Chapter 3 - Chaos and the Complexity of the World

1) Background on Chaos

Chaos theory is a prime example of that rarest of scientific events -- the emergence of something new in a *new sense*. Here we have a breakthrough idea that changes no existing scientific laws, and introduces no new fundamental hypotheses or theories about the world. And yet it is an entirely new approach to physical systems, and to our understanding of them. It permits completely novel interpretations of old data; it expresses new and deeper meaning to terms such as 'random', 'pattern', and 'predictable'; and it clears the way for new articulations about the nature of the human and the cosmos. So profound and far-reaching is its influence that it has been ranked with relativity and quantum mechanics as the most important and lasting scientific advances of the 20th century.

And like relativity and quantum mechanics, it has implications that reach beyond the bounds of the physical sciences. Chaos theory has direct bearing on a number of fields of inquiry, including areas as diverse as economics, social theory, and music theory. Chaotic effects are found in innumerable everyday events, such as the dripping of a faucet, the waving of a flag in the breeze, the patterns of weather, and the activity within the human brain. Importantly, it also addresses issues of philosophical significance, such as determinism and free will, ontological theories of matter and structure, and the nature of evolution in the general sense. Most importantly, it proves very useful in furthering our knowledge and descriptions of the mind, the brain, and their interrelationship.

Chaos theory is part of the larger area of study known as 'dynamical systems theory', which seeks to understand how well-defined systems of mass and energy change over time. It is also a core element of the emerging discipline called 'complexity theory', which incorporates a host of new techniques for studying complicated or unpredictable systems. Chaos theory is rooted in mathematics, so the following discussion will necessarily be of a more mathematical nature. But I hope to show in a non-technical manner something of its deeper philosophical significance.

Finally, let me note a very important point regarding my application of chaos theory to systems of mind. Formally speaking, chaos theory applies only to mathematical models of closed systems. However, the general concepts and principles can be meaningfully applied to real-life situations – including systems that are open to energy exchanges with their surroundings. Here I will be applying the *concepts* rather than a formal analysis.

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The term 'chaos' as used in my context clearly has a specialized meaning, only loosely related to the common sense meaning of 'utter confusion or disorder wholly without organization'. This folk meaning comes from the ancient sense of chaos; in Greek times the word was *khaos,* meaning a void or chasm. The poet Hesiod, writing some 200 years before Plato, described *khaos* as the primordial void that gave birth to the universe: "First of all came the Chasm (*khaos*), and then wide-bosomed Earth …"[1]. In the primordial void, all is unknown; all is random, frenetic disorder. But the new sense of chaos is not a concept of emptiness or unknowing. In fact, chaos today refers specifically to a *unique kind of order*: an order within apparent disorder.

In the simplest terms, chaos is the existence of complicated and unpredictable behavior arising from non-random processes, even very simple ones. It had long been assumed that the only way to model complex physical systems was to use complex equations. Simple equations gave simple, predictable results, and complex equations gave complex results. Complex phenomena of the real world required, therefore, complex descriptions and complex models.

Chaos theory suggests a reversal of this conception. Simple equations can yield very complicated results, and apparently complicated phenomena may be describable, in their essence, with relatively simple models and equations. In other words, *chaos is a method of simplification that allows greater comprehension*.

I begin by looking at the basic mathematics of chaos theory. This semi-formal introduction will serve to explain the concepts I am using. Again, they apply, strictly speaking, only to models of closed systems. But the concepts are useful in examining a wide range of real physical systems.

A mathematical model is describable in terms of mathematical equations. Typically these equations would be either *differential* (for a continuously dynamic model) or *discrete* (for a model that move step-wise through time). Here I consider primarily discrete equations, as these more simply explain the basic concepts involved.

The textbook explanation of chaos theory states that a model requires three characteristics to allow the conditions for chaos: the model must be (1) deterministic, (2) nonlinear, and (3) exhibiting feedback. I will start with a brief examination of each of these.

A *deterministic system* is one that changes with time according to a set of equations (based in laws of science) that have no element of inherent randomness or chance. The equations involve 'ordinary' variables and operations, with no random numbers or other elements. For example, the laws of physics – including, significantly, the wave equation describing quantum particles[2] – are described with deterministic equations. Newton's laws of motion are all deterministic, as are the more accurate relativistic versions of them. Equations like

$$d = rt \quad \text{[distance = rate x time], or}$$
$$F = ma \quad \text{[force = mass x acceleration]}$$

all involve discrete, well-defined quantities, the consequences of which follow by simple calculation.

The opposite of a deterministic system is a *stochastic* system, one that appeals to random events or variables, like the outcome of a coin toss. It is easy to get random-like results when one inserts random variables into a system; deterministic systems, lacking these, were always thought to produce totally predictable outcomes. This is not to imply that stochastic systems cannot display chaos-like behavior; rather, we simply are lacking any well-formulated conception of what 'stochastic chaos' would be.

It is interesting to note that many complex physical processes appear to have elements of randomness. When scientists have sought to describe such systems they often used stochastic models to account for the randomness. But there is an element of fallacy here, and it is based on the need for pragmatism and expedience. Since the laws of physics are deterministic, there is theoretically no need to introduce randomness, other than as a matter of convenience or abbreviation.

Traditionally, deterministic systems are seen to exhibit certain specific properties. One of these is *predictability*. Knowing the current state of any system, and the laws of the forces acting on that system, one was thought to be able, in principle, to calculate the future evolution of the system to an arbitrarily fine degree of accuracy. Laplace is famous for his description of a super-intellect, perhaps God, who, if knowing the current state of all particles in the universe, would therefore know the entire history of the universe, past and present. A related property of deterministic systems is the uniqueness of past and future: at any given time, there occurred only one possible past, and there will be only one possible future. Chaos theory seriously undermines this naïve notion of predictability and replaces it with a more complex and subtle conception: one of predictability in a broad, qualitative sense.

The second characteristic is *nonlinearity*. This refers specifically to the equations that govern a physical system. An equation is nonlinear if any one of the variables in it has an exponent that is not equal to one; a simple example would be $T = 2x^2 + 4y^3 - 7z^{0.3}$. Nonlinear equations may also include the 'transcendental' functions, like sin(x), or e^x. The two equations I noted earlier are linear with respect to each variable on the right-hand side -- there is no power of 2 or 3, a square root, a ‘1/x’, or a sin(x) term.

The formula for the area of a circle, $A=\pi r^2$, is nonlinear with respect to the radius. Some of Newton’s laws of motion are nonlinear: $d=1/2\ a\ t^2$, so the distance traveled by an accelerating object is nonlinear with respect to time. The force due to gravity is nonlinear with respect to the distance between the objects: for example, if a ball of mass ‘m’ is a distance R from the center of a planet of mass ‘M’, then the force on the ball (and, symmetrically, the force on the planet!) is given by

$$F = GmM/R^2$$

where 'G' is the "universal gravitational constant". We say that the gravitational force is 'inverse-square' with respect to distance. The force of gravity, as we know, is one of the four so-called fundamental forces of the universe[3]; the other three are the electromagnetic, the strong nuclear, and the weak nuclear. Importantly, *all four of the fundamental forces, and therefore all forces in the universe, are nonlinear with respect to distance*. The electromagnetic force has the same form as gravity. The strong nuclear has both 'inverse square' and 'inverse' terms with respect to distance. The weak nuclear is not concisely formulated, but it is known to be strongly nonlinear. This nonlinearity of all force is a necessary precondition for the occurrence of chaos in natural systems.

The third essential characteristic is *feedback*. Feedback is a feature of a system in which the forces or effects of a given object pass through some chain of events and circle back to affect the object itself. A common example of feedback occurs in the audio system of an auditorium: some small noise in an amplifier is played through loudspeakers, picked up by a microphone, amplified a little louder, picked up again, and so on, until only a high-pitch squeal can be heard. Another example is the echo-location system of bats: their brain determines that they need to identify things ahead of them, so it signals the voice box to emit a high frequency sound, which bounces off an object in front, feeds back to their ears/brain, and then the bat takes the appropriate course of action. All living systems respond to their environment in a real-time manner that requires feedback.

