $P^{\text{bal}} = Q^{\text{bal}} = L(I - L^2)^{-1/2}$ from Theorem 3.7 we see that this LQG-balanced realization is actually compact LQG-balanced, and the corresponding orthonormal basis is $\{e_n\}$. We note that the projections associated to Lyapunov-balanced truncation and to LQG-balanced truncation are equal, since the orthonormal bases (including order) are identical. We conclude that the system obtained by applying (4) to the truncated Lyapunov-balanced realization is the truncated LQG-balanced realization. From Theorem 3.2, (5), and (7) we now obtain the estimate (6). □

4 Resolvent linear systems

In this section we recall the concept of a distributional resolvent linear system introduced in [12].

A finite-dimensional linear system is usually described by specifying four matrices A, B, C, D and defining for a given initial state x_0 and an input function $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^m)$ the state $x \in C(0,\infty;\mathbb{C}^n)$ and the output $y \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^p)$ as the unique solutions of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t). \tag{8}$$

As is well known, these unique solutions are given explicitly by

$$\begin{aligned} x(t) &= \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,ds, \\ y(t) &= C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-s)}Bu(s)\,ds + Du(t). \end{aligned} \tag{9}$$

If we Laplace-transform (8) and solve for x and y, we obtain

$$\begin{aligned} \hat{x}(s) &= (sI-A)^{-1}x_0 + (sI-A)^{-1}B\hat{u}(s), \\ \hat{y}(s) &= C(sI-A)^{-1}x_0 + \big(C(sI-A)^{-1}B + D\big)\,\hat{u}(s). \end{aligned} \tag{10}$$

Our approach to infinite-dimensional systems will be to generalize situation (10) rather than situation (8) or (9).

We first study the generalizations of the matrix-valued functions $(sI-A)^{-1}$, $(sI-A)^{-1}B$, $C(sI-A)^{-1}$, and $C(sI-A)^{-1}B+D$.

Definition 4.1. *A* resolvent linear system *on a triple of Hilbert spaces* $(\mathcal{U},\mathcal{X},\mathcal{Y})$ *consists of a nonempty connected open subset* Λ *of the complex plane and four operator valued functions* $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}$ *satisfying*

$\mathfrak{a} : \Lambda \to \mathcal{L}(\mathcal{X})$ *satisfies*

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha-\beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \textit{for all } \alpha,\beta \in \Lambda; \tag{11}$$

$\mathfrak{b} : \Lambda \to \mathcal{L}(\mathcal{U},\mathcal{X})$ *satisfies*

$$\mathfrak{b}(\beta) - \mathfrak{b}(\alpha) = (\alpha-\beta)\mathfrak{a}(\beta)\mathfrak{b}(\alpha) \quad \textit{for all } \alpha,\beta \in \Lambda; \tag{12}$$

$\mathfrak{c} : \Lambda \to \mathcal{L}(\mathcal{X},\mathcal{Y})$ *satisfies*

$$\mathfrak{c}(\beta) - \mathfrak{c}(\alpha) = (\alpha-\beta)\mathfrak{c}(\alpha)\mathfrak{a}(\beta) \quad \textit{for all } \alpha,\beta \in \Lambda; \tag{13}$$

$\mathfrak{d} : \Lambda \to \mathcal{L}(\mathcal{U},\mathcal{Y})$ *satisfies*

$$\mathfrak{d}(\beta) - \mathfrak{d}(\alpha) = (\alpha-\beta)\mathfrak{c}(\beta)\mathfrak{b}(\alpha) \quad \textit{for all } \alpha,\beta \in \Lambda. \tag{14}$$

The function $\mathfrak{a}$ *is called the* pseudoresolvent, $\mathfrak{b}$ *is the* incoming wave function, $\mathfrak{c}$ *is the* outgoing wave function, *and* $\mathfrak{d}$ *is the* characteristic function *of the resolvent linear system.*

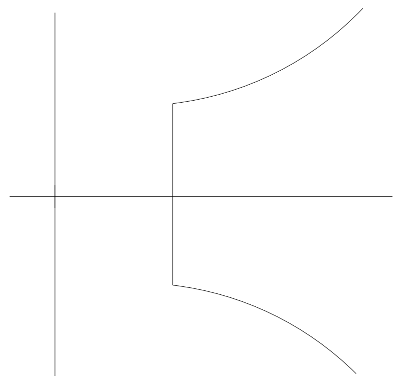

Figure 1: A typical example of the boundary of an exponential region

The motivation for introducing this class of systems is the following connection with discrete-time systems.

