UNIVERSITY OF BATH

Department of Economics

MSc in Economics & Finance

ES50106 – Financial Investment Management

Lecture slides

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ES50106 - Financial investment management

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Lecture 1 - Utility Theory

Structure of this lecture

Preferences

- Preferences allow to order any possible choices, i.e. rank them
- In economics a set of axioms have been developed that achieve this aim

Axiom 1 (Reflection)

- Any choice is at least as good as itself
- $a \succeq a$

Axiom 2 (Transitivity)

- If $a \succeq b$ and $b \succeq c$, then $a \succeq c$
- In experiments, transitivity of choices has been found to be frequently violated

Axiom 3 (Completeness)

- All choices can be compared
- $\forall a, b : a \succeq b$ or $b \succeq a$

Utility function

- A utility function $U : S \to \mathbb{R}$ assigns a real number to each choice such that
  - $a \succ b \Leftrightarrow U(a) > U(b)$
  - $a \sim b \Leftrightarrow U(a) = U(b)$
- The absolute value of these numbers is irrelevant, they can also be negative

The problem

- When outcomes are uncertain, the simple utility axioms do not hold any more
- We can compare individual possible outcomes, but how to aggregate them?
- A set of axioms has been developed to deal with this problem

Notation

- A choice a_i has a set of potential outcomes $\{c_{i1}, c_{i2}, \ldots, c_{iM}\}$
- These outcomes corresponds to the different states of assets
- Each outcome (state) has a certain probability of occurring, p_{ij}
- A choice we also write as $a_i = [p_{i1} c_{i1}, \ldots, p_{iM} c_{iM}]$

Axiom 1 (Completeness)

- All choices can be compared
- $\forall a_i, a_j : a_i \succeq a_j$ or $a_j \succeq a_i$

Axiom 2 (Decomposition)

- Let $a_i = [p_{i1} b_{i1}, \ldots, p_{iM} b_{iM}]$ with $b_{ij} = [q_{ij1} c_1, \ldots, q_{ijL} b_L]$
- If we define $p^*_{ik} = \sum_{l=1}^{M} p_{il} q_{ilk}$ then $a_i \sim [p^*_{i1} c_1, \ldots, p^*_{iL} c_L]$
- If the possible outcomes of a choice are themselves random, we can generate an equivalent choice based only on final payoffs

Axiom 3 (Composition)

- Let $a_i = [p_{i1} b_{i1}, \ldots, p_{iM} b_{iM}]$ and $b_{ij} \sim [q_{ij1} c_1, \ldots, q_{ijL} b_L]$
- Then $a_i \sim [p^*_{i1} c_1, \ldots, p_{ij} [q_{ij1} c_1, \ldots, q_{ijL} b_L], \ldots, p^*_{iL} c_L]$
- We can generate more complex outcomes, but they do not affect the preferences

Axiom 4 (Monotonicity)

- Let $a_i = [p_{i1}c_1, p_{i2}c_2]$ and $b_i = [q_{i1}c_1, q_{i2}c_2]$ with $c_1 \succ c_2$
- If $p_{i1} > q_{i1}$ then $a_i \succ b_i$
- Given the same possible payoffs, the choice with the higher probability on the better payoff is preferred

Axiom 5 (Continuity)

- If $a_i \succ b_i$ and $b_i \succ c_i$ then there exists a $d_i = [p_1 a_i, p_2 c_i]$ such that $d_i \sim b_i$
- This axiom ensures that we always can generate intermediate outcomes from its extremes.

The proof

- Let $a_i = [p_{i1}c_{i1}, \ldots, p_{iM}c_{iM}]$ with $c_1 \succ c_2 \succ \ldots \succ c_M$
- Such an order exists by axiom 1

Using the extreme outcomes

- We can rewrite c_i as $c_i \sim [u_i c_1, (1 - u_i) c_M] \equiv c_i^*$
- This uses axiom 5

Substitute in the choice

- $a_i = [p_{i1} c_1^*, \dots, p_{iM} c_M^*]$
- This uses axiom 3 and replaces the outcomes with stochastic outcomes
- Each c_i^* consists of the same two possible outcomes, c_1 and c_M, only with different probabilities

Simplifying the outcomes

- $a_i \sim [p_i c_1, (1 - p_i) c_M]$
- $p_i = \sum_{j=1}^{M} u_{ij} p_{ij}$
- $p_i = E[u_i]$
- The choice has been reduced to two possible outcomes

Comparing outcomes

- For another choice we get equivalently $a_j \sim [p_j c_1, (1 - p_j) c_M]$ with $p_i = E[u_i]$
- If $p_i > p_j$ we find $a_i \succ a_j$ using axiom 4
- Or $E[u_i] > E[u_j]$

Interpreting u_i

- From our definition we know from axiom 4 that $c_i \succ c_j \Leftrightarrow u_i > u_j$
- The ordering of u_i therefore reflects the order if the preferences
- This is exactly the same as the definition of the utility function
- u_i can be interpreted as utility

Expected utility principle

- Choices should be made according to the expected utility of the choices
- $a_i \succ a_j \Leftrightarrow E[U(a_i)] > E[U(a_j)]$
- Determine the utility of each possible outcome and then calculate the expected value

Definition

Individuals are risk averse if they always prefer to receive a fixed payment to a random payment of equal expected value.

Certainty equivalent

- In order to be indifferent between a fixed and random payment, the fixed payment must be lower by an amount of π
- $E[U(x)] = E\left[U\left(E[x] - \pi\right)\right] = U(E[x] - \pi)$
- $E[x] - \pi$ is called the cash equivalent or certainty equivalent of x
- π is called the risk premium of x

Approximating the random payoff

- Using a 2^{nd} order Taylor series expansion around $E[x]$ we get
- $$E[U(x)] = E\left[U(E[x]) + U'(E[x])(x - E[x]) + \frac{1}{2}U''(E[x])(x - E[x])^2\right]$$
$$= U(E[x]) + \frac{1}{2}U''(E[x])Var[x]$$

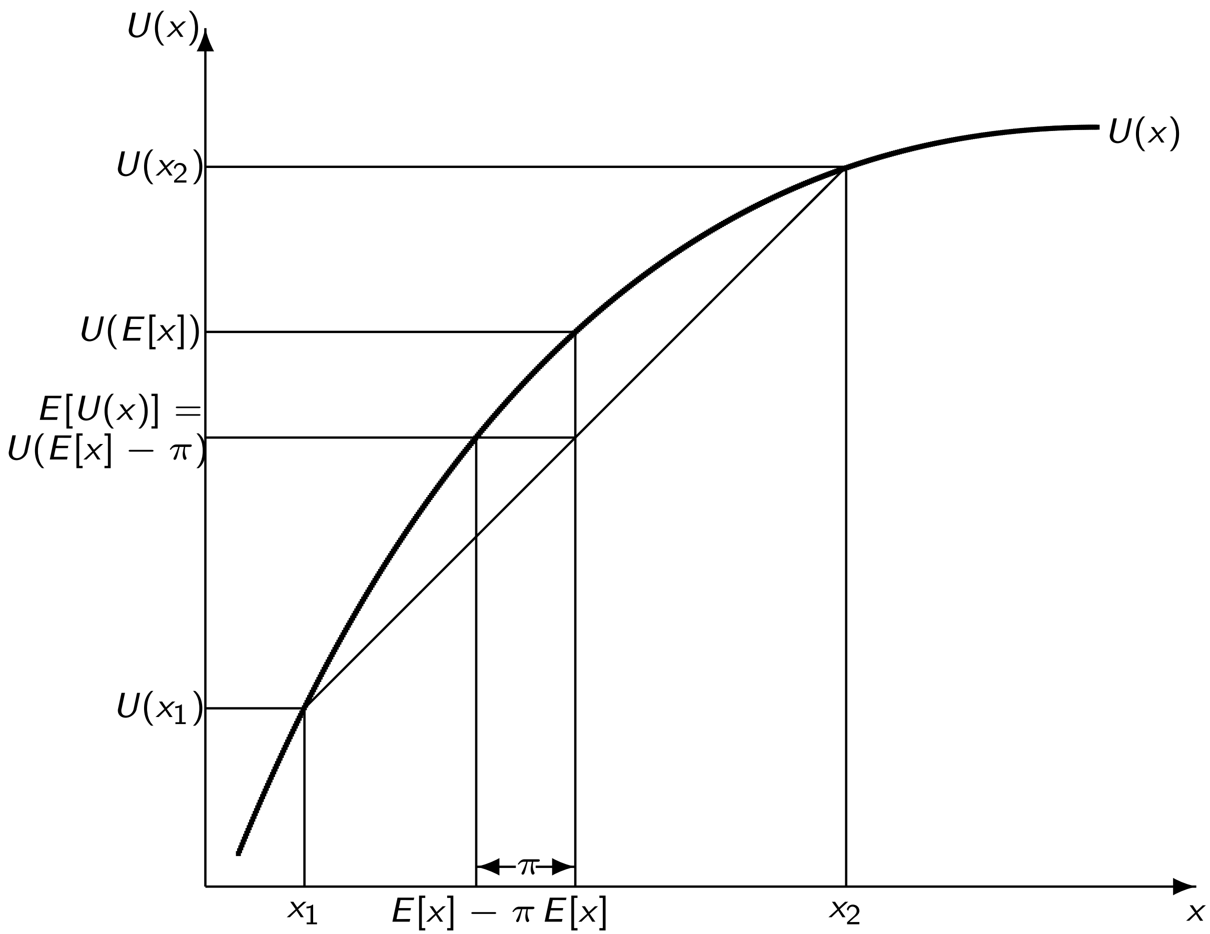
Approximating the fixed payoff

- Using a 1^{st} order Taylor series expansion around $E[x]$ we get
- $U(E[x]-\pi) = U(E[x]) + U'(E[x])\pi$

Risk premium

- Combining the two equations we get $\pi = \frac{1}{2}\left(-\frac{U''(E[x])}{U'(E[x])}\right) Var[x]$
- $z = -\frac{U''(E[x])}{U'(E[x])}$ is known as the absolute risk aversion
- The more concave the utility function is, the more risk averse the individual
- The relative risk aversion is defined as zV

Risk aversion and expected utility



The idea

- Relative utility level should depend only on the difference of wealth associated with two choices
- $\frac{U(V)}{U(V')} = f(V - V')$

Deriving the utility function

- $\frac{U'(V)}{U(V')} = f'(V - V')$
- $\frac{U'(V')}{U(V')} = f'(0) \equiv -z$
- $[\ln U(V')]' = -z$
- $\ln U(V') = -zV' + c$
- $U(V) = Ce^{-zV} = -e^{-zV}$
- Set $C = -1$ to ensure $U'(V) \geq 0$

Risk aversion

- Using $U(V) = -e^{-zV}$ we can show that $-\frac{U''(V)}{U'(V)} = z$
- Hence the utility function has Constant Absolute Risk Aversion

The idea

- Relative utility levels of two choices depend on the relative change in wealth the choices produce
- $\frac{U(V)}{U(V')} = f\left(\frac{V}{V'}\right)$

Deriving the utility function

- $\frac{U(V)}{U(V')} = \frac{U(\exp(\ln V))}{U(\exp(\ln V'))} = \frac{\widehat{U}(\ln V)}{\widehat{U}(\ln V')}$
  $= f\left(\frac{V}{V'}\right) = f\left(\exp(\left(\ln \frac{V}{V'}\right)\right) = \widehat{f}(\ln V - \ln V')$
- Log-levels of wealth follow a CARA utility function

Deriving the utility function (ctd.)