Feedback is typically described as either 'negative' or 'positive'. Negative feedback leads to a diminution of the 'error signal', and thus is useful for control systems; positive feedback results in an amplification of error, leading to often rapid growth of the signal strength (as with the auditorium example). This idea of feedback being either positive or negative is, however, more relevant for artificial systems than natural ones. More generally, it is simply a continual input of information (energy) from the body or the environment, based on the changing conditions.

In a system of equations, feedback refers to the output of an equation affecting the input of one or more equations. This is typically done by the process called ‘iteration’, in which the output of an equation is then used as the input to that same equation for the next calculation. The process is repeated a number of times, always feeding the output back as input. Iteration is the mathematical analog of feedback. The following example will clarify.

Chaos is a nearly universal, or potentially universal, property of nonlinear systems and their underlying equations. As such, it can appear in even very simple cases. The most well-known example is the ‘logistic equation’, which is instructive because it demonstrates the three essential aspects of chaos. Consider the equation $f(x) = c\ (x - x^2)$, where ‘c’ is a constant. The equation is deterministic, as there are no strange random variables in it. It is nonlinear, due to the x^2 term. We induce feedback via iteration, as follows. Take some initial value of x between 0 and 1, call it x_o. Calculate the ‘output’ value $f(x_o)$. Use this as the new value of x, call it x_1. Calculate the next output $f(x_1)$. Repeat the process indefinitely. You will have calculated a sequence of numbers: x_o, x_1, x_2, x_3, ...

Depending on the value of ‘c’ that you use, you will get vastly different results. In general, three different sorts of behavior can arise. One, the sequence will start at, say, 0.1, bounce around a bit, and then settle down quickly to a single value – the ‘solution’ – from which it will not move. Interestingly, no matter where you start with your x_o, the sequence will converge to the same solution. Thus, the solution point acts as an ‘attractor’ to all initial values of x. Since the solution point is a fixed (i.e. single) point, this case is called a ‘fixed point attractor’.

In the logistic equation, you will get a fixed point attractor for any ‘c’ such that $0 < c < 3$. Different values of ‘c’ will yield different solution points. As a specific example, if we take $c = 2$ and $x_o = 0.1$, the values will progress as follows:

$x_o = 0.1$

$x_1 = 0.18$

$x_2 = 0.295$

$x_3 = 0.416$

$x_4 = 0.486$

$x_5 = 0.499$

$x_6 = 0.5$

$x_7 = 0.5$ ...

at which point the sequence is fixed at 0.5. This sequence is easily confirmed on any spreadsheet or pocket calculator. A somewhat more interesting case occurs when c = 2.8. Here, the fixed point attractor has a value x = 0.64285... The graph shown in Fig. 1 shows how the points of the sequence are drawn to the fixed point solution.

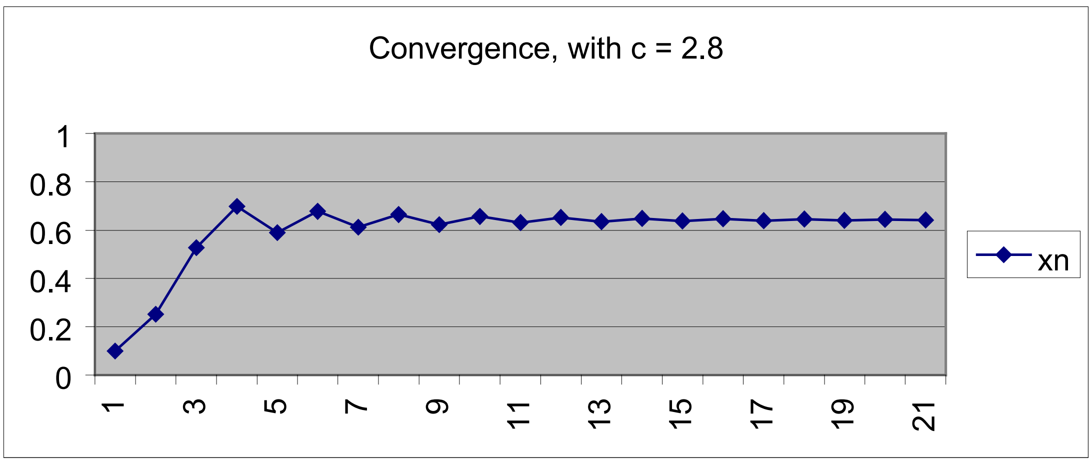

Figure 1 – Convergence of the logistic equation (c = 2.8)

In the second type of general behavior, called a 'limit cycle', the sequence settles in not on a single value, but instead oscillates between two or more values. In our example, this happens when 'c' crosses the threshold value of 3. Figure 2 shows this behavior for c = 3.2. Thus, the pair of points (0.799, 0.513) acts as the attractor.

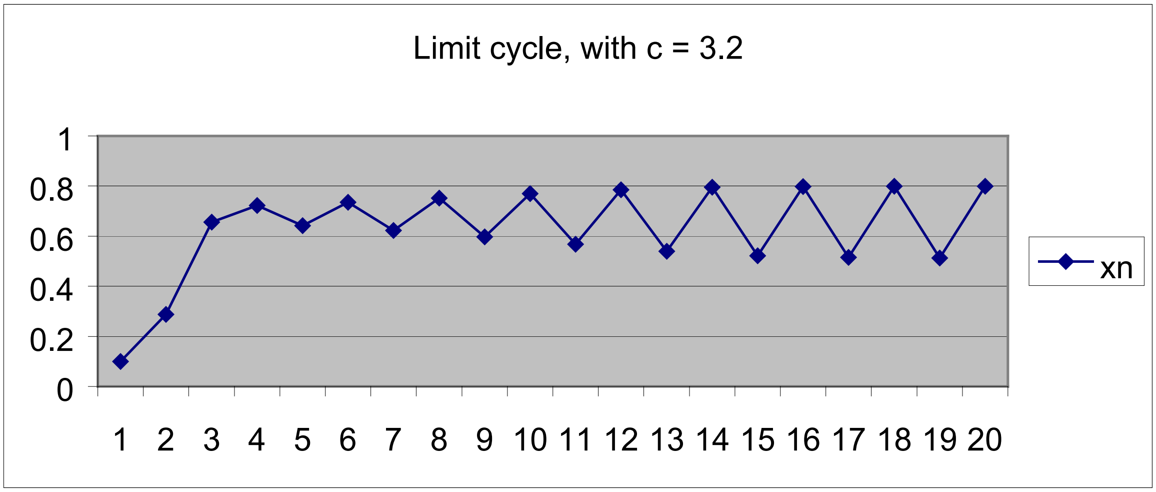

Figure 2 – Period 2 limit cycle (c = 3.2)

The limit cycle of two points only holds for a small range of 'c'. If we increase it further, we find that as we cross the threshold c = 3.4495 we get a 4-point attractor. Taking c = 3.5, for example, gives the results shown in Fig. 3. As we further increase 'c', we get successive thresholds, closer and closer together, each of which cause a *doubling* of the number of attractor points.

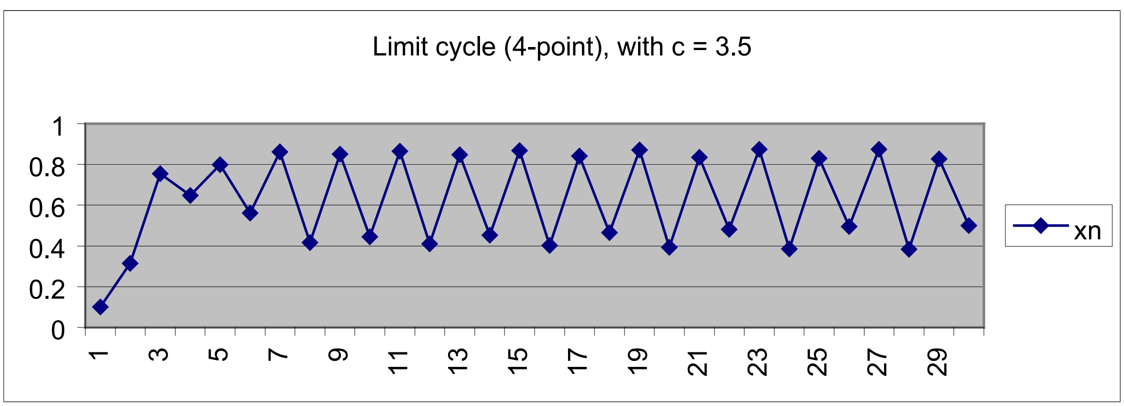

Figure 3 – Period 4 limit cycle (c = 3.5)

Then, we reach a critical value of 'c' (here, c_{crit} = 3.5699...), in which we get, literally, an infinite number of attractor points[4]. This is the third type of behavior -- chaos. Of course, nothing in the equations stops us from using a c-value equal or greater than this critical number. What happens is that the sequence values spend 'forever' bouncing from point to point on this infinitely large attractor, never (in theory) repeating themselves. This infinitely large attractor, being rather strange, is appropriately known as a "strange attractor"[5].