Definition 4.2. *Let $\alpha > 0$. The Cayley transform with parameter α of a resolvent linear system with $\alpha \in \Lambda$ is the discrete-time system*

$$A_d := -I + 2\alpha \ \mathfrak{a}(\alpha), \qquad B_d := \sqrt{2\alpha}\ \mathfrak{b}(\alpha), \tag{15}$$

$$C_d := \sqrt{2\alpha}\ \mathfrak{c}(\alpha), \qquad D_d := \mathfrak{d}(\alpha). \tag{16}$$

Remark 4.3. The Cayley transform with parameter α gives a one-to-one correspondence between the set of resolvent linear systems with $\alpha \in \Lambda$ and the set of discrete-time systems.

The following relation between the characteristic function of a resolvent linear system and that of its Cayley transform is easily proven: $G(s) = G_d(z)$, where $z := (\alpha - s)/(\alpha + s)$.

We define two subclasses of resolvent linear systems for which one can make sense of the dynamical system (10).

Definition 4.4. *A* distributional resolvent linear system *is a resolvent linear system with the additional property that there exist constants $\alpha, \beta > 0$ and a polynomial p such that*

$$\Lambda_E := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq \beta, \quad |\mathrm{Im}\ s| \leq \mathrm{e}^{\alpha \mathrm{Re}\ s}\} \subset \Lambda \tag{17}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \textit{for all } s \in \Lambda_E. \tag{18}$$

A region Λ_E as above is called an exponential region (see Figure 1 for a sketch of the boundary of such a region).

It is easily seen using the functional equations from Definition 4.1 that the functions $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ of a distributional resolvent linear system are also bounded in norm by a polynomial on the exponential region Λ_E.

Definition 4.5. *A distributional resolvent linear system is called* exponentially bounded *if there exist a constant $\gamma > 0$ and a polynomial p such that*

$$\Lambda_H := \{s \in \mathbb{C} : \text{Re } s \geq \gamma\} \subset \Lambda \tag{19}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \textit{for all } s \in \Lambda_H. \tag{20}$$

Note that the difference between Definitions 4.4 and 4.5 is in the region considered.

Remark 4.6. The term "exponentially bounded" stems from time-domain properties of this subclass. In [12] exponentially bounded distributional resolvent linear systems were called integrated resolvent linear systems. In view of the time-domain results in [13] the term exponentially bounded distributional resolvent linear system, however, seems to be more appropriate.

Remark 4.7. In what follows we will need the following well-known characterization of Laplace transformable Banach space valued distributions by Schwartz. The image of the Schwartz–Laplace transformable Banach space valued distributions is exactly the set of polynomially bounded analytic functions defined on some right half-plane. For details see [18]. A generalization of this Laplace transform was given by Kunstmann. He defined the Laplace transform in such a way that the image of the set of Laplace transformable distributions is exactly the set of functions that are analytic and polynomially bounded on an exponential region (see Kunstmann [8]).

Definition 4.8. *The state x and output y of a distributional resolvent linear system corresponding to the initial state $x_0 \in \mathcal{X}$ and the input u (a $\mathcal{U}$-valued Kunstmann–Laplace transformable distribution) are defined through their Kunstmann–Laplace*
transforms as

$$\hat{x}(s) := \mathfrak{a}(s)x_0 + \mathfrak{b}(s)\hat{u}(s), \quad \hat{y}(s) := \mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s). \tag{21}$$

For the case of exponentially bounded distributional resolvent linear systems, if we restrict u to be Schwartz–Laplace transformable, then x and y are Schwartz–Laplace transformable.

For a distributional resolvent linear system we define the set of *stable input-output pairs*

$$\mathcal{V}(x_0) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{bmatrix} : y \text{ satisfies (21)} \right\}.$$

Definition 4.9. *We say that a distributional resolvent linear system satisfies the* finite cost condition *if for every $x_0 \in \mathcal{X}$ the set $\mathcal{V}(x_0)$ is nonempty.*