- $\widehat{U}(\ln V) = U(V) = Ce^{-\widehat{z}\ln V} = CV^{-\widehat{z}} = \frac{V_{-\widehat{z}}}{-\widehat{z}}$
- $z = \frac{U''(V)}{U'(V)} = \frac{(\widehat{z}+1)}{V}$ or $\widehat{z} = zV - 1$
- More commonly written with $\gamma = \widehat{z} + 1$ as $U(V) = \frac{V^{1-\gamma}}{1-\gamma}$
- Hence $\gamma = zV$ which is the relative risk aversion
- Utility function has Constant Relative Risk Aversion
- for $\gamma = 1$ ($\widehat{z} = 0$) this becomes $U(V) = \ln V$

1. Utility axioms
2. Expected utility
3. Risk aversion
4. Classes of utility functions
   - CARA utility functions
   - CRRA utility function
5. Outlook

Outlook

We will use utility functions widely to determine the optimal portfolio in the coming lecture.

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Lecture 2 - Portfolio selection theory

Structure of this lecture

1. Mean-Variance criterion
2. The Markowitz frontier
3. Outlook

Investment decisions

When investing you have to make three decisions

1. How much to invest into risky assets
2. Which risky assets to invest in
3. How much to invest into each of the selected assets

An optimal portfolio

[An optimal portfolio] is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies. The investor should build toward an integrated portfolio which best suites his needs.

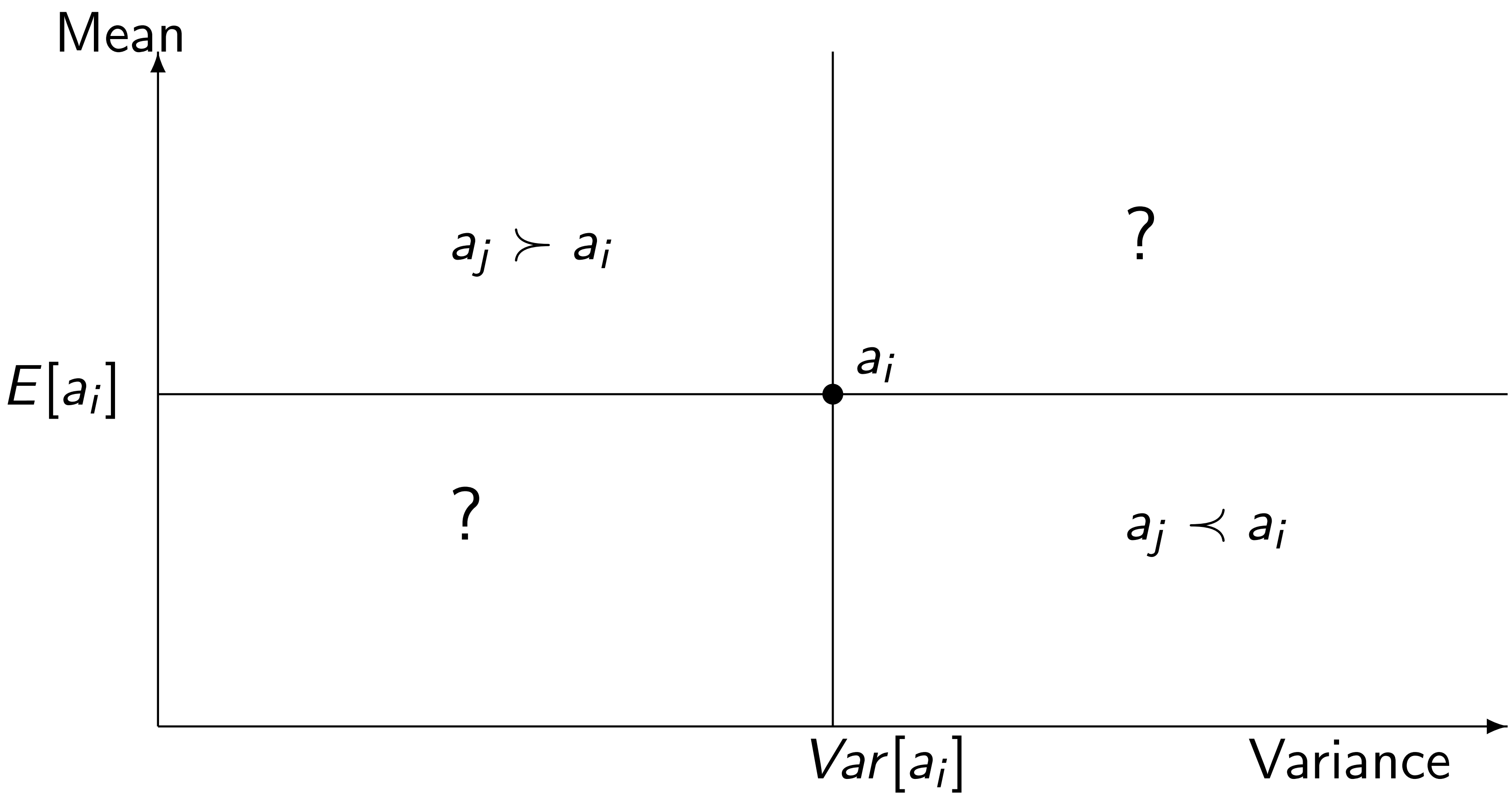
Using two moments

- Ideally we would like to determine the portfolio without the utility function
- We can use the two moments that emerge from the utility functions: mean and variance
- Investors like a high mean and dislike a high variance

The decision criterion

$$a_i \succeq a_j \Leftrightarrow \begin{cases} Var[a_i] < Var[a_j] \quad \text{and} \quad E[a_i] \geq E[a_j] \\ \text{or} \\ Var[a_i] \leq Var[a_j] \quad \text{and} \quad E[a_i] > E[a_j] \end{cases} .$$

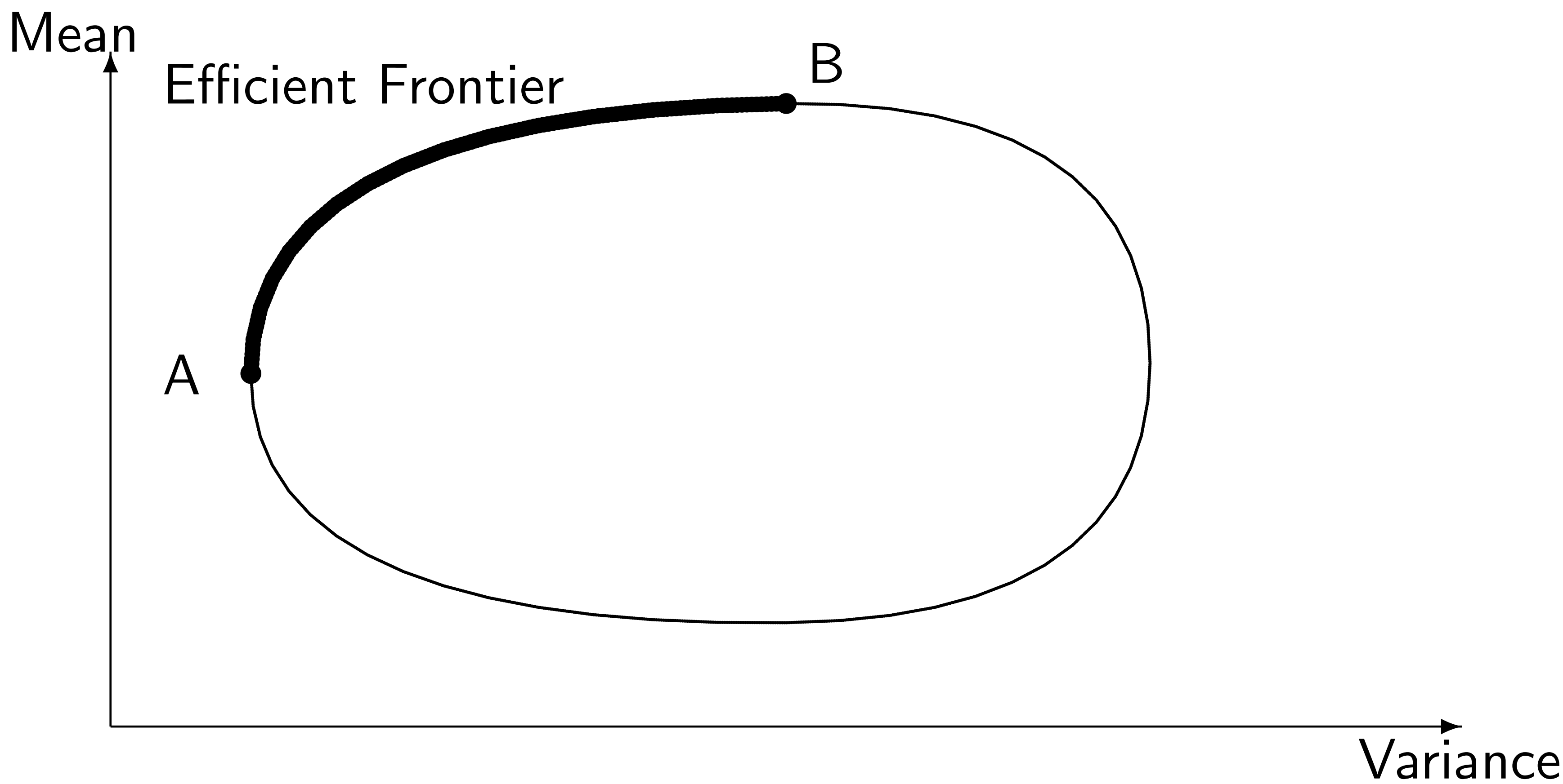
Decision-making using the mean-variance criterion



Efficient frontier

- Alternatives not dominated by another alternative are called efficient
- All efficient alternatives are forming the efficient frontier

Efficient frontier



Utility function

- To distinguish between elements of the efficient frontier we need the utility function
- Use quadratic utility function: $U(x) = x + bx^2$

Properties of the utility function

- Risk aversion: $z = \frac{2b}{1+2b\mu}$
- For risk averse investors we need $b < 0$
- Marginal utility: $U'(x) = 1 + 2bx > 0 \Leftrightarrow x < -\frac{1}{2b}$

Problem with quadratic utility functions

- For large outcomes, the marginal utility is negative
- We have to ensure that we only consider situations before the bliss point