Figure 4 shows the sequence for a value of c = 3.58, just into the chaotic region; I will call this 'shallow chaos'. Every value is unique, and the long-term behavior is technically unpredictable -- if one wanted to know the value (to the same order of precision) at step #1000, there is no procedure for determining it other than to literally calculate every value up to 1000.

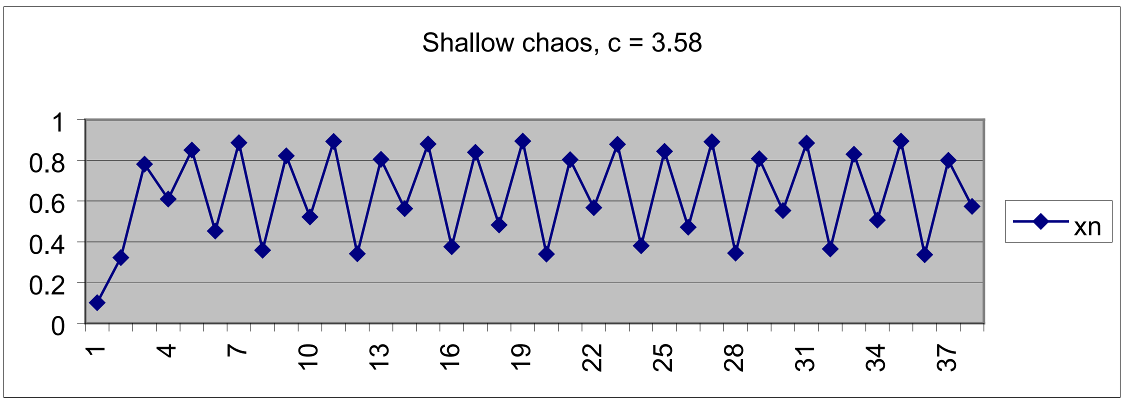

Figure 4 – 'Shallow' chaos (c = 3.58)

And yet, we see that there is some pattern to this chaos. We see that the points are scattered, yet bounded: they tend to fall between 0.3 and 0.6, and between 0.8 and 0.9. Also, successive points will alternate between the lower band and the upper band, i.e. all odd steps are in the lower and all even steps are in the upper. So, we can say *something* about, for example, step #1000: it will lie in the upper band, and have a value between 0.8 and 0.9. Is this a prediction? Not in the traditional scientific sense, which assumes that any quantity can be calculated to an arbitrarily high degree of precision. But clearly it has some useful value. *It is more of a qualitative prediction than a quantitative prediction.* This is a trademark of chaos. And it indicates something of the flavor in which chaos can be used to explore what Goodwin calls the "science of qualities" – see Goodwin (1994, 1999a, 1999b) or Reason and Goodwin (1999).

We can press further into the chaotic region. Take c = 3.92. Call this 'deep chaos'. Here is true disorder -- see Fig. 5. The values are well-distributed between 0 and 1. No pattern or 'banding' appears. Consecutive points may be very close or very distant. We can make virtually no prediction about the value at step #1000. This behavior tells us more about the structure of the equation, i.e. the value of 'c', than anything else.

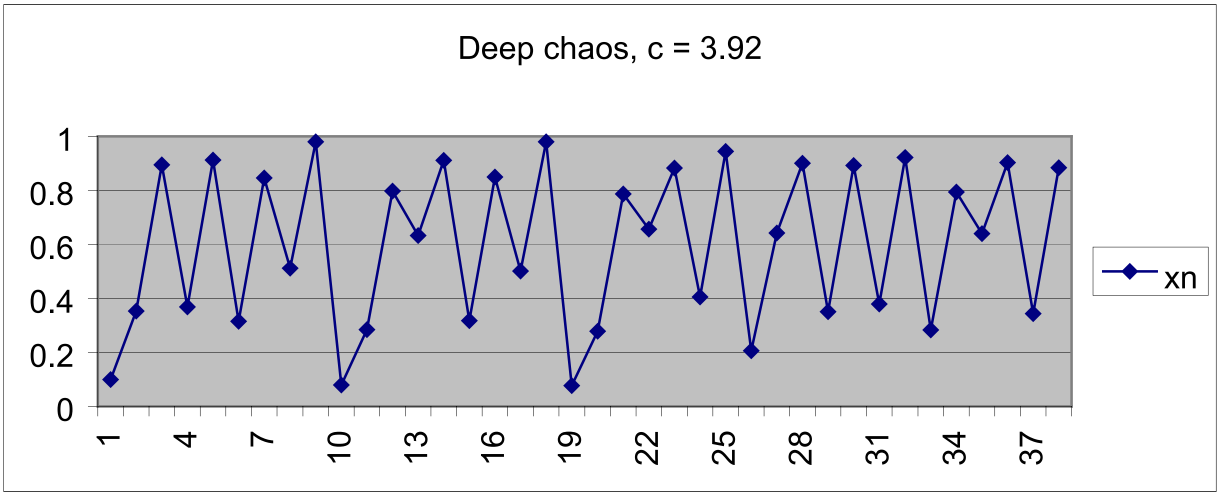

Figure 5 – 'Deep' chaos (c = 3.92)

There is one important bit of order that emerges from this deep chaos. It is usually possible to recreate the 'original conditions' of the system from the data themselves. Consider if the points in the sequence were not calculated, but were obtained by some empirical measurement -- say, percentage humidity on consecutive days, or the time interval between drips of a faucet. A group of researchers at UC-Santa Cruz[6] discovered a clever way to depict the underlying pattern, the essence of the strange attractor. They simply plotted consecutive points on an x-y graph. So we can plot the sequence of points (x_o, x_1), (x_1, x_2), (x_2, x_3), ... We see the results in Fig. 6 – an inverted parabola. In this case, of course, the plot was predictable, since the logistic equation itself is just this parabola. But even when lacking such *a priori* information, we do have techniques for uncovering patterns in chaos.

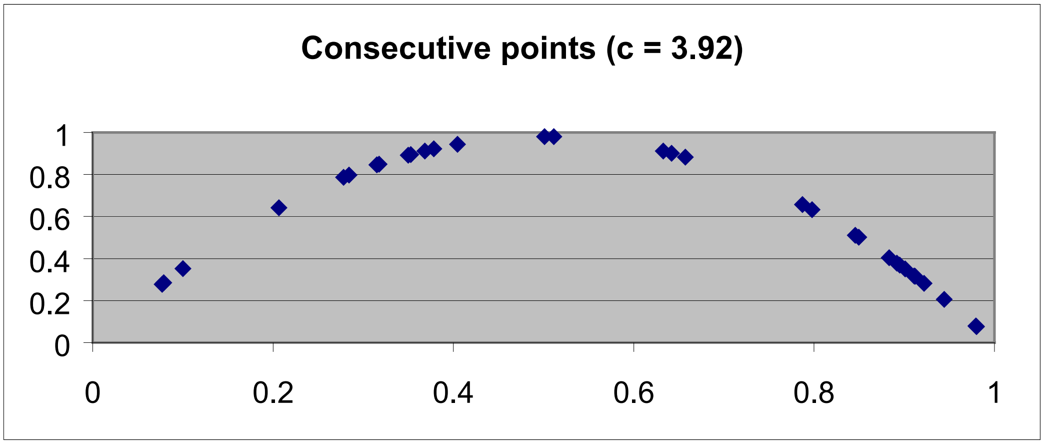

Figure 6 – Characterizing a chaotic system, based on empirical data

2) History of Chaos Theory

Mathematics aside, I want to take a moment to look at how chaos theory evolved. Traditionally chaos theory is considered to originate in the work of the French physicist, mathematician, and philosopher, Henri Poincare (1854-1912). However, some of the key ideas were anticipated earlier by the American mathematician and philosopher Charles Sanders Peirce (1839-1914). Peirce was aware of the importance of nonlinearity in natural systems, and that this fact may have significant consequences. In an important 1892 article, "Man's Glassy Essence", Peirce examines the nature of life and the phenomenon of sensitivity or awareness in living systems. He observes a fundamental tenet of chaos theory: that almost imperceptibly small influences can have large effects in a suitably dynamic system, like a living cell:

> If, then, we suppose that matter never does obey its ideal laws with absolute precision, but that there are almost insensible fortuitous departures from regularity, these will produce, in general, equally minute effects. But protoplasm is in an excessively unstable condition; and it is the characteristic of unstable equilibrium, that near that point excessively minute changes may produce startling large effects. (1892: 18)

This insightful passage anticipates not only the central point of chaos theory, but also the notion of a living system as an 'excitable medium' (see Goodwin, 1994), and as situated in an 'edge-of-chaos' condition (see Kaufmann, 1995). And it predates the more well-known passage by Poincare, cited below, by 16 years.

Poincare had strong intuitive feelings about the nature of the physical world, and these led him to anticipate a number of developments that would occur later in the 20th century. His philosophy of ‘conventionalism’ presaged concepts in relativity theory. He had a feel for the interconnectedness of the cosmos; in 1902 he wrote, “[N]o system exists which is abstracted from all external action; every part of the universe is subject, more or less, to the action of the other parts.” (1902: 103). He made extensive use of dynamical systems theory, which, as we will see, is crucial for understanding chaos.

Importantly, Poincare’s work on resolving the ‘3-body problem’ led him to insights on the nature of nonlinear systems. This problem refers to predicting the movement through space of three mutually-gravitating objects, like a planet with two moons. Newton’s equations of gravitational force are easily solved for two bodies, and there was no inherent reason to expect issues with three bodies. As it turns out, the problem is very difficult, and in most cases has no analytic solution. If one were to attempt an actual physical experiment, one must take measurements. Any physical measurement inevitably has a small amount of error or uncertainty. The reigning sensibilities claimed that ‘small errors in measurement yield small errors in prediction’. Poincare understood that in a dynamic, nonlinear system, this was simply not true. Even assuming perfect knowledge of physical laws,

> we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation *with the same approximation*, that is all we require, and we should say that the phenomenon had been predicted... But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible... (1908: 67-8)

Here we have the second early and succinct description of the phenomenon that we call chaos. I will demonstrate this susceptibility to error shortly.

In spite of these two remarkable insights and hints of complexity residing in simple dynamical systems, it took more than 50 years before the scientific community grasped the importance. In the early 1960's, MIT researcher Edward Lorenz recorded the first known instance of chaos. He was using a simple computer model to study conditions related to the Earth's climate. The model addressed the interrelationship of three nonlinear factors in convective flow (as in a liquid) under conditions of a temperature gradient. Lorenz's model was a system of three nonlinear equations that he found to be extraordinarily sensitive to 'error' in the initial conditions. He ran the model with one set of input values, got a prediction, and then compared it to the prediction arising from a very small change in one of the input values. Like most scientists, Lorenz expected to see nearly identical predictions. Instead he found exponentially increasing variation between the two. After ruling out problems with the computer or the software, Lorenz decided that this type of response was inherent in his model. He published his conclusions in a now-famous paper, "Deterministic, nonperiodic flow" (1963).

At about the same time, French researcher Michel Henon was working on a sophisticated variation of the three-body problem, namely, the movement of stars within globular clusters. Like Lorenz, Henon sought to model the movement of stars on a computer, using a set of nonlinear equations. The model was set up such that the average energy level of the stars was controllable, to account for the difference between dispersed, slow-moving star groups and compact, fast-moving ones. At low energy, the model predicted regular, stable, periodic orbits, as was expected. As the energy level was increased, though, the picture became more complicated. At first, areas of stability mixed with areas of apparent randomness. Then, greater and greater randomness appeared (see Henon and Heiles, 1964, for detailed analysis). Chaos was emerging, dependent upon the level of energy.

The behavior of Henon's system is strongly reminiscent of the logistic equation example, in which chaos emerged as we increased the value of 'c'. In fact, we can interpret 'c' as the *energy level of the logistic equation*, or the 'proliferative drive' of the system.

Biologists, when using the logistic equation to model population fluctuations, refer to 'c' as the "intrinsic growth rate" or "net rate of increase" of the given species (see May, 1974, or Rayner, 1997). Such examples as these argue for a deep connection between chaos and energy.

The next decade began a series of milestones in chaos theory, occurring almost annually. In 1971, two mathematicians, David Ruelle of Belgium and Floris Takens of the Netherlands published a paper, "On the Nature of Turbulence" (Ruelle and Takens, 1971), in which they coined the term "strange attractor". In 1974, Princeton biologist Robert May published the first analysis of chaos in the logistic equation, and introduced the term 'chaos' (attributed to mathematician James Yorke) for the first time; his paper was "Biological Populations with Non-overlapping Generations: Stable Points, Stable Cycles, and Chaos" (May, 1974). Yorke published his own paper on chaos shortly thereafter, "Period Three Implies Chaos" (Yorke and Li, 1975). Henon developed his set of equations further, and by 1976 came up with a concise 2-equation system that produced a well-known strange attractor, referred to now as the 'Henon attractor'.

By this time, chaos theory was beginning to attract attention in the wider scientific community. In 1977 the first international conference dedicated to chaos theory was held in Como, Italy. Also in that year, Benoit Mandelbrot published his first book on the study of 'fractals', titled <u>Fractals: Form, Chance and Dimension</u> (1977). Fractal patterns are very complex computer-generated pictures arising from the same type of simple equations that produce chaos; most notable of these patterns is the famous 'Mandlebrot set'. An important advance came in 1978 when Mitchell Fiegenbaum proved that the 'period doubling' pattern found in the logistic equation is not unique to it, and is in fact found in *every* nonlinear system (Fiegenbaum, 1978); this had the effect of showing that chaos was *universal*, and not limited to some small class of equations.

Chaos theory has been further refined in the past 20 years, but without significant revision to the basic theory. The most important developments have been in the applications of chaos theory to various areas of science, psychology, and philosophy. People like Sally Goerner put forth ideas linking chaos to evolution (see her 1994). Chaos was sought in the movements of the planets (Parker, 1996; Frank, 1998) – with results that I discuss shortly. The linkage between chaos and mind was examined by some (see for example Kaufmann (1993), Goertzel (1993), Combs (1995), Kelso (1995), or Kelso and Fuchs (1995)), and this anticipated some of my ideas, as I will explain. And such philosophers as Peter Smith (1998) spelled out in great detail some of the philosophical significance of chaos theory, including what can and cannot be claimed of it. All these works have expanded and deepened our understanding of the importance of chaos theory.

3) The Concept of Phase Space

We have seen, in some detail, the example of the logistic equation and how it can yield chaotic results under the proper conditions. I now want to examine the method in which we describe real-world dynamical systems. This will introduce the concept of '*phase space'*, and it is critical to understanding the nature of chaos.

Every physical system that changes in time can be described mathematically in terms of its 'state variables'. These variables are the quantities that capture the essence of the system. They describe how it changes with time. Most importantly, the state variables embody the *energy dynamics* of the system.