For $\alpha > 0$ the mapping $\mathcal{H}_d : \mathbf{H}^2(\mathbb{C}_0^+;\mathcal{H}) \to \mathbf{H}^2(\mathbb{D};\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is unitary. Here $\mathcal{H}_d$ is defined by

$$(\mathcal{H}_d g)(z) = \frac{\sqrt{2\alpha}}{1+z} \; g\left(\alpha \, \frac{1-z}{1+z}\right), \tag{22}$$

with its inverse given by

$$(\mathcal{H}_d^{-1} f)(s) = \frac{\sqrt{2\alpha}}{\alpha + s} \;\; f\left(\frac{\alpha - s}{\alpha + s}\right). \tag{23}$$

$\mathbb{C}_0^+$ is the right half-plane and $\mathbf{H}^2$ is a Hardy space. The following theorem shows that, for a suitably chosen parameter α, there is a one-to-one relationship between the stable input-output pairs of a distributional resolvent linear system and those of its Cayley transform.

Theorem 4.10. *Let* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d})$ *be a distributional resolvent linear system with* $\alpha \in \Lambda_E$*, where* $\alpha > 0$*. Let* $[A_d, B_d; C_d, D_d]$ *be its Cayley transform with parameter* α*. Then* $(u; y) \in \mathcal{V}(x_0)$ *if and only if* $(\mathcal{H}_d u; \mathcal{H}_d y) \in \mathcal{V}_d(x_0)$*.*

The following is [12, Lemma 9].

Lemma 4.11. *For a distributional resolvent linear system on a triple of Hilbert spaces for which the finite cost condition is satisfied there exists a nonnegative operator* $Q \in \mathcal{L}(\mathcal{X})$ *such that the optimal cost for the cost function*

$$\int_0^\infty \|u(t)\|^2 + \|y(t)\|^2 \, dt \tag{24}$$

is given by $\langle Qx_0, x_0 \rangle$*. This* Q *satisfies the Riccati equation*

$$\begin{aligned}
&-\mathfrak{a}(\alpha)^* Q - Q\mathfrak{a}(\alpha) + 2\alpha\mathfrak{a}(\alpha)^* Q\mathfrak{a}(\alpha) + \mathfrak{c}(\alpha)^*\mathfrak{c}(\alpha) \\
&\quad = (\mathfrak{c}(\alpha)^*\mathfrak{d}(\alpha) - Q\mathfrak{b}(\alpha) + 2\alpha\mathfrak{a}(\alpha)^* Q\mathfrak{b}(\alpha)) \\
&\quad\quad (I + \mathfrak{d}(\alpha)^*\mathfrak{d}(\alpha) + 2\alpha\mathfrak{b}(\alpha)^* Q\mathfrak{b}(\alpha))^{-1} \\
&\quad\quad (\mathfrak{d}(\alpha)^*\mathfrak{c}(\alpha) - \mathfrak{b}(\alpha)^* Q + 2\alpha\mathfrak{b}(\alpha)^* Q\mathfrak{a}(\alpha))
\end{aligned}$$

for all $\alpha \in \Lambda_E$*.*

The operator Q mentioned above is called the *optimal cost operator* of the distributional resolvent linear system. We now study admissible state feedbacks.

Definition 4.12. *An* admissible state feedback pair *for a* distributional resolvent linear system *is a pair* $[\mathfrak{k}, \mathfrak{f}] : \Lambda_E \to \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{U})$ *that satisfies*

$$\begin{aligned}
\mathfrak{k}(\beta) - \mathfrak{k}(\alpha) &= (\alpha - \beta)\mathfrak{k}(\alpha)\mathfrak{a}(\beta), \\
\mathfrak{f}(\beta) - \mathfrak{f}(\alpha) &= (\alpha - \beta)\mathfrak{k}(\beta)\mathfrak{b}(\alpha),
\end{aligned}$$

and such that $(I - \mathfrak{f}(s))^{-1}$ *exists and is polynomially bounded on some exponential region.*

The closed-loop system of a distributional resolvent linear system with an admissible state feedback pair is the distributional resolvent linear system

$$\begin{aligned}
\mathfrak{a}^{\mathrm{cl}} &:= \mathfrak{a} + \mathfrak{b}(I - \mathfrak{f})^{-1}\mathfrak{k}, \quad \mathfrak{b}^{\mathrm{cl}} := \mathfrak{b}(I - \mathfrak{f})^{-1}, \\
\mathfrak{c}^{\mathrm{cl}} &:= \begin{bmatrix} (I - \mathfrak{f})^{-1}\mathfrak{k} \\ \mathfrak{c} + \mathfrak{d}(I - \mathfrak{f})^{-1}\mathfrak{k} \end{bmatrix}, \quad \mathfrak{d}^{\mathrm{cl}} := \begin{bmatrix} (I - \mathfrak{f})^{-1} \\ \mathfrak{d}(I - \mathfrak{f})^{-1} \end{bmatrix}.
\end{aligned}$$

It can be easily checked that this is indeed a distributional resolvent linear system. The exponential region on which this closed-loop system is defined is the largest exponential region contained in the intersection of the exponential region on which the original system was defined and the exponential region on which $(I-\mathfrak{f})^{-1}$ exists and is polynomially bounded.