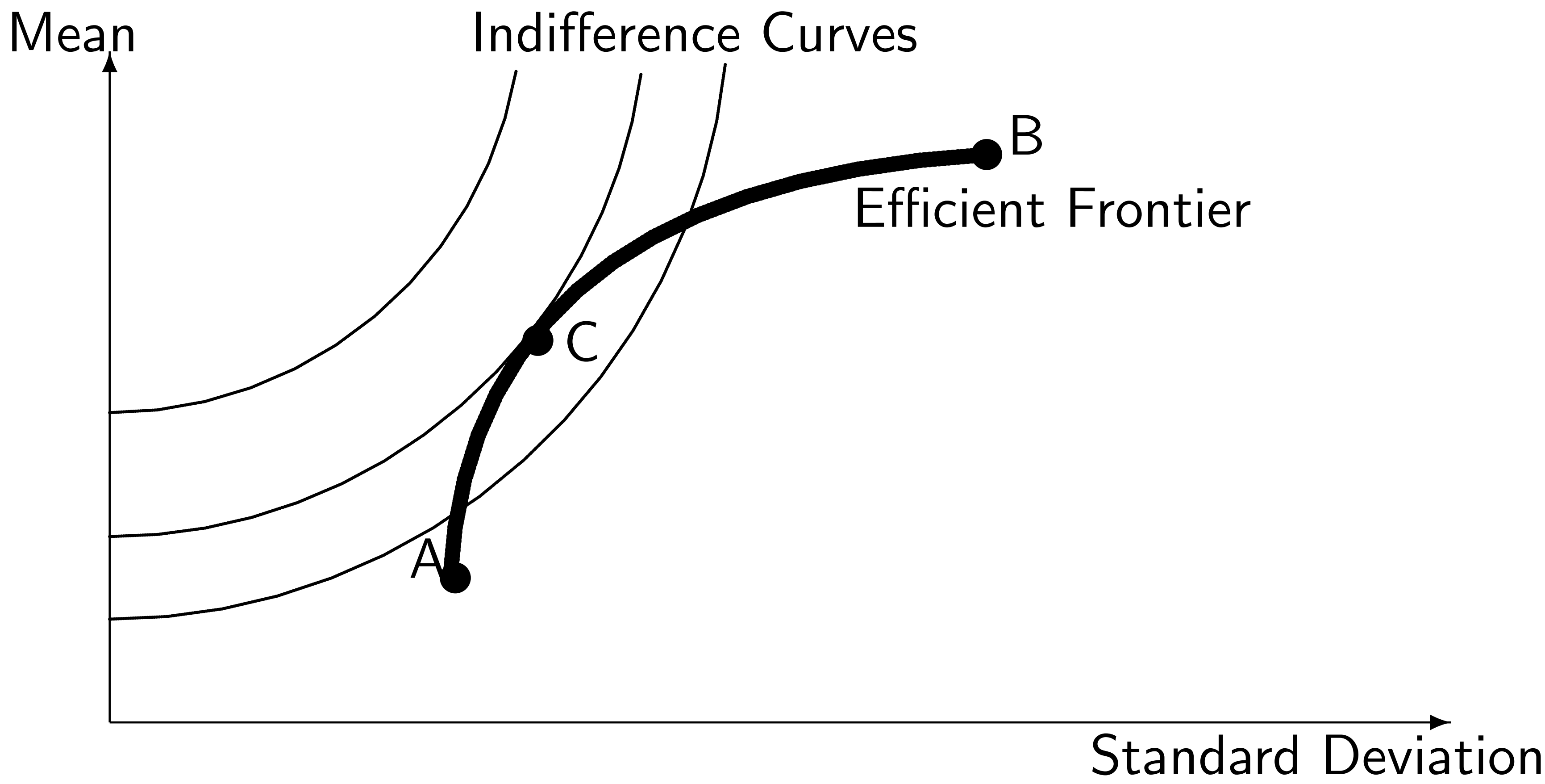
Indifference curves

- $E\left[U(x)\right] = E\left[x + bx^2\right] = \mu + b(\mu^2 + \sigma^2)$
- $dE\left[U(x)\right] = (1 + 2b\mu)d\mu + 2\sigma d\sigma = 0$
- $\frac{d\mu}{d\sigma} = -\frac{2b\sigma}{1+2b\mu} = z\sigma > 0$

Functional form

- $E\left[U(x)\right] = E\left[x + bx^2\right] = \mu + b(\mu^2 + \sigma^2)$
- $\mu^2 + \frac{1}{b}\mu + \sigma^2 = \frac{E[U(x)]}{b}$
- $\left(\mu + \frac{1}{2b}\right)^2 = \frac{E[U(x)]}{b} + \frac{1}{4b^2}$
- Indifference curves are circles with center $\left(0, -\frac{1}{2b}\right)$

Determining the optimal alternative



Optimality of mean-variance criterion

- $a_i \succ a_j \Leftrightarrow E\left[U(a_i)\right] > E\left[U(a_j)\right]$
- $\mu_i + b\mu_i^2 + b\sigma_i^2 > \mu_j + b\mu_j^2 + b\sigma_j^2$
- $\mu_i - \mu_j + b(\mu_i^2 - \mu_j^2) + b(\sigma_i^2 - \sigma_j^2) = (\mu_i - \mu_j)\left(1 + b(\mu_i + \mu_j)\right) + b(\sigma)i^2 - \sigma_j^2 > 0$

Optimality of mean-variance criterion (ctd.)

- Divide by $-2b > 0$:
- $(\mu_i - \mu_j)\left(-\frac{1}{2b} - \frac{\mu_i+\mu_j}{2}\right) - \frac{\sigma_i^2-\sigma_j^2}{2} > 0$
- This condition is fulfilled iff the mean-variance criterion is fulfilled

Relation to portfolio theory

- These principles can now be applied to portfolio theory
- We only have to use the portfolio characteristics and apply the theory

Assumptions

- No transaction costs and taxes
- Assets are infinitely divisible
- No restrictions on investments
- Investors maximize expected utility using the mean-variance criterion
- Prices are competitive
- Single time period for investments

Portfolio characteristics

- $\mu_p = \mu' \omega$
- $\sigma_p^2 = \omega' \Sigma \omega$
- $\omega' \iota = 1$

The two asset case

- $\mu_p = \omega\mu_1 + (1-\omega)\mu_2 = \mu_2 + \omega(\mu_1 - \mu_2)$
- $\sigma_p^2 = \omega^2\sigma_1^2 + (1-\omega)^2\sigma_1^2 + \omega(1-\omega)\sigma_1\sigma - 2\rho$
- $\qquad\qquad = \sigma_2^2 + \omega^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) + 2\omega(\sigma_1\sigma_2\rho - \sigma_2^2)$

Minimum risk portfolio

- $\frac{\partial \sigma_p^2}{\partial \omega} = 2\omega(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) + 2(\sigma_1\sigma_2\rho - \sigma_2^2) = 0$
- $\omega_{MRP} = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$
- $\sigma_{MRP}^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$
- We can show that the MRP is decreasing as the correlation reduces

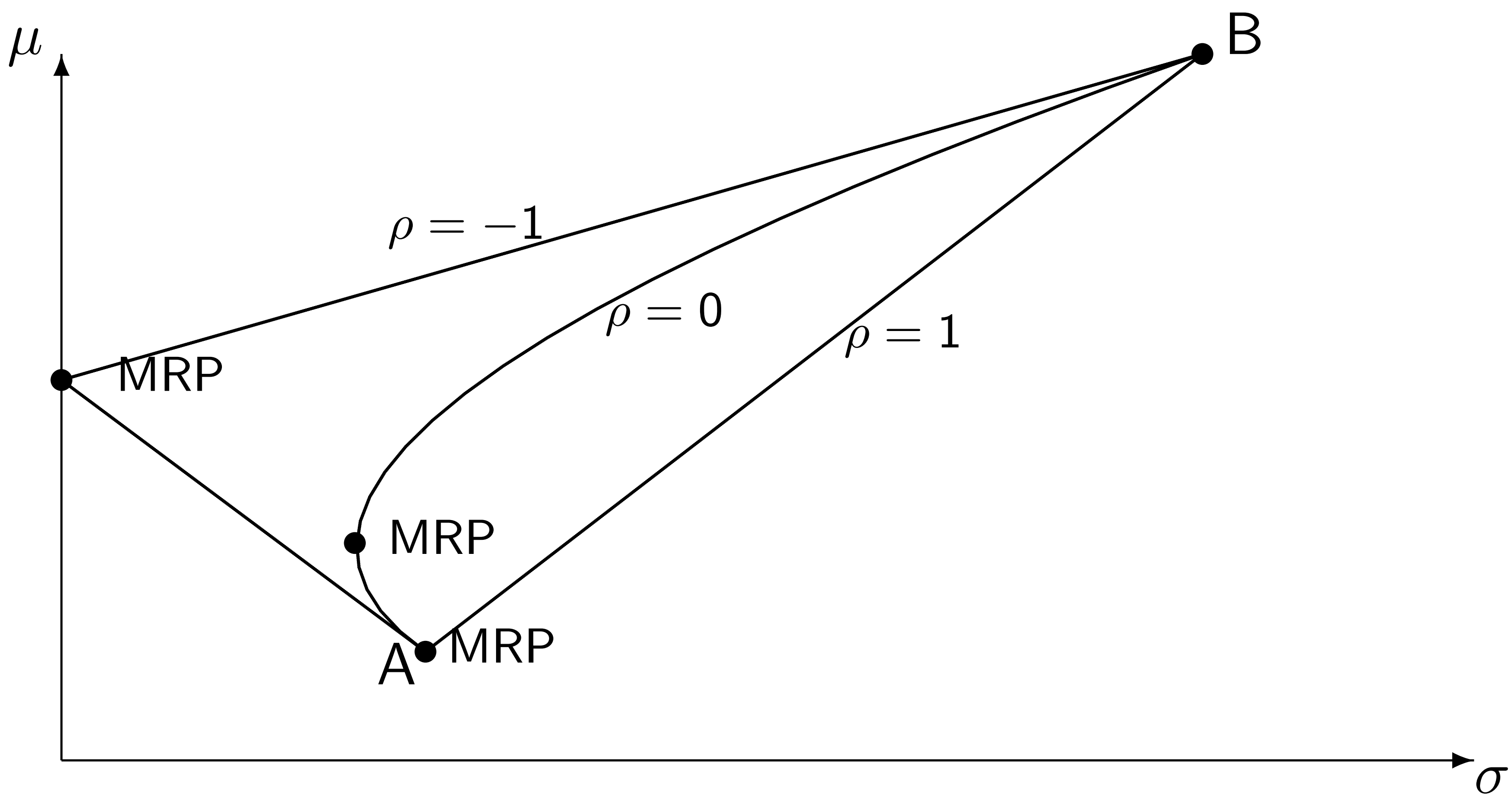
Shape of portfolio sets

- $\sigma_p^2 - \sigma_{MRP}^2 = \sigma_2^2 + \omega^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) + 2\omega(\sigma_1\sigma_2\rho - \sigma_2^2) - \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho}$
- $= (\omega - \omega^{MRP})^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho)$
- $\mu_p - \mu_{MRP} = (\omega - \omega^{MRP})(\mu_1 - \mu_2)$
- We can now combine these two expressions