As before, an example offers the best explanation. The standard example is a simple, free-swinging pendulum. Let me note first of all that this example is intended only to describe 'phase space diagrams', not chaos; later examples will consider chaos in phase space. Assume first of all an ideal pendulum, with a bob of mass 'm', arm length of 'L', in a frictionless environment. To describe it mathematically we need to overlay the pendulum on an x-y graph -- see Fig. 7a. The fulcrum is attached at y = L, and the bob swings from a point of maximum displacement, down to the origin (0, 0), over to its opposite peak, and back again.

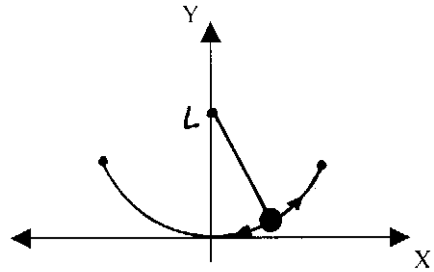

FIGURE 7a – Simple pendulum

We now start the pendulum swinging. To describe its motion mathematically we have at least three options: One, we can note that it swings with a period T (the time required for one round-trip swing), given by the standard formula $T^2 = 4\pi^2 L/g$ (where g = 32 feet/sec^2, the gravitational acceleration of the Earth); this applies only under conditions of 'linearity', i.e. ordinary back-and-forth motion. Two, we can find two equations to describe the x and y coordinates of the bob, in the form $x = f_1(t)$, $y = f_2(t)$. Three, we can create a 'phase space diagram', which is the method of interest here.

The phase space pattern is drawn in a Cartesian grid system, similar to, but different from, the grid on which we sketched the actual pendulum. Since the bob moves in a 2-dimensional plane and is constrained by the lever arm, we need only two state variables to fully describe its motion: its displacement (angle) from the y-axis, 'α', and its velocity (i.e. 'speed') 'v', defined as tangent to the circle in a clockwise direction[7]. Therefore, the phase space diagram can be drawn in 2-D, like a normal x-y coordinate system. The horizontal axis 'α' indicates the displacement, and the vertical axis 'V' indicates the velocity.

As the pendulum swings back and forth, both 'α' and 'v' are continuously changing. At each moment in time we can identify a particular value of position and velocity. These two values are then plotted as a single point on the phase space picture. For a simple frictionless pendulum, this plot will form a circle -- see Fig. 7b. Start the bob at the right-most position, on Fig. 7a. This point has $\alpha = \alpha_m$, and v = 0. Plot this as point P1 on the phase space graph, Fig. 7b. As the bob swings down to the origin, 'α' decreases smoothly while 'v' increases smoothly, until it reaches the point of zero angular displacement ($\alpha = 0$) and maximum velocity ($v = v_m$). In phase space, this appears as a smooth arc from point P1 up to point P2. As the bob moves on to the left end of its path, the phase space plot curves down to the point P3. On the return trip, velocity becomes negative (counter-clockwise), the phase plot passes through point P4, and then ends up back at the starting point.

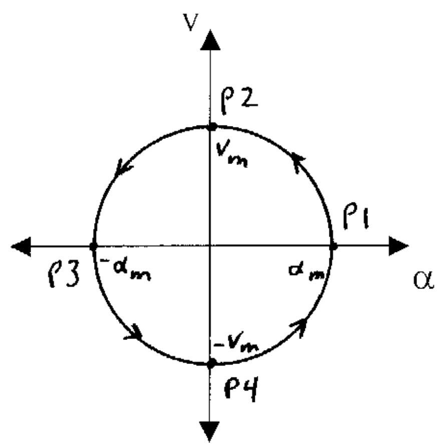

FIGURE 7b – The ideal pendulum in phase space

The picture is really quite simple and elegant. As the bob swings back and forth, there exists a corresponding point in phase space that travels around the circle shown in Fig. 7b. No matter what the 'state' of the bob – momentarily stopped at the right, moving with some velocity 'v', momentarily stopped at the left – we can capture the essence of the pendulum by *a single point* in phase space. This point describes the instantaneous energy of the pendulum. Insofar as the physics of the pendulum is concerned, the phase space point essentially *is* the pendulum.

This issue must be emphasized. There are many other characteristics of a real pendulum that would serve to create a true 'total description': The material and size of the bob, the color of the arm, the sound it might make, and so on. *None of these things are essential to its operation as a pendulum*. As far as how fast it swings, there is only one essential feature, namely, the length of the arm (as per the formula given earlier). As far as its description as a dynamic system, there is only one essential feature, namely, the energy state of the bob -- and this requires us to track two variables, position and velocity. All other features, while being important to a total physical description, are unimportant to its dynamics. It functions as *one system*, and this is captured in the *one point* moving in phase space. And the *trajectory*, or pattern of movement, of this point tells us much about the dynamic nature of the system.

Now consider a real pendulum, swinging free as before, but this time with friction. Each sweep of the bob loses a small amount of energy, and the displacement 'x' becomes a little smaller with each pass. What happens in phase space? Our point continuously 'loses energy' also, moving not in a circle but in an inward spiral. The phase point spirals in to the origin (zero 'x' and zero velocity) as the bob grinds to a halt -- see Fig. 8:

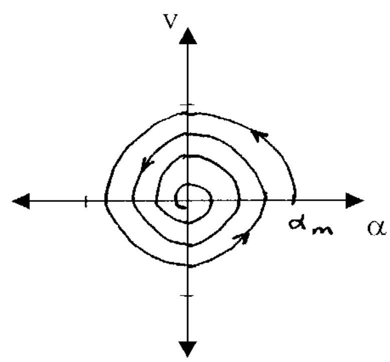

FIGURE 8 – Frictional pendulum in phase space

As with the ideal pendulum, there is no chaos here either[8]. What we do have is a clear example of how to use phase space descriptions for dynamic systems. The spiral shows us, at one glance, the whole history of the pendulum. At any point in time, we can locate its current state, know where it was, and know where it's going. It is a highly deterministic, highly predictable system.

Consider a third version of the pendulum example, in which we are able to supply a small 'kick' of energy once per cycle, perhaps with a small electromagnet. Starting at some given angle, the pendulum now swings a little *higher* with each pass, rising first to a horizontal level (Fig. 9a), then approaching a vertical position on each side (Fig. 9a):

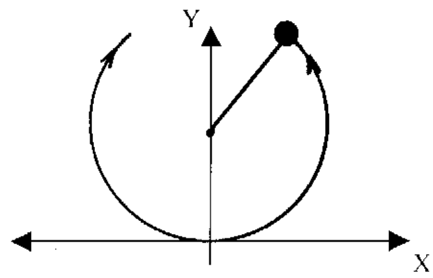

FIGURE 9a – Pendulum driven progressively higher

In phase space, the point spirals outward, until displacement approaches a maximum of $+180^o$ and -180^o (Fig. 9b):

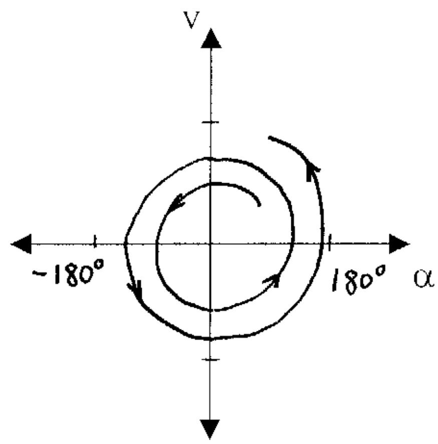

FIGURE 9b – Progression in phase space

Now add just a bit more energy, and the pendulum goes over the top, such that it now travels only in the (say) clockwise direction. There is no more 'return trip'. Consequently, there is a sudden change to the phase space pattern. The bottom lobe vanishes, since velocity is now only 'positive' (clockwise) – see Figure 10. The phase space point now travels up the semi-circle, hits a peak velocity (with $\alpha = 0$, i.e. at the bottom of its swing), travels down the other side, and then instantly jumps back across[9] to the point at $\alpha = +180^o$.