The following is [12, Lemma 8].

Lemma 4.13. *For a distributional resolvent linear system on a triple of Hilbert spaces for which the finite cost condition is satisfied there exists an admissible state feedback pair such that the optimal control u^{opt} for the cost function* (24) *is given by $\hat{u}^{\mathrm{opt}}(s) = (I-\mathfrak{f}(s))^{-1}\mathfrak{k}(s)x_0$ for $s \in \Lambda_E$.*

Remark 4.14. A proper proof of Lemma 4.13 is given in [12, Lemma 8]. We do want to mention the main idea of the proof. First, one Cayley-transforms the system with a suitable parameter α. One then defines $\mathfrak{k}(\alpha)$ and $\mathfrak{f}(\alpha)$ in terms of the optimal admissible state feedback pair $[K, F]$ of the Cayley-transformed system. The functions $\mathfrak{k}$ and $\mathfrak{f}$ are then extended to Λ_E using the functional equations from Definition 4.12. In the specific case of the optimal state feedback it is simple to prove that $(I-\mathfrak{f})^{-1}$ exists and is polynomially bounded on an exponential region: its Cayley transform equals the denominator M_d of the normalized right factor mentioned in Theorem 3.6. So the Cayley transform of $(I-\mathfrak{f})^{-1}$ is in H^∞ of the unit disc, from which it follows that $(I-\mathfrak{f})^{-1}$ is in H^∞ of the right half-plane.

Definition 4.15. *An* admissible state feedback pair *for an* exponentially bounded distributional resolvent linear system *is a pair $[\mathfrak{k}, \mathfrak{f}] : \Lambda_H \to \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{U})$ that satisfies*

$$\begin{aligned}\mathfrak{k}(\beta) - \mathfrak{k}(\alpha) &= (\alpha-\beta)\mathfrak{k}(\alpha)\mathfrak{a}(\beta),\\ \mathfrak{f}(\beta) - \mathfrak{f}(\alpha) &= (\alpha-\beta)\mathfrak{k}(\beta)\mathfrak{b}(\alpha),\end{aligned}$$

and such that $(I-\mathfrak{f}(s))^{-1}$ exists and is polynomially bounded on some right half-plane.

The closed-loop system of an exponentially bounded distributional resolvent linear system and an admissible state feedback pair in the sense of Definition 4.15 is easily seen to be an exponentially bounded distributional resolvent linear system. Theorem 4.13 holds for exponentially bounded distributional resolvent linear systems with admissible state feedback operator now understood in the stronger sense of Definition 4.15.

Definition 4.16. *The* dual *of a resolvent linear system* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ *is the resolvent linear system*

$$\mathfrak{a}^d(s) := \mathfrak{a}(\bar{s})^*, \quad \mathfrak{b}^d(s) := \mathfrak{c}(\bar{s})^*, \quad \mathfrak{c}^d(s) := \mathfrak{b}(\bar{s})^*, \quad \mathfrak{d}^d(s) := \mathfrak{d}(\bar{s})^*.$$

Note that the dual of a distributional resolvent linear system is a distributional resolvent linear system and that the dual of an exponentially bounded distributional resolvent linear system is an exponentially bounded distributional resolvent linear system.

The concept of approximate observability has a natural generalization to distributional resolvent linear systems.

Definition 4.17. *A distributional resolvent linear system is said to be* approximately observable *if for zero input the output is zero if and only if the initial state is zero.*

Note that a distributional resolvent linear system is approximately observable if and only if $\mathfrak{c}(s)x_0 = 0$ for all $s \in \Lambda_E$ implies $x_0 = 0$.

Definition 4.18. *A distributional resolvent linear system is said to be* approximately controllable *if its dual system is approximately observable. It is called* minimal *if it is both approximately controllable and approximately observable.*

It is easily seen that a distributional resolvent linear system is approximately controllable (observable) if and only if its Cayley transform with a parameter $\alpha \in \Lambda_E$ is.