Efficient set as a hyperbola

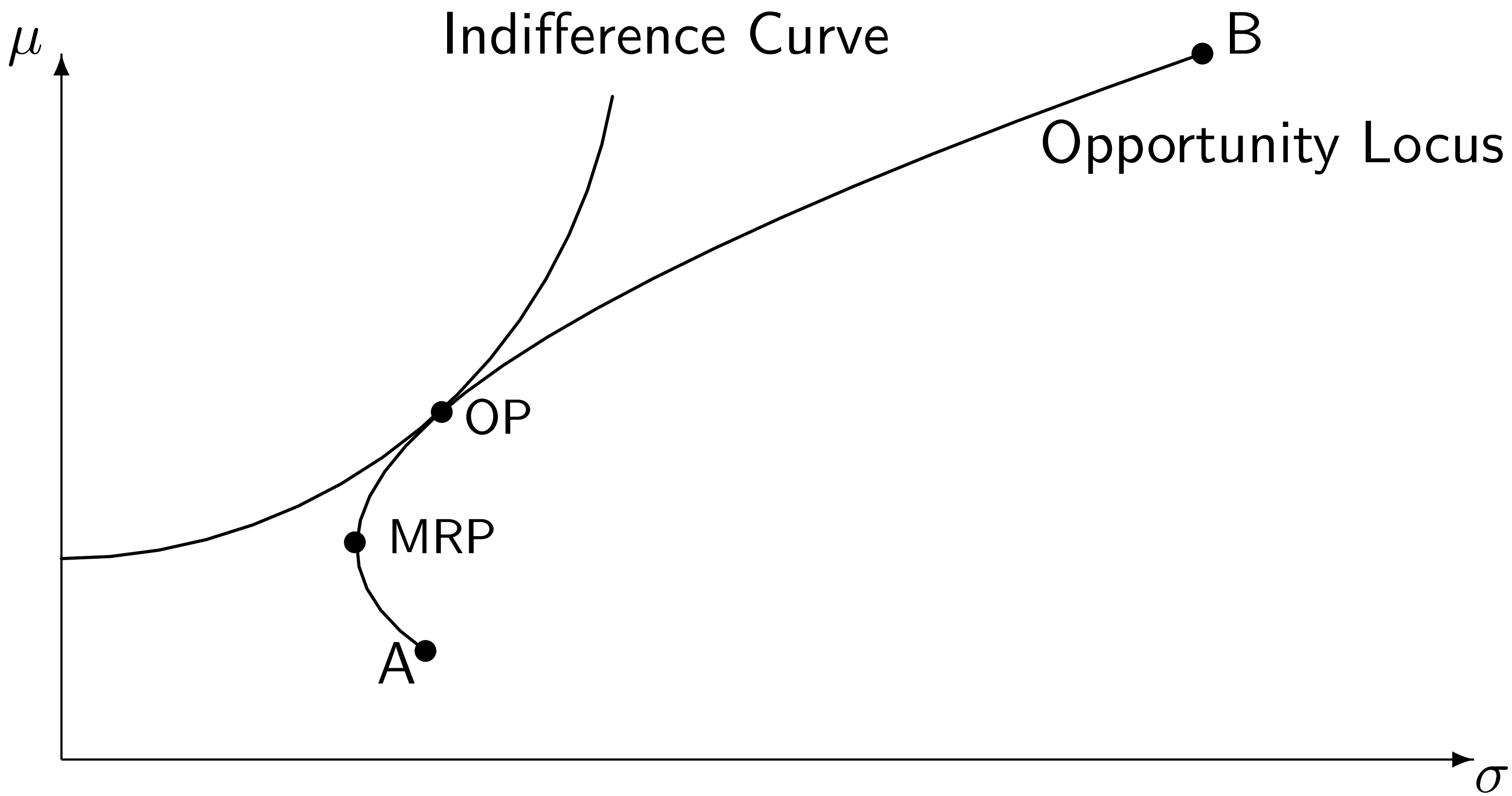
- $(\mu_p - \mu_{MPR})^2 = \frac{\sigma_p^2 - \sigma_{MRP}^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}(\mu_1 - \mu_2)^2$
- This is a hyperbola with axes
- $\mu_p = \mu_{MRP} \pm \frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}}\sigma_p$

Efficient portfolios



Determining the optimal portfolio

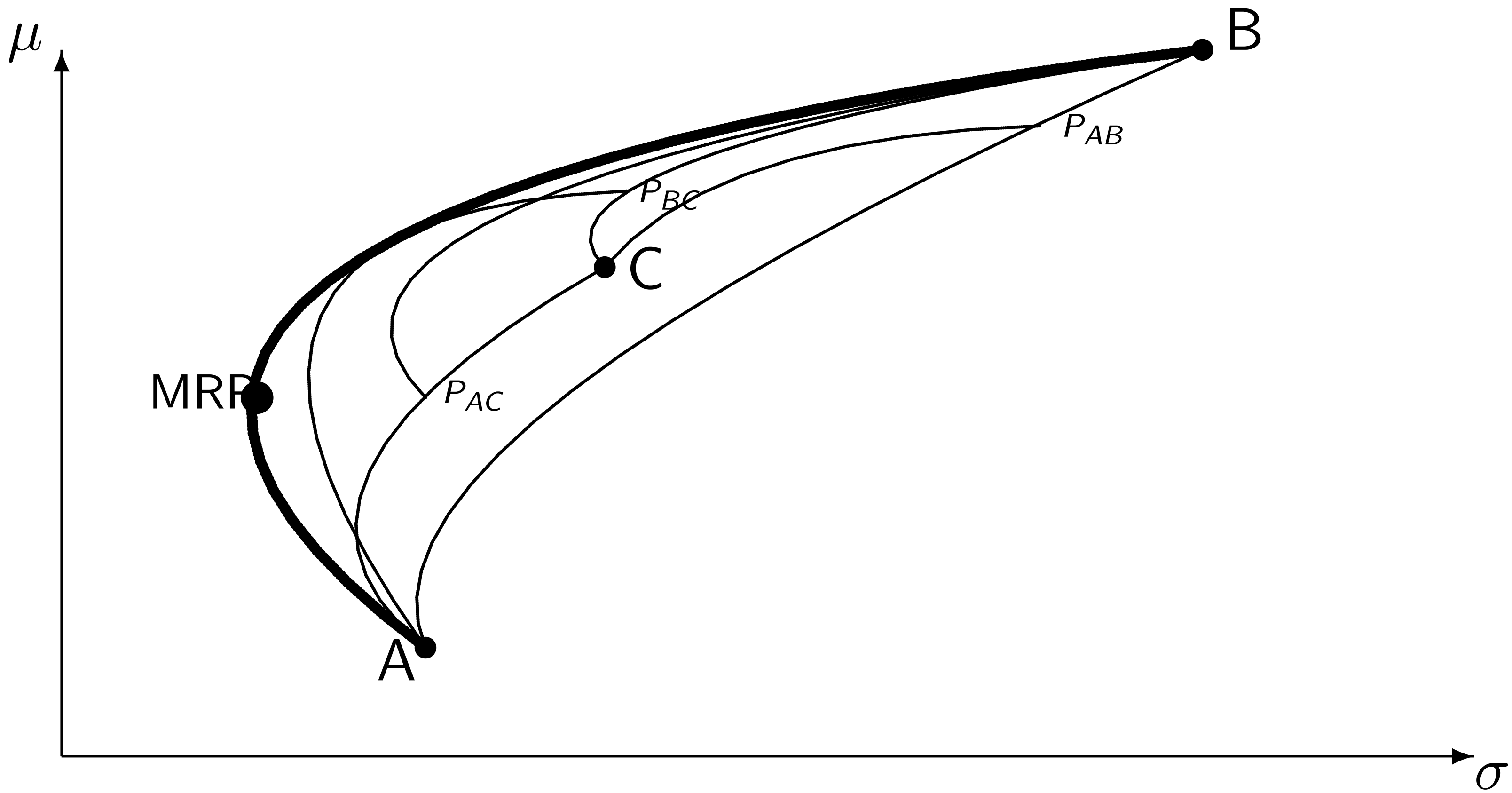
We can now introduce the indifference curves to determine the optimal portfolio


μ
Indifference Curve
B
Opportunity Locus
OP
MRP
A
σ

Extension to multiple assets

- We can now extend this idea of portfolio determination to multiple assets
- we only need to combine all possible portfolios of any 2 assets
- Portfolios of assets are then treated as assets themselves

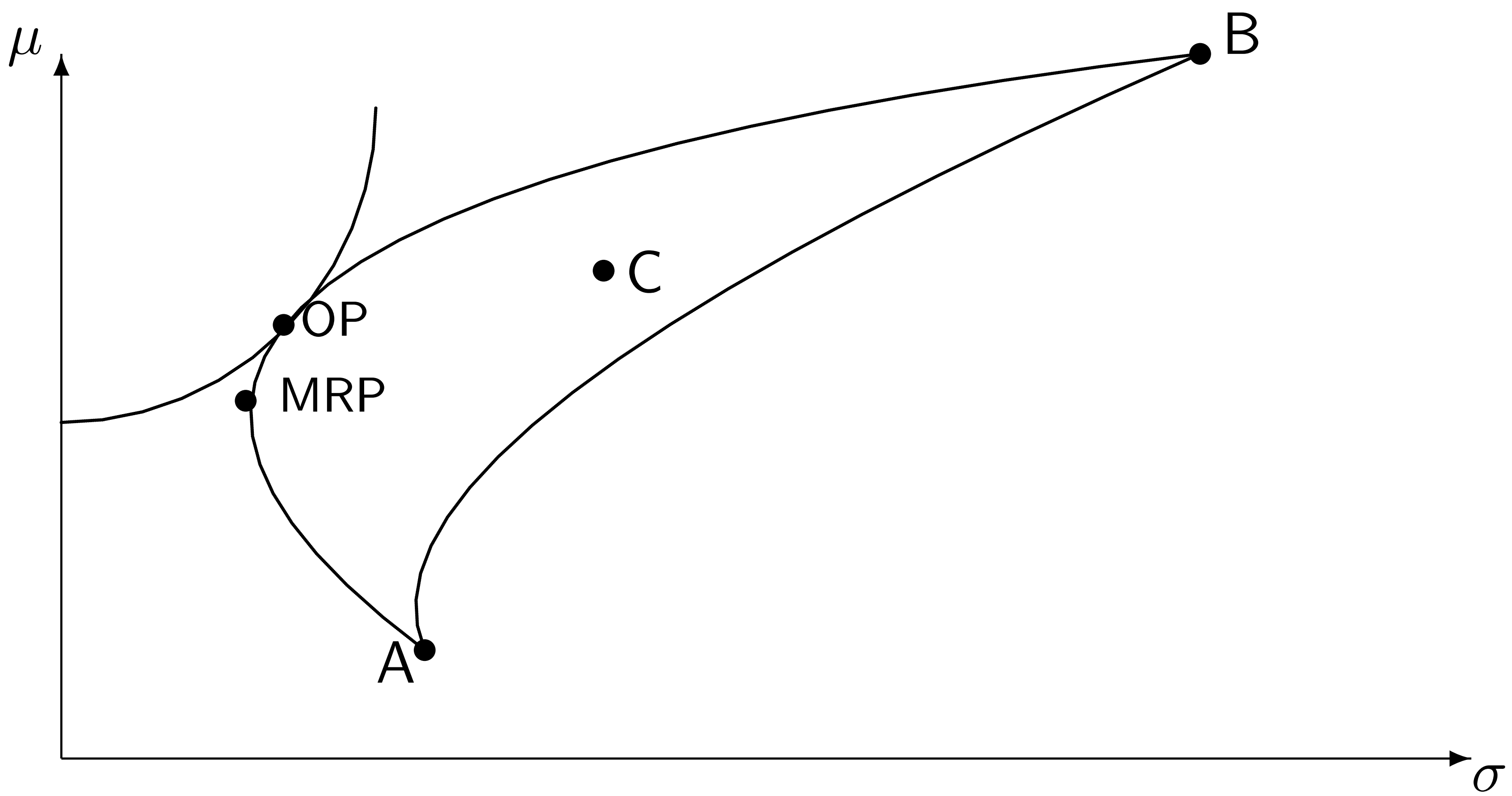
Portfolio selection with multiple assets