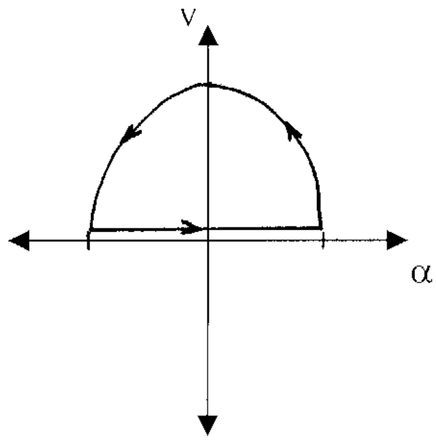

FIGURE 10 – Phase space 'over the top'

With further energy, the upper lobe ‘rises’ above the origin, because now velocity never equals zero. As we drive it even harder, the velocity difference between ‘bob-vertically-up’ and ‘bob-vertically-down’ becomes proportionately smaller, so the semi-circle begins to flatten out as it rises. Eventually the lobe gets completely compressed and approaches a flat line segment (Figures 11a-c):

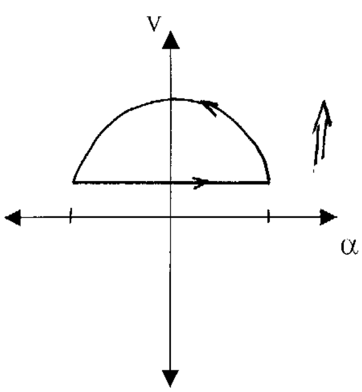

FIGURE 11a

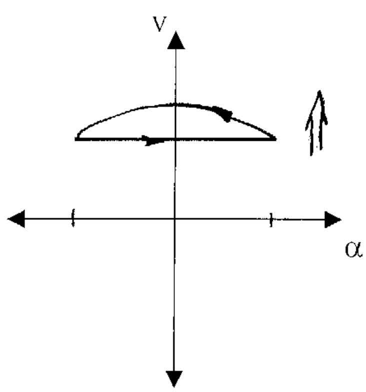

FIGURE 11b

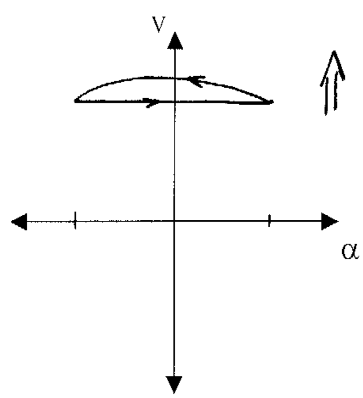

FIGURE 11c

This sequence of diagrams clearly shows the ‘energy evolution’ of the pendulum. There is a one-to-one correspondence between the ‘actual pendulum’ sequence and the phase space sequence.

The pendulum was an example of a dynamic system in which a physical object was moving in space. Phase space analysis, though, applies to any situation where mass or energy are changing with time. Consider a second example of a camera flashbulb firing. The situation can be described, in simplified form, as follows. We have a circuit consisting of a voltage source (a capacitor), a resistor (R1) representing the resistance of the wiring, a switch, and light bulb (modeled as another resistor, Rb) -- see Fig. 12a. For sake of simplicity, assume R1 = Rb. Electrical energy is stored in the capacitor. At the desired time, the switch is closed, allowing electricity to flow through the bulb. The current rushes in very fast, causing the bulb to emit light; the voltage quickly reaches a peak value (V_{max}), and then decays somewhat more slowly down to zero.

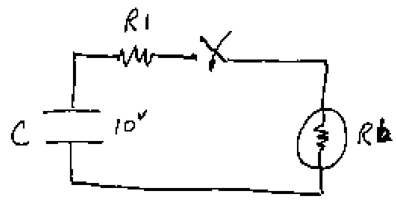

FIGURE 12a

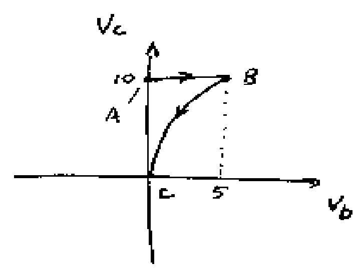

FIGURE 12b

Our simplified circuit has three basic elements: C, R1, and Rb. We can describe the dynamics of the circuit by looking at the voltage over any two of these, since the third voltage is fixed by the other two[10]. So take capacitor voltage Vc and bulb voltage Vb as the two state variables.

Voltage Vc starts 'charged' at, say, 10 volts. When the switch is closed, current flows out of the capacitor and through the two resistances until the voltage drains down to zero. Voltage Vb, initially at zero, jumps quickly up to 5 volts, emits a bright light, and then decays also to zero. If we plot the phase space, using Vb and Vc as our two axes, we get Fig. 12b: starting (before switch closure) at the point A(0, 10v), moving quickly across to point B(5v, 10v), then exponentially decaying down to the origin point C(0, 0).

This is not often done in analyzing electrical circuits, because it is typically more useful to know the individual voltages and currents than to depict a phase space point moving in a multi-dimensional space. I would suggest that this is an underutilized method of analysis, and perhaps new insights into electronic systems could be gained by studying phase space descriptions.

4) Phase Space in More Complex Systems

So where does chaos enter phase space? Potentially, whenever the three key conditions -- nonlinearity, feedback, and deterministic processes -- occur. Typically this occurs with two (for a map) or three (for differential) equations that are interlinked. The classic example of chaos is the very first one, discovered by Edward Lorenz in the early 1960's. The Lorenz system consisted of three differential equations, with three variables (x, y, z) each varying with a parameter that may be considered 'time'.

The phase space point thus traces out a trajectory in a 3-dimensional phase space, and can be plotted with the help of a computer. The plot is not a simple pattern as with the pendulum, but a complex, multi-path 'figure-8' pattern -- see Figure 13. Another example is the 'Duffing attractor', shown in Figure 14. These systems are chaotic because one cannot predict how the trajectory will progress, short of simply performing the calculation.

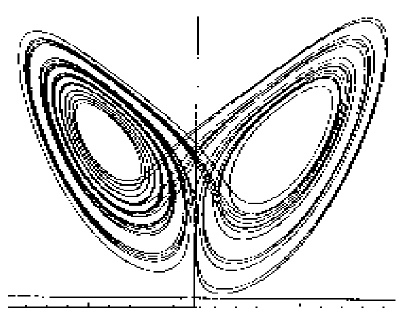

FIGURE 13 – Lorenz attractor

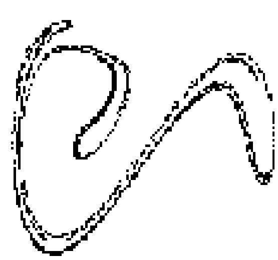

FIGURE 14 – Duffing attractor

The pattern that forms is a visual representation of the strange attractor of the system. It depicts the system moving through an infinite number of states, never exactly repeating itself, yet staying within a defined region of phase space. As in the earlier logistic equation example, the system is drawn into the attractor region. One may start the system (or set of equations) in any particular configuration, and this corresponds to a particular point in phase space. If this initial point is outside the attractor region, the system will evolve such that its system states, and corresponding phase space points, are drawn toward the strange attractor[11]. The system inherently 'prefers' only a certain subset of the possible states and will not enter other states unless driven there by external forces. If an outside force does disturb the system, it will tend to return to the attractor region.

Chaotic systems display three important characteristics: One, they are unpredictable *in detail*. Two, they show *large-scale stability and 'predictability'* by staying within the strange attractor. Three, they are *very sensitive to small change*. I have discussed the first two, and I now want to say a few words about the third.

One typically hears of chaotic systems having "sensitive dependence on initial conditions". Assume, for example, that we start the Lorenz equations at some point A(1, 2, 3) and then watch the movement through phase space. Call this "Run A". If we now calculate a "Run B" starting from the point B(1, 2, 3.0001), one would have typically assumed that the progression of the system would be virtually identical with that of Run A. A small change in initial conditions would not be expected to make a big difference, and in fact, this is true for many *linear* systems (including linear approximations of nonlinear systems), as well as all compact nonlinear systems with only fixed-point or limit-cycle attractors.

But this is decidedly not the case for a chaotic system. Due to the nonlinear feedback, a small difference gets multiplied by a factor greater than '1'; this difference increases by being multiplied by itself with each time increment. Thus, for a difference of .0001 in the logistic equation, with a power of 2, the 'error' of 0.01% becomes fully 10% after only 10 iterations[12].