We denote by $\mathbf{H}^\infty(\mathbb{C}_0^+, \mathcal{E})$ the Hardy space of uniformly bounded $\mathcal{E}$-valued analytic functions defined on the right half-plane, where $\mathcal{E}$ is a Banach space.

Definition 4.19. *Let Λ_E be an exponential region. A function $G : \Lambda_E \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$, is said to have a* right factorization *if there exist $N \in \mathbf{H}^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$ and $M \in \mathbf{H}^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathcal{U}))$ such that $M(s)^{-1}$ exists for all $s \in \Lambda_E$ and $G = NM^{-1}$ on Λ_E.*

This factorization is called normalized *if $[M; N]$ is inner, i.e., if for almost all $\omega \in \mathbb{R}$ we have*

$$M_b(i\omega)^* M_b(i\omega) + N_b(i\omega)^* N_b(i\omega) = I,$$

where M_b and N_b are the boundary functions of M and N, respectively.

This factorization is called right coprime *if there exist $\tilde{X} \in \mathbf{H}^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathcal{U}))$, $\tilde{Y} \in \mathbf{H}^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathcal{Y}, \mathcal{U}))$ such that*

$$\tilde{X}M - \tilde{Y}N = I \quad \text{on } \mathbb{C}_0^+. \tag{25}$$

Using the Cayley transform, we obtain from Theorem 3.6 the following theorem.

Theorem 4.20. *If a distributional resolvent linear system satisfies the finite cost condition, then its characteristic function has a normalized right factor. If, in addition, the dual finite cost condition is satisfied, then this factor is right coprime.*

The above theorem is a slight generalization of [3, Theorem 8.9]. The proof is almost identical; one simply replaces the reciprocal transform used there by the Cayley transform with a suitable parameter (i.e., positive and in Λ_E). The relation between characteristic functions mentioned in Remark 4.3 is of course essential.

5 Well-posed linear systems

We now show how the well-known class of well-posed linear systems fits into our framework.

Definition 5.1. *A resolvent linear system is called* well-posed *if*

1. *the pseudoresolvent is the resolvent of the generator of a strongly continuous semigroup T;*

2. *for every $x \in \mathcal{X}$ the function $\mathfrak{c}x$ restricts to a function in $\mathbf{H}^2(\mathbb{C}_\omega^+;\mathcal{Y})$, where ω is some real number strictly larger than the growth bound of T;*

3. *for every $x \in \mathcal{X}$ the function $\mathfrak{b}^d x$ restricts to a function in $\mathbf{H}^2(\mathbb{C}_\omega^+;\mathcal{U})$, where ω is some real number strictly larger than the growth bound of T; and*

4. *$\mathfrak{d}$ restricts to a function in $\mathbf{H}^\infty(\mathbb{C}_\omega^+;\mathcal{L}(\mathcal{U},\mathcal{Y}))$, where ω is some real number strictly larger than the growth bound of T.*

The above definition is equivalent to the usual time-domain definition.

Definition 5.2. *An* admissible state feedback pair *for a* well-posed linear system *is a pair $[\mathfrak{k},\mathfrak{f}] : \mathbb{C}_\omega^+ \to \mathcal{L}(\mathcal{X} \times \mathcal{U},\mathcal{U})$ that satisfies*

$$\mathfrak{k}(\beta) - \mathfrak{k}(\alpha) = (\alpha - \beta)\mathfrak{k}(\alpha)\mathfrak{a}(\beta),$$
$$\mathfrak{f}(\beta) - \mathfrak{f}(\alpha) = (\alpha - \beta)\mathfrak{k}(\beta)\mathfrak{b}(\alpha),$$

and such that for every $x \in \mathcal{X}$ the function $\mathfrak{k}x$ restricts to a function in $\mathbf{H}^2(\mathbb{C}_\omega^+;\mathcal{U})$, the function $\mathfrak{f}$ restricts to a function in $\mathbf{H}^\infty(\mathbb{C}_\omega^+;\mathcal{L}(\mathcal{U}))$, and $(I - \mathfrak{f}(s))^{-1}$ exists and is uniformly bounded on some right half-plane.

The above definition is equivalent to the time-domain definition in [19]. In [19] it is shown that the closed-loop system of a well-posed linear system with an admissible state feedback in the sense of Definition 5.2 is a well-posed linear system.