Properties of the efficient set

- The efficient set will form a convex hull of the opportunity set
- The more assets the more to the upper left it will move
- Allowing short sales increases opportunity set

The optimal portfolio



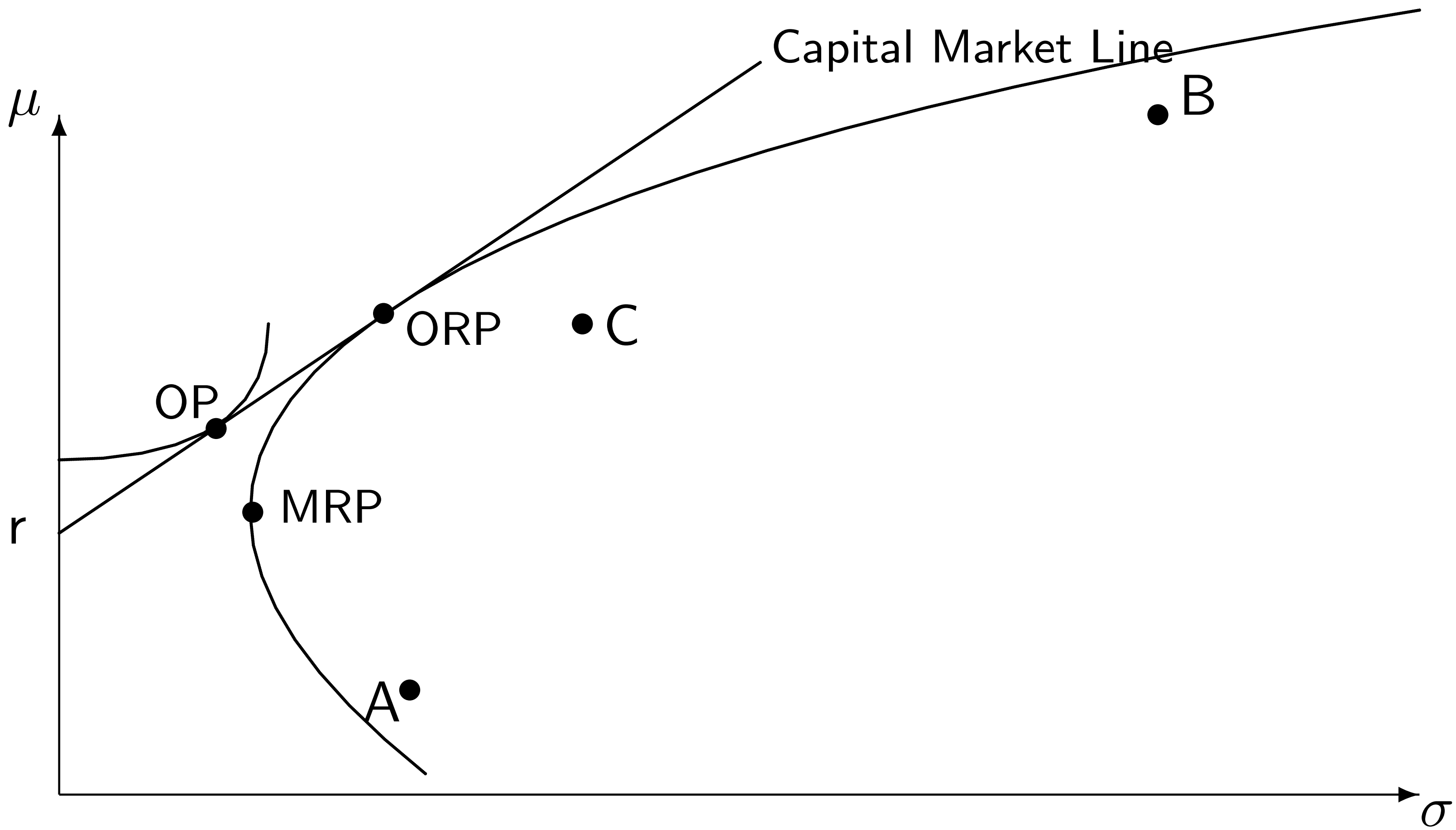
Riskless asset

- A risk less asset has $\sigma = 0$ and $\mu = r$, hence
- $\mu_p = \omega\mu_1 + (1-\omega)r = r + \omega(\mu_1 - r)$
- $\sigma_p^2 = \omega^2\sigma_1^2$
- $\mu_p = r + \frac{\mu_1 - r}{\sigma_1}\sigma_p$

Opportunity set with a riskless asset

- The opportunity set is a straight line with slope $\frac{\mu_1 - r}{\sigma_1}$
- This allows to combine the decisions on risky and riskless assets

Optimal portfolio with a riskless asset



Tobin's separation theorem

- The ORP is independent of preferences
- The optimal portfolio is a combination of the ORP and the riskless asset
- Only this combination depends on the preferences
- This two step combination is called Tobin's separation theorem

Diversification

- Diversification reduces the risk of an investment
- The marginal benefits are decreasing with the number of assets
- Random selection of portfolios hardly ever get close to the efficient frontier

Stability of portfolio selection

- The ORP is very sensitive to the estimated means and covariance matrix
- Therefore a more robust method of estimating the efficient frontier is desirable

Resampling

- A way of getting a more robust estimate of the efficient frontier is to vary the estimates of the parameters within the sampling distribution
- Averaging the resulting weights gives a more robust estimate of the efficient frontier

Outlook

- Portfolio selection theory provides the basic idea for designing an investment strategy
- In the coming lectures we will look at the long-run investment strategy and then short-run deviations to exploit additional information

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Lecture 3 - Myopic portfolio choice

Structure of this lecture

1. Moments of portfolios
2. Short-term portfolio choice
3. Long-term portfolio choice without re-balancing
4. Long-term portfolio choice with re-balancing
5. The effect of diversification
6. Resampling the efficient frontier
7. Outlook

Vector notation

1. $\widehat{\mu}_P = \sum_{i=1}^{N} \omega_i \mu_i = \omega' \mu$
2. $\widehat{\sigma}_P^2 = \sum_{i,j=1}^{N} \omega_i \omega_j \sigma_{ij} = \omega' \Sigma \omega$

Riskless asset

1. A risk less asset is in principle included in the above
2. It is a special case with $\mu_i = r$ and $\sigma_{ij} = 0$
3. Due to the importance of the ORP the riskfree asset is often treated separately

Portfolios with riskless asset

1. invest a fraction α into the risky assets (ORP)
2. $\mu_P = \alpha\widehat{\mu}_P + (1-\alpha)r = \alpha(\widehat{\mu}_P - r) + r$
3. $\sigma_P^2 = \alpha^2\widehat{\sigma}_P^2$

Total portfolio moments

1. We can now combine these two representations
2. Define $\widehat{\omega} = \alpha\omega$
3. $\mu_P = \alpha(\omega'\mu - r) + r = \widehat{\omega}'(\mu - r\iota) + r$
4. $\sigma_P^2 = \alpha^2\omega'\Sigma\omega = \widehat{\omega}'\Sigma\widehat{\omega}$

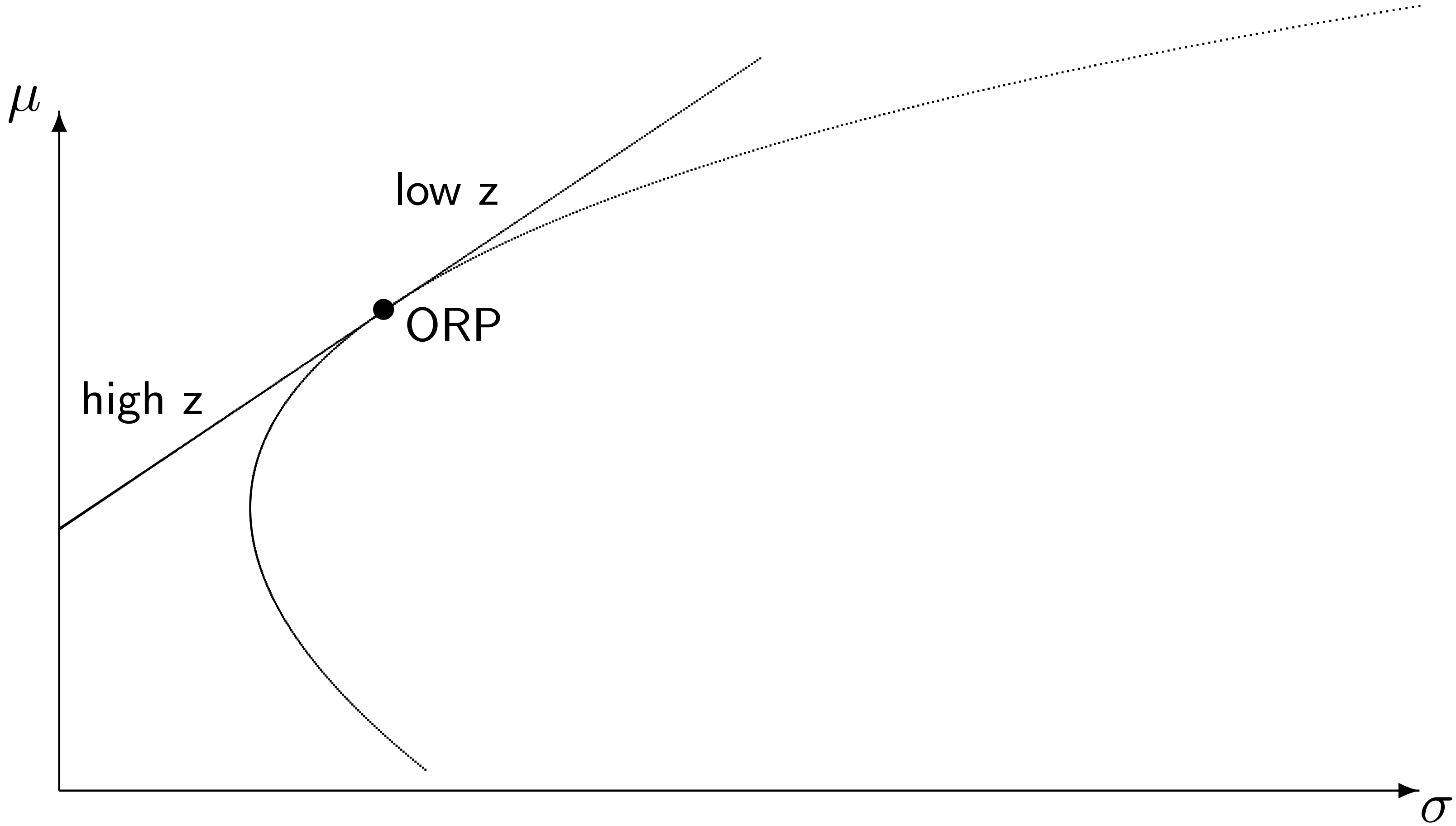
Objective function

1. The u optimal portfolio maximizes expected utility
2. $L = \mu_P - \frac{1}{2} z \sigma_P^2 \rightarrow \max_{\widehat{\omega}}$
3. $\frac{\partial L}{\partial \widehat{\omega}} = \mu - r\iota - z\Sigma\widehat{\omega} = 0$
4. $\widehat{\omega} = \frac{1}{z}\Sigma^{-1}(\mu - r\iota)$

Composition of the optimal portfolio

1. Portfolio composition is essentially independent of preferences (ORP): $\Sigma^{-1}(\mu - r\iota)$
2. Preferences only affect the level of investments into the risky assets via $\frac{1}{z}$, i.e. α

OP and risk aversion



Long-term investors

1. So far we assumed that investor have a time horizon of one time period
2. In reality many investors seek investments for multiple time periods
3. For now let us assume that such investors cannot re-balance their portfolio after each time period

Moments over multiple time periods

1. Assuming uncorrelated returns we have
2. $\mu_{i,T} = T\mu_i$
3. $\sigma_{ij,T} = T\sigma_{ij}$
4. Assumes implicitly that moments are not expected to change

Objective function

1. $L_T = \mu_{P,T} - \frac{1}{2} z \sigma^2_{P,T} = T\mu_P - \frac{1}{2} z T \sigma^2_P = TL$
2. The constant factor T is irrelevant for the optimization problem
3. The optimal portfolio will be identical regardless of the time horizon

Myopic portfolio choice

A portfolio choice is said to be myopic if short-term investors and long-term investors make the same portfolio choice.