Again, it may help to look at a numeric example. Take the logistic equation sequence, with 'c' = 3.58 (shallow chaos), and $x_o = 0.7$. Call this the baseline sequence. Then compare this to four other sequences, each with increasing variation in initial conditions: sequence A ($x_o = 0.701$), sequence B ($x_o = 0.7001$), and sequence C ($x_o = 0.70001$). Though they all start relatively close together, each 'error' sequence gradually diverges from the baseline, until eventually there is no correlation at all to the baseline. If we sum up the magnitude of the error at each iteration, we get a nice picture of the accelerating divergence -- see Figure 15. Note that the sequence with the biggest initial error diverges first (A), then the next biggest error (B), and then the last (C). If we were to put the system into deep chaos, say c = 3.7, we get even faster divergence of all three series – see Figure 16. The actual values of the three series are shown in Figure 17. Note that after about 25 iterations all sequences have diverged from the baseline and from each other, and appear as independent, random sequences.

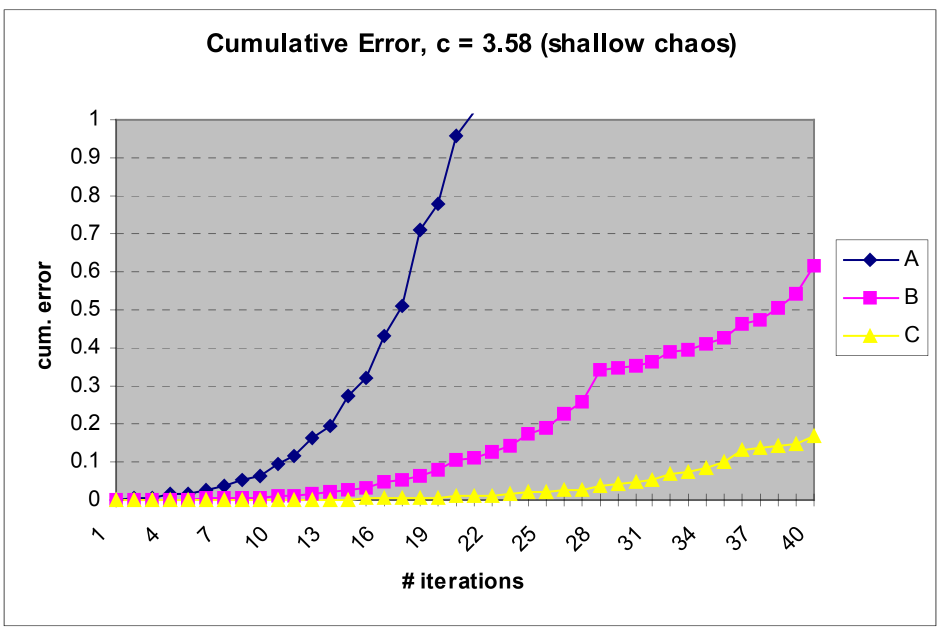

Figure 15 – <u>Divergence rates based on 'initial difference'</u> (A>B>C).

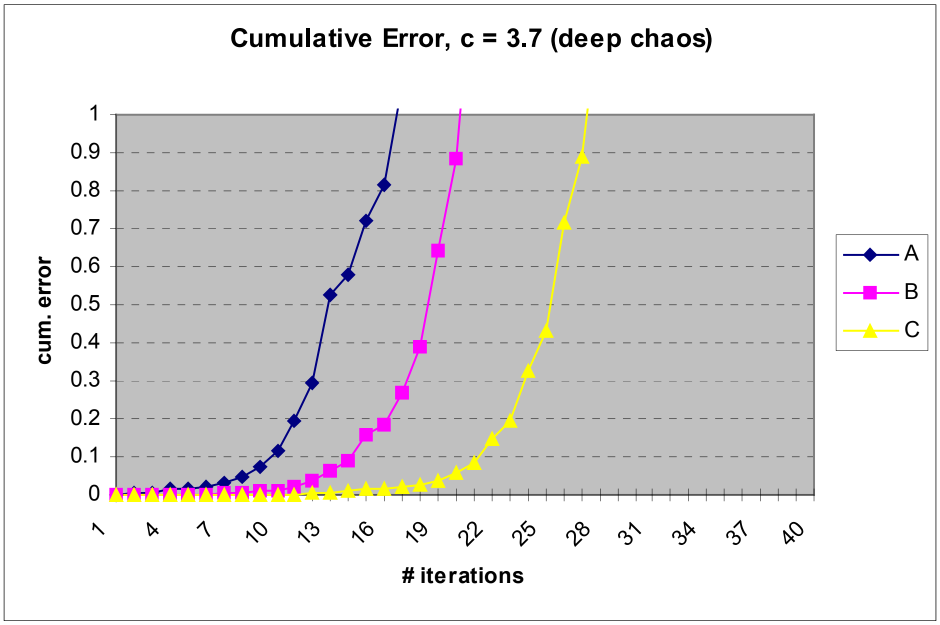

Figure 16 – Divergence rates, 'deep chaos'.

	Baseline	A	B	C
Iteration	**0.7**	**0.701**	**0.7001**	**0.70001**
1	0.777	0.77552	0.77685	0.77699
2	0.6411	0.64414	0.64141	0.64113
3	0.85133	0.84813	0.85102	0.8513
4	0.46829	0.47658	0.46911	0.46837
5	0.92128	0.92297	0.92147	0.9213

6	0.26834	0.26306	0.26774	0.26828
7	0.72643	0.71728	0.72541	0.72633
8	0.7353	0.75033	0.73701	0.73547
9	0.72014	0.69314	0.71716	0.71984
10	0.74569	0.78698	0.75052	0.74618
11	0.70166	0.62029	0.6928	0.70077
12	0.77453	0.87146	0.78747	0.77586
13	0.64614	0.41445	0.61923	0.64343
14	0.84598	0.89792	0.8724	0.84888
15	0.4821	0.33914	0.41188	0.47465
16	0.92381	0.82925	0.89627	0.92262
17	0.26041	0.52389	0.34399	0.26414
18	0.71261	0.92289	0.83495	0.71918
19	0.75775	0.26331	0.5099	0.74726
20	0.67919	0.71772	0.92464	0.6988
21	0.80619	0.74961	0.25783	0.77877
22	0.57811	0.69447	0.708	0.63746
23	0.90243	0.78508	0.76492	0.85509
24	0.32579	0.62431	0.66533	0.45847
25	0.81271	0.86783	0.82386	0.91862
26	0.56318	0.42441	0.53692	0.27661
27	0.91023	0.90386	0.91996	0.74036
28	0.30233	0.32153	0.27246	0.71124
29	0.78042	0.80715	0.73343	0.75989
30	0.63404	0.57595	0.72339	0.67509

Figure 17 – Divergence data points.

There are three lessons to learn here: (1) large variation causes divergence faster than small variation, (2) all variation, no matter how small, eventually causes divergence, and (3) it's not just "initial conditions", but *any* small change at *any* point in time. To expand on this last point: take any 3-body system in space -- e.g. a planet with two orbiting moons. Observe the system to evolve as it will. Then cause a 'small variation', say, by firing an explosive charge on one of the moons. The system will now evolve differently than it would have otherwise. At first, imperceptibly, then gradually more and more -- maybe years, maybe millennia -- the two futures would appear ‘completely different’. Completely different, that is, in terms of the positions and velocities of the three bodies[13]. It is still the same system in either case, just with two different 'future orbits'.

How different is 'completely different'? The answer depends on the *sensitivity of the observer*. Technically speaking, the difference is there immediately after the effect; it just may be very small or hard to detect. A sufficiently sensitive organism or measuring device could conceivably detect virtually any change. We of course measure change by our own human standards, and a great many changes are below the level of our sensitivity. We either cannot detect a given change, or see it as 'inconsequential' or 'trivial'. This is our human bias. Other systems, or other organisms, may not be so disposed toward a particular change.