6 LQG-balanced realizations

In this section we prove the continuous-time analogues of the discrete-time results of section 3.2.

Definition 6.1. *For a distributional resolvent linear system that satisfies both the finite cost condition and the dual finite cost condition the nonzero elements of the set $\sqrt{\sigma(PQ)}$, where Q is the optimal cost operator of the system and P is the optimal cost operator of the dual system, are called* LQG-characteristic values*.*

The following theorem shows that the LQG-characteristic values depend only on the characteristic function.

Theorem 6.2. *Two distributional resolvent linear systems that both satisfy both the finite cost condition and the dual finite cost condition and whose characteristic functions are equal on an exponential region have the same set of LQG-characteristic values.*

Proof. Cayley-transform both distributional resolvent linear systems with a parameter which is in the exponential region of both. The transfer functions of the Cayley-transformed systems then agree in some neighborhood of zero. It follows from Theorem 3.8 that the LQG-characteristic values of these Cayley-transformed systems are equal. The LQG-characteristic values of a distributional resolvent linear system and its Cayley transform are equal, since the optimal cost operators are equal (which follows from Theorem 4.10). Hence it follows that the LQG-characteristic values of the two distributional resolvent linear systems are equal. ☐

Definition 6.3. *A distributional resolvent linear system is called* LQG-balanced *if it and its dual both satisfy the finite cost condition and if the optimal cost operators of the system and that of its dual are equal. It is called* compact LQG-balanced *if, in addition, this operator is compact.*

The following theorem gives a necessary and sufficient condition for the existence of LQG-balanced realizations.

Theorem 6.4. *An $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued holomorphic function, defined and polynomially bounded on an exponential region, has a normalized right coprime factor if and only if it has an LQG-balanced realization.*

Proof. Assume the given function $\mathfrak{d}$ has a normalized right coprime factor $[M;N]$. It follows from [11] or [19, section 9.5] that $[M;N]$ has a minimal well-posed Lyapunov-balanced realization $\mathfrak{a}_L$, $\mathfrak{b}_L$, $[\mathfrak{c}_L^1;\mathfrak{c}_L^2]$, $[M;N]$. Consider the well-posed linear system $\mathfrak{a}_L$, $\mathfrak{b}_L$, $\mathfrak{c}_L^2$, N and the feedback pair $[\mathfrak{k},\mathfrak{f}] := [-\mathfrak{c}_L^1, I-M]$. This feedback pair is admissible for the given system: the algebraic relations easily follow from the fact that the Lyapunov-balanced system is a resolvent linear system (even a well-posed linear system). Since $(I-\mathfrak{f}(s))^{-1} = M^{-1}$ it remains to show that M^{-1} is polynomially bounded on some exponential region. This follows from the equation $M^{-1} = X - Y\mathfrak{d}$ on Λ_E, which follows from the Bezout equation (25). The closed-loop system of the above system with the given feedback pair is $\mathfrak{a}_L - \mathfrak{b}_L M^{-1}\mathfrak{c}_L^1$, $\mathfrak{b}_L M^{-1}$, $[-M^{-1}\mathfrak{c}_L^1; \mathfrak{c}_L^2 - NM^{-1}\mathfrak{c}_L^1]$, $[M^{-1}; NM^{-1}]$. It follows that this is a distributional resolvent linear system. We drop one of the components and obtain the following distributional resolvent linear system:

$$\mathfrak{a}_s := \mathfrak{a}_L - \mathfrak{b}_L M^{-1}\mathfrak{c}_L^1, \quad \mathfrak{b}_s := \mathfrak{b}_L M^{-1}, \quad \mathfrak{c}_s := \mathfrak{c}_L^2 - NM^{-1}\mathfrak{c}_L^1, \quad \mathfrak{d}_s := NM^{-1}. \tag{26}$$

Now choose $\alpha > 0$ in the intersection of the exponential regions of all the systems considered above and Cayley-transform these systems with this parameter. It is obvious from the constructions and Theorem 3.7 that the system (26) has L as its optimal cost operator and $L(I-L^2)^{-1}$ as its dual optimal cost operator, where L is the gramian of the Lyapunov-balanced realization. Define $S := (I-L^2)^{-1/4}$, and define $\mathfrak{a}_l := S\mathfrak{a}_s S^{-1}$, $\mathfrak{b}_l := S\mathfrak{b}$, $\mathfrak{c}_l := \mathfrak{c}S^{-1}$, $\mathfrak{d}_l = \mathfrak{d}_s$. We conclude that $\mathfrak{a}_l$, $\mathfrak{b}_l$, $\mathfrak{c}_l$, $\mathfrak{d}_l$ is LQG-balanced. Since $\mathfrak{d}_l = NM^{-1} = \mathfrak{d}$ this distributional resolvent linear system is an LQG-balanced realization of $\mathfrak{d}$.