Two time period model

1. Assuming identical moments for both time periods
2. $R_P = (1 + R_{P,1})(1 + R_{P,2}) - 1 \approx R_{P,1} + R_{P,2}$
3. $\mu_P = \widehat{\omega}_1'(\mu_1 - r_1\iota) + r_1 + \widehat{\omega}_2'(\mu_2 - r_2\iota) + r_2$
4. $\quad\;\; = (\widehat{\omega}_1 + \widehat{\omega}_2)'(\mu - r\iota) + 2r$
5. $\sigma_P^2 = \widehat{\omega_1}'\Sigma_1\widehat{\omega_1} + \widehat{\omega_2}'\Sigma_2\widehat{\omega_2} = \widehat{\omega_1}'\Sigma\widehat{\omega_1} + \widehat{\omega_2}'\Sigma\widehat{\omega_2}$

Optimal portfolio

1. $L = \mu_P - \frac{1}{2} z \sigma_P^2 \rightarrow \max_{\alpha_1, \alpha_2}$
2. $\frac{\partial L}{\partial \alpha_i} = \mu - r\iota - z\Sigma\alpha_i = 0$
3. $\alpha_i = \frac{1}{z}\Sigma^{-1}(\mu - r\iota)$

Properties of the portfolio

1. The α_i are identical and we choose a constant portfolio over time
2. The portfolio is also identical to that of short-term investors
3. Hence portfolio choice is myopic

Assumption about risk aversion

1. In deriving the result it has been assumed that the risk aversion remains constant
2. As the wealth changes due to the return received, the risk aversion might change
3. This can only be prevented when using a CARA-utility function

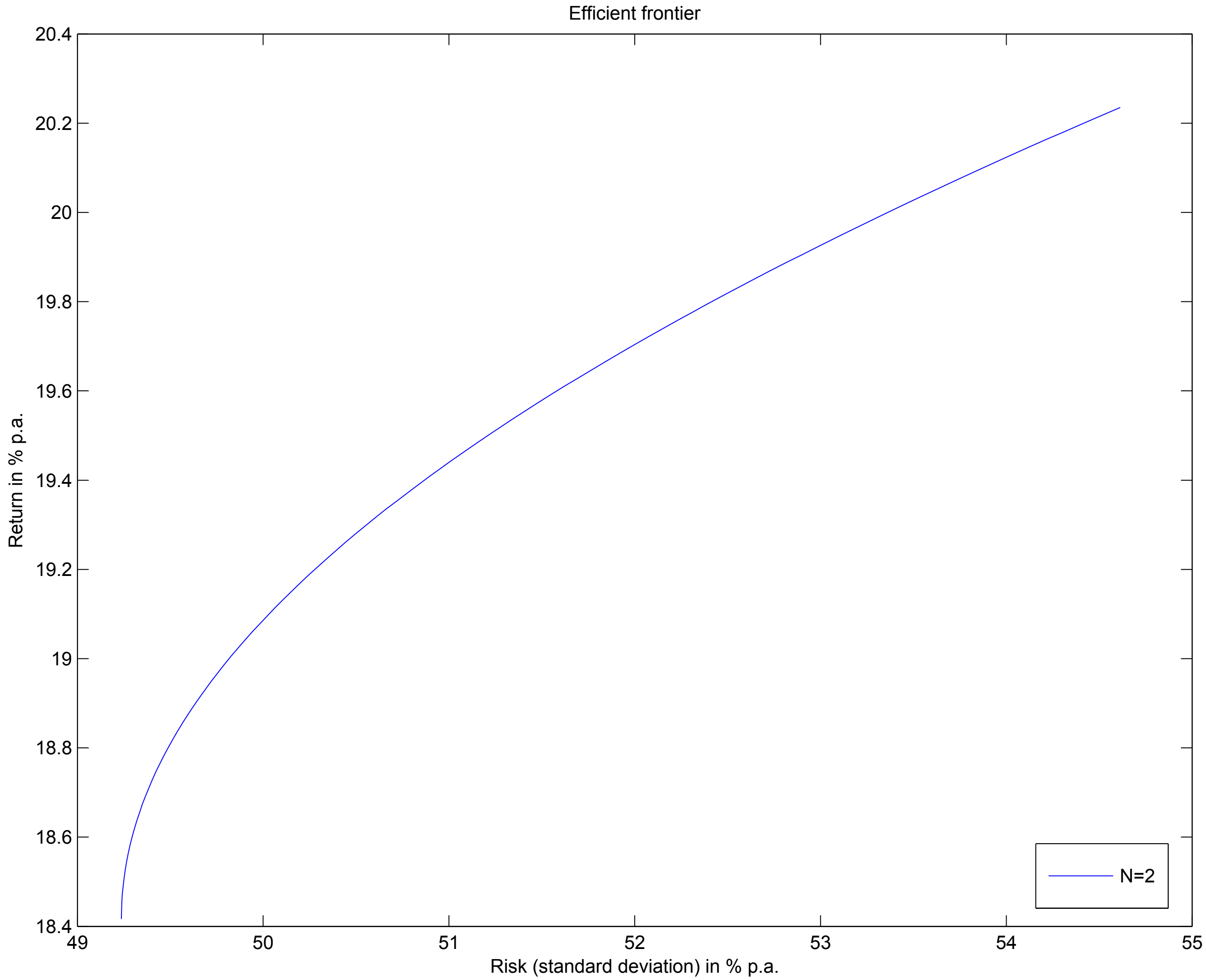
Effect of increasing assets in portfolio selection

- Increasing the number of assets increases the opportunity set
- Efficient frontier moves to the upper left
- Shape of efficient frontier dependent on assets actually chosen
- Once diversified benefits of diversification are limited

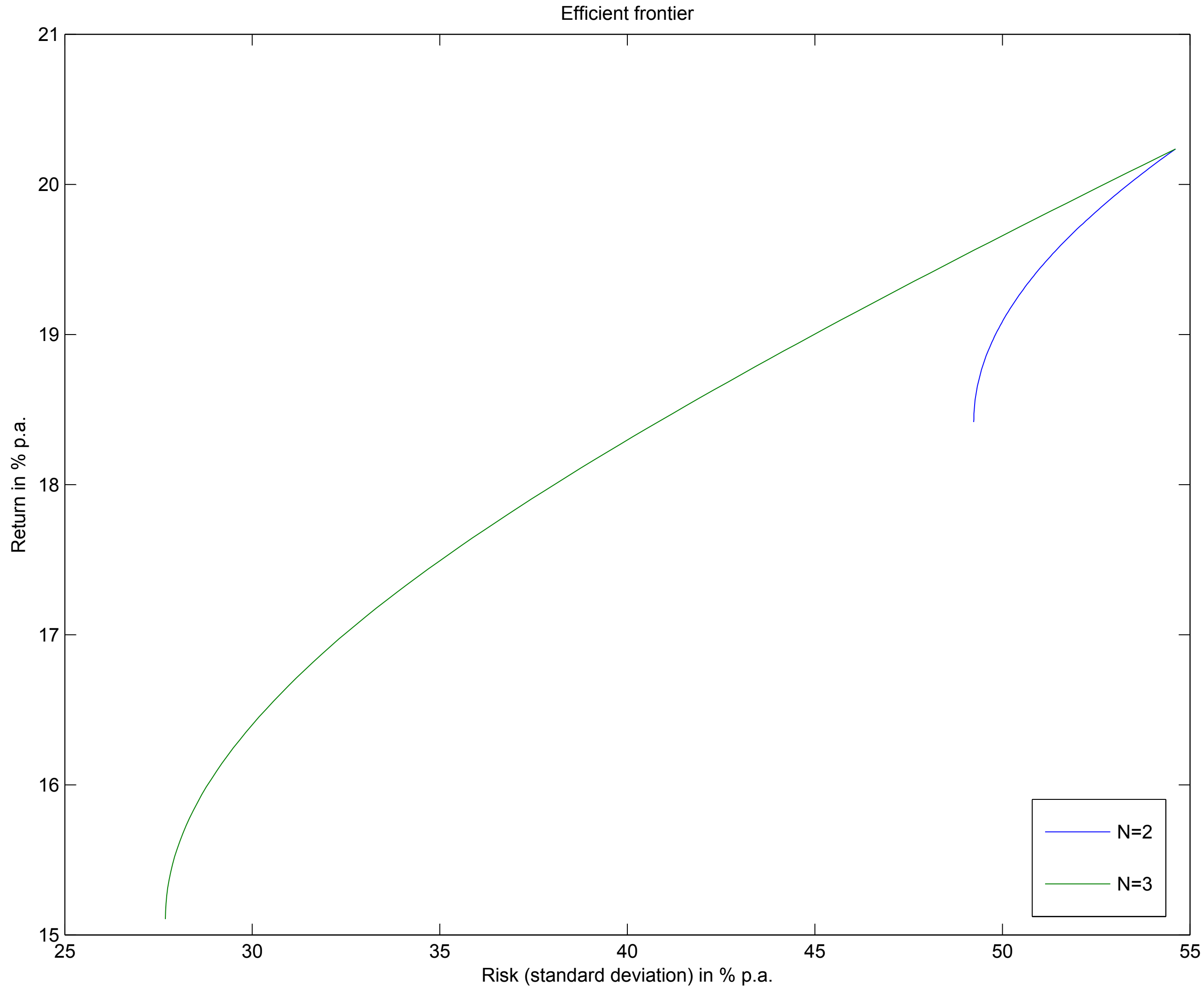
Risk of (un)diversified portfolios

- Assume an equally weighted portfolio
- $\sigma_P^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij} = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\sigma_{ij} = \frac{1}{N}\overline{\sigma}_i + \frac{N(N-1)}{N^2}\overline{\sigma}_{ij}$
- $\rightarrow_{N\rightarrow+\infty} \overline{\sigma}_{ij}$
- Marginal benefits of diversification reduce as N increases