The explosion in my previous example may be thought of as an *intervention* in which energy is inserted into the system. This ‘energy insert’ could be anything from a single firecracker to an atomic bomb. Clearly, different levels of energy will produce different effects. Let me offer a new idea here: For any given system, I submit that there exists a *critical threshold of energy*. Below this threshold, the system's future is inevitably altered, but to a minor degree: the overall appearance of the system, i.e. the overall pattern of the virtual strange attractor, remains unchanged. In real physical systems, virtual strange attractors are robust, stable patterns. A firecracker will not cause a moon to fly off its orbit in any human lifetime, if ever. And yet the energy insert is not to be ignored. The path through phase space is changed, and the effect, *no matter how small*, is eventually felt: the “positions and velocities” are different than they would have been otherwise.

But above the threshold, the energy insert is high enough to structurally change the system, resulting in a readily apparent reordering and redistribution of the system boundaries. The strange attractor undergoes a fundamental change, much as the pendulum pattern changed when it went ‘over the top’; this might correspond to the detonation of a nuclear explosive on one of the moons. The system is now at a higher energy level, still chaotic, and now even more unpredictable, because without the strange attractor as a guide we have no information on what will happen.

What about our real solar system? This system of a sun and nine planets – so regular and unchanging that it was the very inspiration of the Cartesian/Newtonian 'clockwork universe' – seems an unlikely place for chaos. This question has been debated for years, ever since the emergence of chaos theory. And it is a critical test case. The solar system is nearly, but not totally, frictionless. Small bits of dust, asteroid debris, and innumerable particles continually rain down on the planets. The gravitational field from the other planets acts as a small drag or boost. The planetary system is a 'dissipative structure' (a system which continually dissipates energy, such as via friction), but of the smallest degree. Chaos appears readily in highly dissipative systems, but weakly dissipative systems were thought to be immune. Chaos found here could mean chaos everywhere.

The most recent analyses indicate that, in fact, the solar system is chaotic. Not only is the movement of each planet unpredictable (to a greater or lesser degree, depending on time scale), but so too is the actual *presence* of planets over the long term. The first indication of chaos came in the 1980's, when MIT researchers J. Wisdom and G. Sussman created computer models to project planetary motion millions of years into the future. Said Wisdom, "After 845 million years of evolution, we saw clear signs that Pluto's orbit was chaotic" (Frank, 1998: 57). More recent analysis has focused on the inner planets. Calculations by French astrophysicist Jacques Lasker, confirmed by American researcher Tom Quinn, show that "the motion of the inner planets -- including Earth -- is chaotic in the technical sense." (ibid, p. 58). Thus, the more accurate picture of the planetary movement is, in the words of astrophysicist Adam Frank, to see "each orbit...[as a] tightly woven bundle of orbits, an infinitely tangled web of paths." (ibid, p. 56).

Interestingly, Wisdom distinguishes between "wild chaos" and 'normal' chaos -- corresponding roughly to my concepts of 'deep' and 'shallow' chaos[14]. Under conditions of wild chaos, there is sufficient sensitivity to cause major variations in the orbital path of a planet, and perhaps even result in it being flung out of the solar system. The present number of nine planets is probably only a chance number. According to George Lake, director of NASA's high performance computing program, "Now we understand that planets may not get permanent membership in their solar systems. Is it possible that a billion years ago there were 12 planets and, perhaps, a billion years from now there may only be six?" (ibid). Lasker's projections indicate roughly a 0.1% chance that Mercury will be thrown out of the solar system sometime in the next 5 billion years. The orbit of Mars, too, could become gradually more elliptical, causing it to come into close proximity to Earth. If this happens, either Mars or the Earth could be ejected into deep space. The one thing working in our favor is that Earth is more massive, thus less likely to be the one thrown out of its orbit.

So it seems that chaos is always with us, even in the most stable, least dissipative systems we know of. Therefore, the most reasonable conclusion is that *chaos exists in every dynamic system*, to a greater or lesser degree[15]. One can speak of a '*degree of chaocity*', being low for the apparently stable systems and high for the unstable ones. And again, I use the term 'system' here to mean any persistent structure of mass/energy, in particular any structure that embodies a continuous internal movement of mass/energy – which is, after all, every physical structure. Every physical object or collection of objects – a cell, a crystal, a tree, a human being, an ocean, an ecosystem – is dynamic, chaotic (in some sense), and is subject to investigation through the methods of nonlinear dynamical systems, i.e. phase space analysis.

Since all real physical systems are dynamic (if only at the atomic level), and all appear to act in a manner corresponding to the chaotic dynamics we find in our mathematical models, then it is reasonable to assume that *for each real system there exists something corresponding to a strange attractor pattern in its phase space description*. And in fact this is borne out by experience. Everywhere we see real systems, from the solar system to weather patterns to the flow of water down a river bed, that act in a manner that corresponds to a strange attractor: they exhibit unpredictability in precise detail, and yet display large scale stability.

Thus, even though we may not attribute to them the formal, technical definition of a strange attractor (which is defined only for closed systems), there nonetheless exists a strange-attractor-like quality within all physical systems. To differentiate from the formal definition, I will henceforth refer to this attractor-like behavior of real systems as a *'quasi-attractor'*, or 'virtual strange attractor'. I find this concept very useful in describing the nature of physical systems, and it permits a deeper investigation into their dynamics and interactions with the world.

The question is how to go about performing this sort of investigation. How does one define the state variables in such a way as to capture the essential information about the energy dynamics of the system? I offer an answer to this question by beginning, somewhat paradoxically, with the most complex and evolved biological system we know of: the human brain.

NOTES:

[1] Aristotle, *De Anima*, 984 b28.

[2] There is an element of randomness to the wave equation, but this is only relevant when a measurement is being taken. Some see this as a form of indeterminism in the basic structure of matter, and in fact is points to a universe in which a hidden indeterminism lies behind the deterministic equations of physics.

[3] Two of the forces, electromagnetic and weak nuclear, have been theoretically unified into a single 'electro-weak' force, but I will keep with the traditional usage and refer to them as distinct forces.

[4] For most values of x_o. Some x_o will give rise to ordinary (finite) limit cycles

[5] Technically, the points are called a 'Poincare map', and represent a cross-section of phase space which contains the actual strange attractor. More on this later.

[6] See Shaw (1984), Packard (1980), and Takens (1981).

[7] This implies use of 'polar coordinates'. The reference point $\alpha = 0$ will be defined as 'straight down', with positive angles to the counter-clockwise and negative angles clockwise (thus ranging from $-\pi$ radians to $+\pi$ radians).

[8] Or better, very constrained chaos -- more on this later

[9] This jump appears because of the 2-D depiction shown here. The actual phase space trajectory would be three-dimensional (e.g. cylindrical). I am showing a 2-D projection, which more readily serves the purpose at hand.

[10] The sum of the three voltages must be zero, by Kirchoff's Voltage Law.

[11] Not all initial points in phase space will be drawn to the same attractor. Each attractor has a 'basin of attraction' that defines a region from which all initial points will be drawn.

[12] Take 1.0001 and keep squaring it ten times; one gets 1.1078..., representing more than 10% variation.

[13] In reality, of course, this can be difficult or impossible. A large-scale system like the planets can not, properly speaking, be 'experimented' on, simply because there is no basis for comparison. If we were to detonate an explosive on the moon, we have no basis for knowing to what degree the system (path through phase space) has been altered. All we know is the long-term stability of system strange attractor (i.e. moon moving regularly around the Earth), and with a sufficient blast this could be clearly altered.

The same problem occurs on the Earth. Global warming is a kind of 'experiment'. We see global temperatures rising, but we have no way of knowing if, or to what degree, this is due to human actions – because we have no 'unwarmed' Earth to compare to. In a laboratory, where conditions are *relatively* isolated and *relatively* repeatable, one can more easily confirm chaotic effects.

[14] Sufficiently 'deep' chaos will approach the critical threshold of energy, resulting in a fundamental restructuring of the system.

[15] It is interesting to observe that, even quite recently, the common view among experts in complexity theory has been that 'some physical systems are chaotic' and 'some are not'. For example, Bak and Chen (1991) write: "In nonchaotic systems, such as the earth orbiting around the sun, the uncertainty [of the orbital path] remains constant at all times" (p. 51).