The converse trivially follows from Theorem 4.20. $\square$

Theorem 6.4 can be rephrased in terms of realizations as follows. Here Theorem 4.20 is used.

Corollary 6.5. *For a distributional resolvent linear system that satisfies both the finite cost condition and the dual finite cost condition there exists an LQG-balanced distributional resolvent linear system such that the characteristic functions of these two systems are equal on some exponential region.*

The following two corollaries show that in the special cases of exponentially bounded distributional resolvent linear systems and well-posed linear systems the LQG-balanced realization belongs to the same class.

Corollary 6.6. *For an exponentially bounded distributional resolvent linear system that satisfies both the finite cost condition and the dual finite cost condition there exists an LQG-balanced exponentially bounded distributional resolvent linear system such that the characteristic functions of these two systems are equal on some right half-plane.*

Proof. This follows from the proof of Theorem 6.4 noting that the Bezout equation now shows that the feedback is admissible in the sense of Definition 4.15. $\square$

Corollary 6.7. *For a well-posed linear system that satisfies both the finite cost condition and the dual finite cost condition there exists an LQG-balanced well-posed linear system such that the characteristic functions of these two systems are equal on some right half-plane.*

Proof. This follows from the proof of Theorem 6.4 noting that the Bezout equation now shows that the feedback is admissible in the sense of Definition 5.2. $\square$

Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ be an LQG-balanced distributional resolvent linear system, and let $U \in \mathcal{L}(\mathcal{X})$ be unitary. Then obviously $U\mathfrak{a}U^*$, $U\mathfrak{b}$, $\mathfrak{c}U^*$, $\mathfrak{d}$ is also an LQG-balanced distributional resolvent linear system. The next theorem shows that these are all LQG-balanced distributional resolvent linear systems with characteristic function $\mathfrak{d}$ if we assume a minimality assumption on the state space.

Theorem 6.8. *If two distributional resolvent linear systems whose characteristic functions agree on some exponential region are both LQG-balanced, approximately controllable, and approximately observable, then there exists a unitary state-space transformation between them.*

Proof. Choose a parameter that is in the exponential region of both systems, and Cayley-transform both systems with this parameter. The resulting systems are LQG-balanced, approximately controllable, and approximately observable and have the same transfer function. It follows from part 3 of Theorem 3.8 that these discrete-time systems are unitarily equivalent. From this it follows that the distributional resolvent linear systems are unitarily equivalent. □

The gap-metric in continuous time is defined in exactly the same way as was done in discrete time in section 3.2, but with the unit disc $\mathbb{D}$ replaced by the right half-plane $\mathbb{C}_0^+$. It is easily seen that the distance between two systems equals the distance between their Cayley transforms (taken with the same parameter, obviously).

Using the Cayley transform, the following theorem follows immediately from Theorem 3.9.

Theorem 6.9. *Suppose a distributional resolvent linear system satisfies the following assumptions:*

- *the finite cost condition is satisfied,*
- *the finite cost condition for the dual system is satisfied,*
- *the product PQ of the optimal cost operator and the dual optimal cost operator is nuclear, and*
- *the input and output spaces are finite-dimensional.*

Then there exists a compact LQG-balanced distributional resolvent linear system whose characteristic function equals the characteristic function of the original system on some exponential region and

$$\delta_g(\mathfrak{d}, G^n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}, \tag{27}$$

where G^n is the transfer function of an n-dimensional truncated LQG-balanced realization.

7 Conclusions

In this article we have obtained existence and uniqueness results for LQG-balanced realizations for continuous-time infinite-dimensional systems. We also obtained a priori error bounds in the gap-metric for both the continuous-time and the discrete-time cases.

References

[1] C. Bonnet, *Convergence and convergence rate of the balanced realization truncations for infinite-dimensional discrete-time systems*, Systems Control Lett., 20 (1993), pp. 353–359.