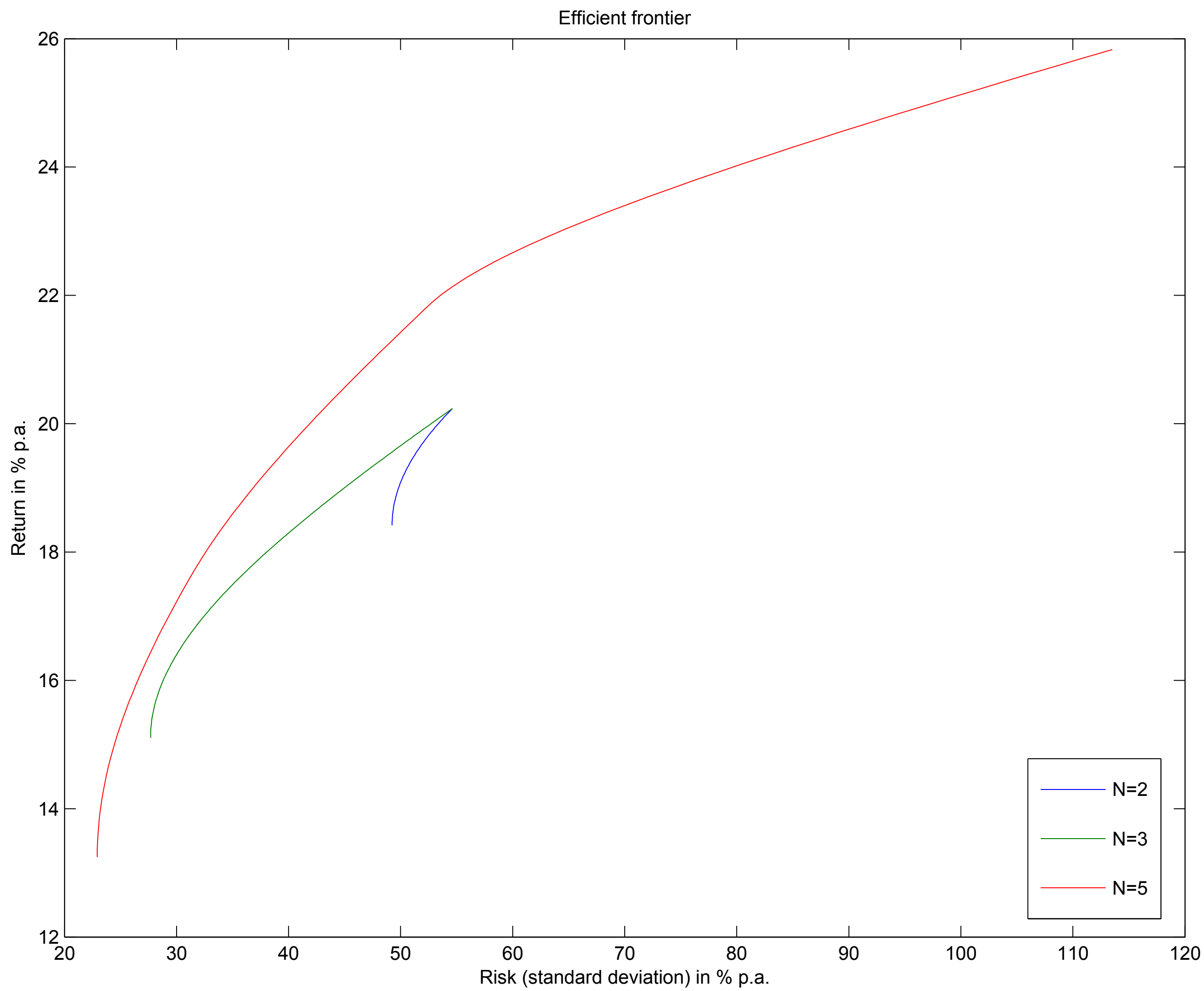
Increasing the number of assets from the S&P500



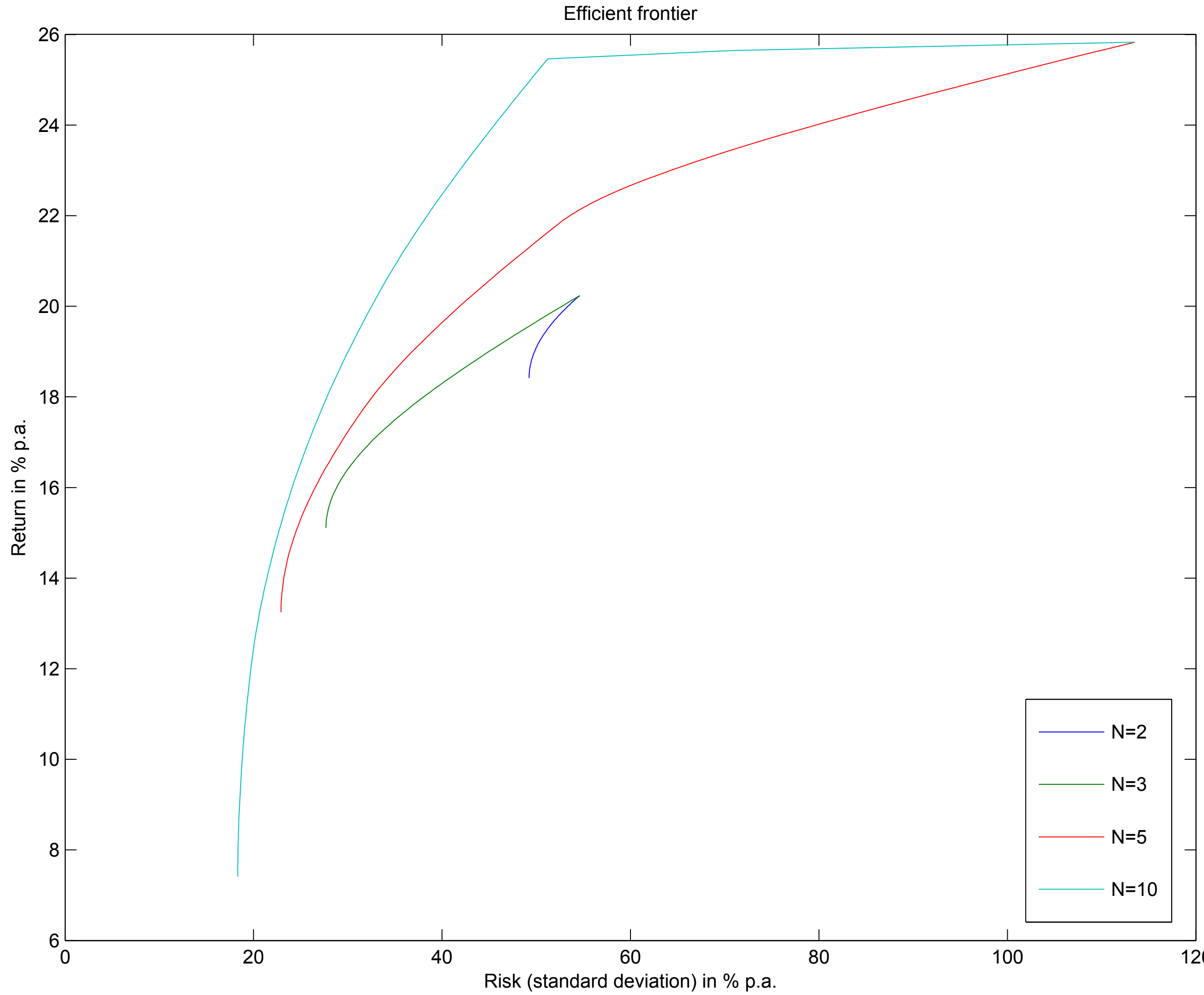
Increasing the number of assets from the S&P500



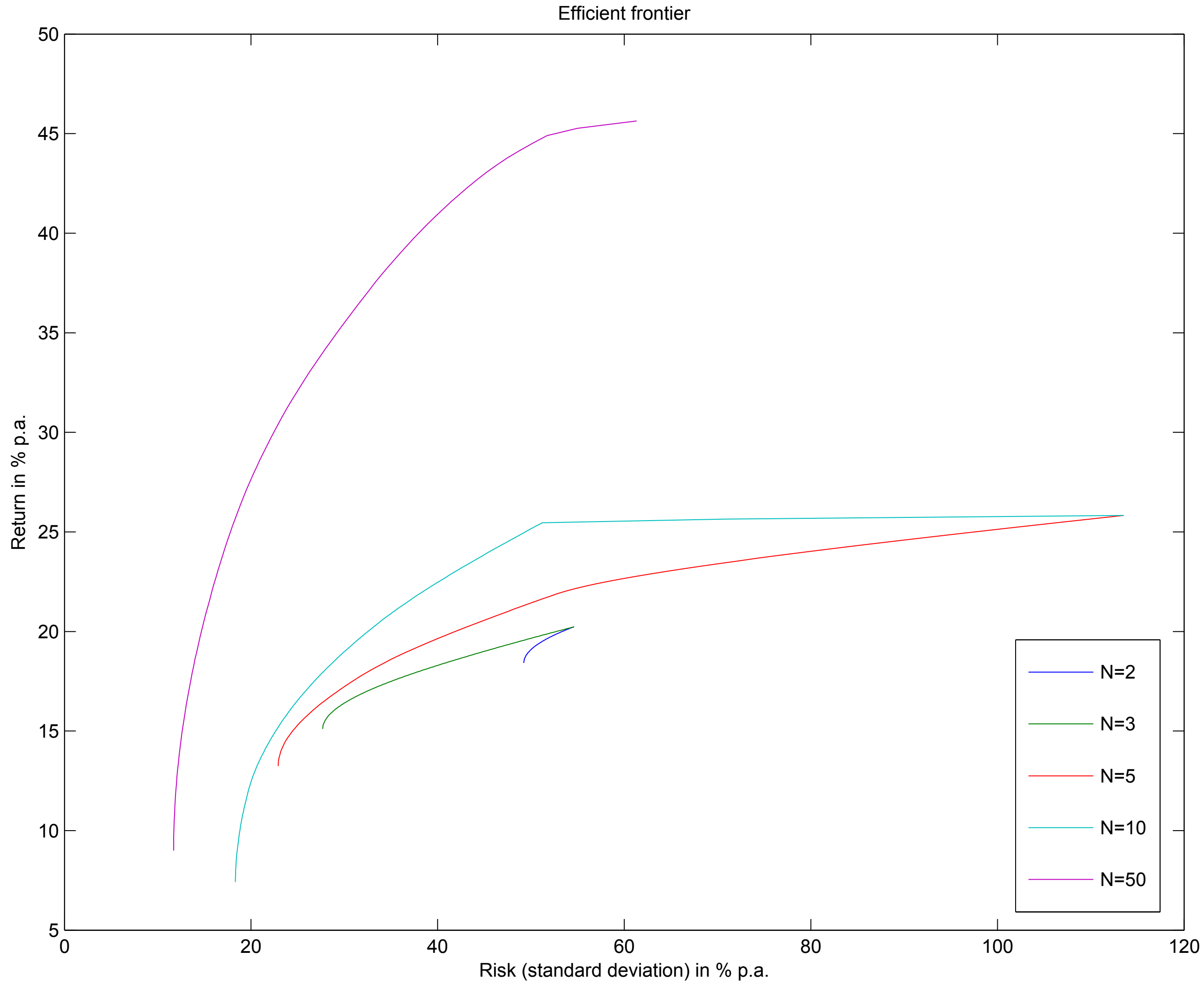
Increasing the number of assets from the S&P500