[2] R. F. Curtain, *Model reduction for control design for distributed parameter systems*, in Research Directions in Distributed Parameter Systems (Raleigh, NC, 2000), Frontiers Appl. Math. 27, SIAM, Philadelphia, 2003, pp. 95–121.

[3] R. F. Curtain and M. R. Opmeer, *Normalized doubly coprime factorizations for infinite-dimensional linear systems*, Math. Control Signals Systems, 18 (2006), pp. 1–31.

[4] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Texts Appl. Math. 21, Springer-Verlag, New York, 1995.

[5] K. Glover, R. F. Curtain, and J. R. Partington, *Realisation and approximation of linear infinite-dimensional systems with error bounds*, SIAM J. Control Optim., 26 (1988), pp. 863–898.

[6] J. Hoffmann, D. Prätzel-Wolters, and E. Zerz, *A balanced canonical form for discrete-time minimal systems using characteristic maps*, Linear Algebra Appl., 277 (1998), pp. 63–81.

[7] E. A. Jonckheere and L. M. Silverman, *A new set of invariants for linear systems—application to reduced order compensator design*, IEEE Trans. Automat. Control, 28 (1983), pp. 953–964.

[8] P. C. Kunstmann, *Laplace transform theory for logarithmic regions*, in Evolution Equations and Their Applications in Physical and Life Sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Appl. Math. 215, Dekker, New York, 2001, pp. 125–138.

[9] B. C. Moore, *Principal component analysis in linear systems: Controllability, observability, and model reduction*, IEEE Trans. Automat. Control, 26 (1981), pp. 17–32.

[10] D. Mustafa and K. Glover, *Minimum Entropy H_∞ Control*, Lecture Notes in Control and Inform. Sci. 146, Springer-Verlag, Berlin, 1990.

[11] R. Ober and S. Montgomery-Smith, *Bilinear transformation of infinite-dimensional state-space systems and balanced realizations of nonrational transfer functions*, SIAM J. Control Optim., 28 (1990), pp. 438–465.

[12] M. R. Opmeer, *Infinite-dimensional linear systems: A distributional approach*, Proc. London Math. Soc., 91 (2005), pp. 738–760.

[13] M. R. Opmeer, *Distribution semigroups and control systems*, J. Evol. Equ., 6 (2006), pp. 145–159.

[14] M. R. Opmeer, *Model Reduction for Controller Design for Infinite-Dimensional Systems*, Ph.D. thesis, University of Groningen, Groningen, The Netherlands 2006; available at http://irs.ub.rug.nl/ppn/296880116.

[15] M. R. Opmeer and R. F. Curtain, *New Riccati equations for well-posed linear systems*, Systems Control Lett., 52 (2004), pp. 339–347.

[16] M. R. Opmeer and R. F. Curtain, *Linear quadratic Gaussian balancing for discrete-time infinite-dimensional linear systems*, SIAM J. Control Optim., 43 (2004), pp. 1196–1221.

[17] V. V. Peller, *Hankel Operators and Their Applications*, Springer Monogr. Math., Springer-Verlag, New York, 2003.

[18] L. Schwartz, *Théorie des distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X, Nouvelle édition, entièrement corrigée, refondue et augmentée, Hermann, Paris, 1966.

[19] O. Staffans, *Well-Posed Linear Systems*, Encyclopedia Math. Appl. 103, Cambridge University Press, Cambridge, UK, 2005.

[20] E. I. Verriest, *Low sensitivity design and optimal order reduction for the LQG-problem*, in Proceedings of the 24th Midwest Symposium on Circuits and Systems, Albuquerque, NM, 1981, pp. 365–369.

[21] E. I. Verriest, *Suboptimal lqg-design via balanced realizations*, in Proceedings of the 20th IEEE Conference on Decision and Control, IEEE Control Systems Society, Piscataway, NJ, 1981, pp. 686–687.

[22] N. J. Young, *Balanced, normal, and intermediate realizations of nonrational transfer functions*, IMA J. Math. Control Inform., 3 (1986), pp. 43–58.

[23] N. J. Young, *Balanced realizations in infinite dimensions*, in Operator Theory and Systems (Amsterdam, 1985), Oper. Theory Adv. Appl. 19, Birkhäuser, Basel, 1986, pp. 449–471.

[24] K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Prentice–Hall, Upper Saddle River, NJ, 1998.

[25] S. Q. Zhu, *Graph topology and gap topology for unstable systems*, IEEE Trans. Automat. Control, 34 (1989), pp. 848–855.