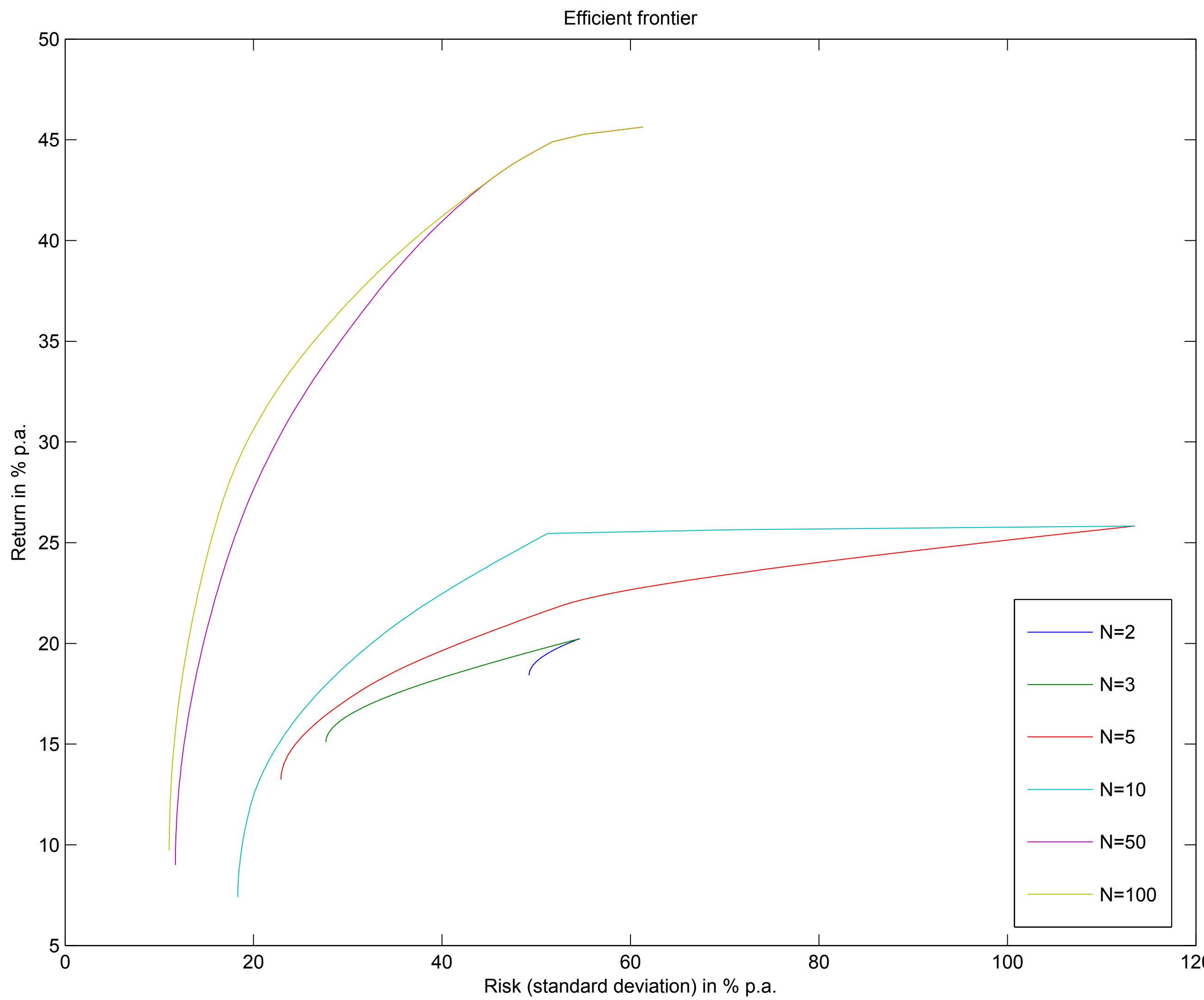
Increasing the number of assets from the S&P500



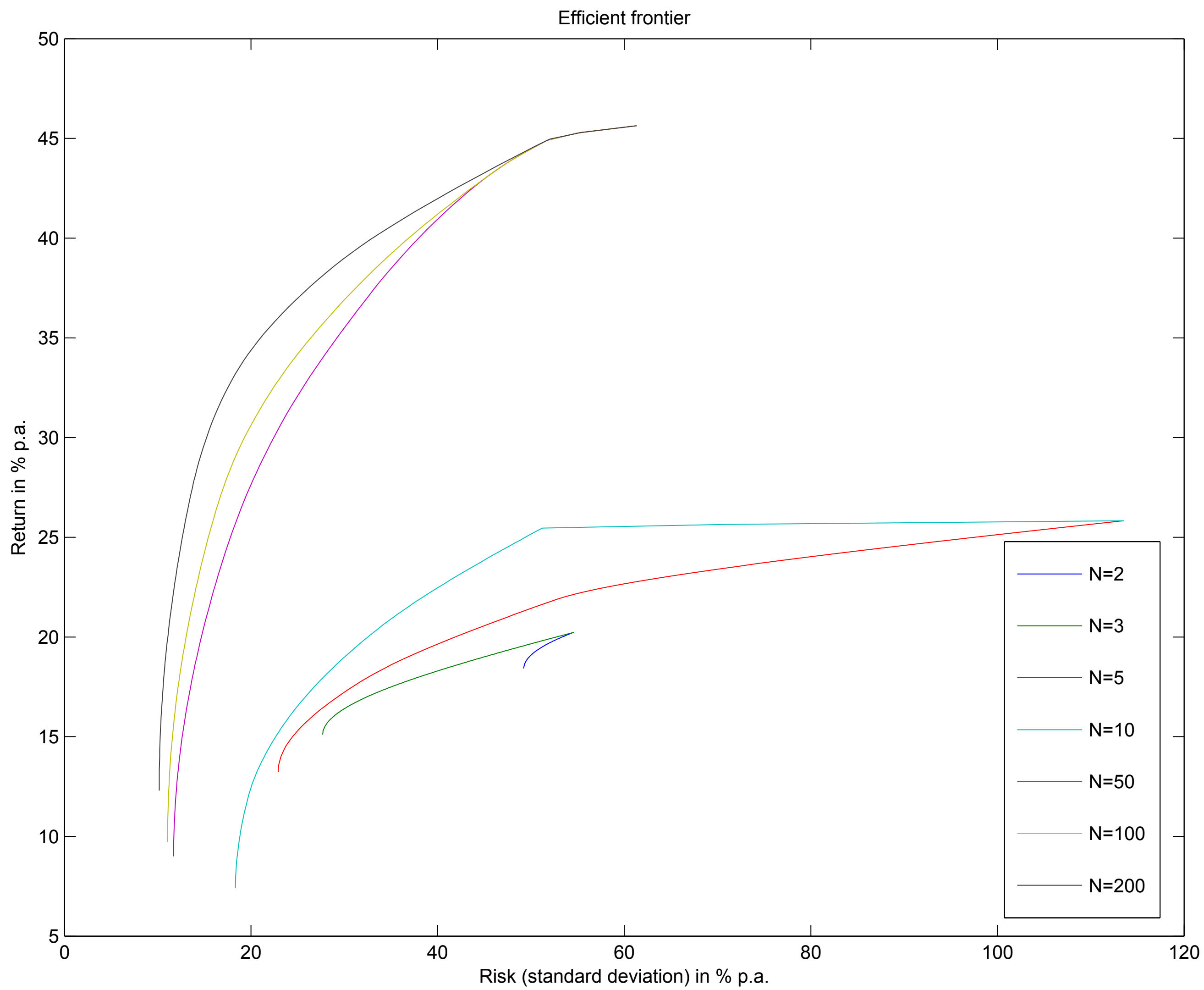
Increasing the number of assets from the S&P500



Increasing the number of assets from the S&P500



Increasing the number of assets from the S&P500



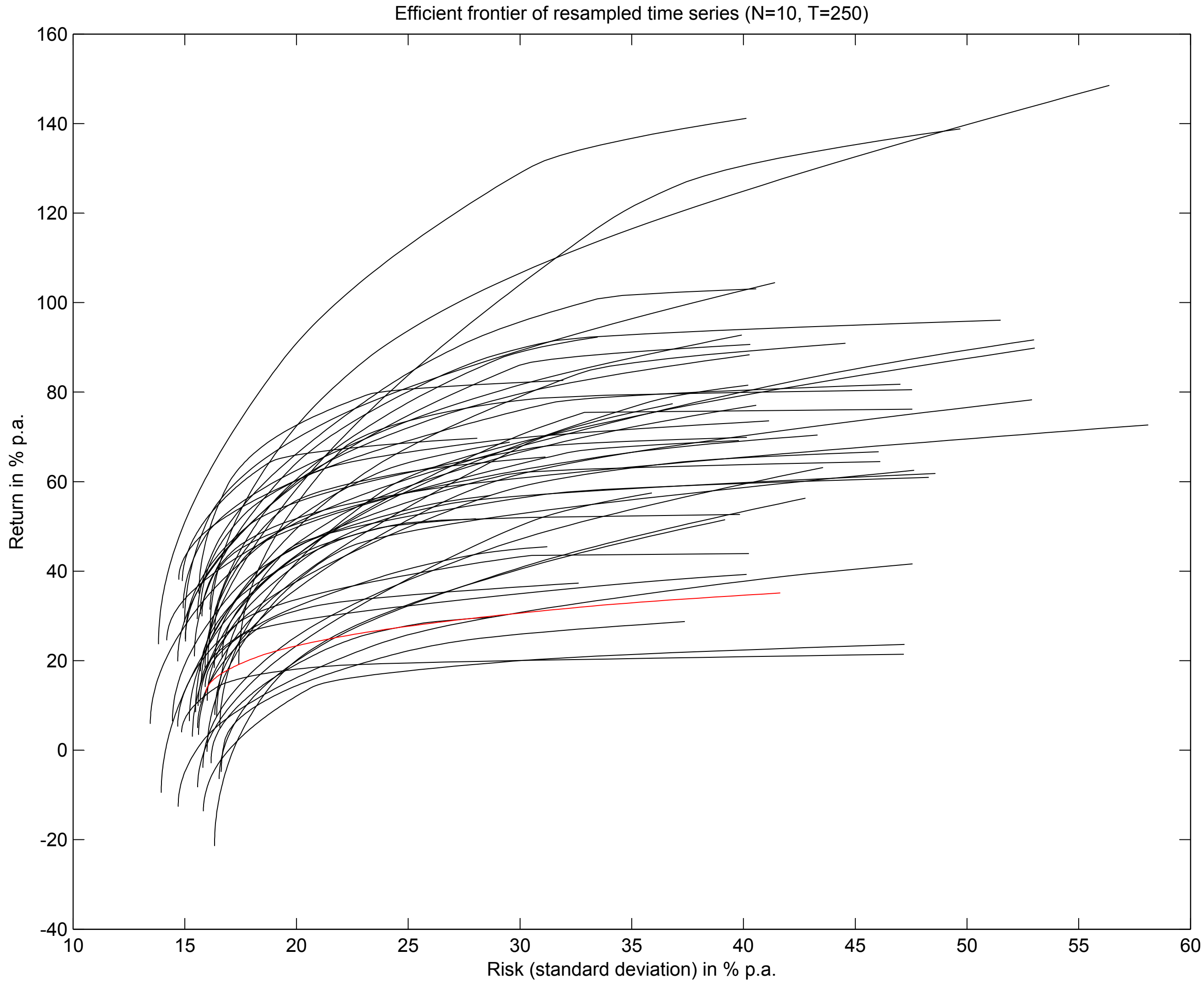
Benefits of portfolio selection theory

- Using portfolio selection theory will improve the utility of investors
- Excessive diversification does not add much benefits
- These benefits need to be weighed against costs

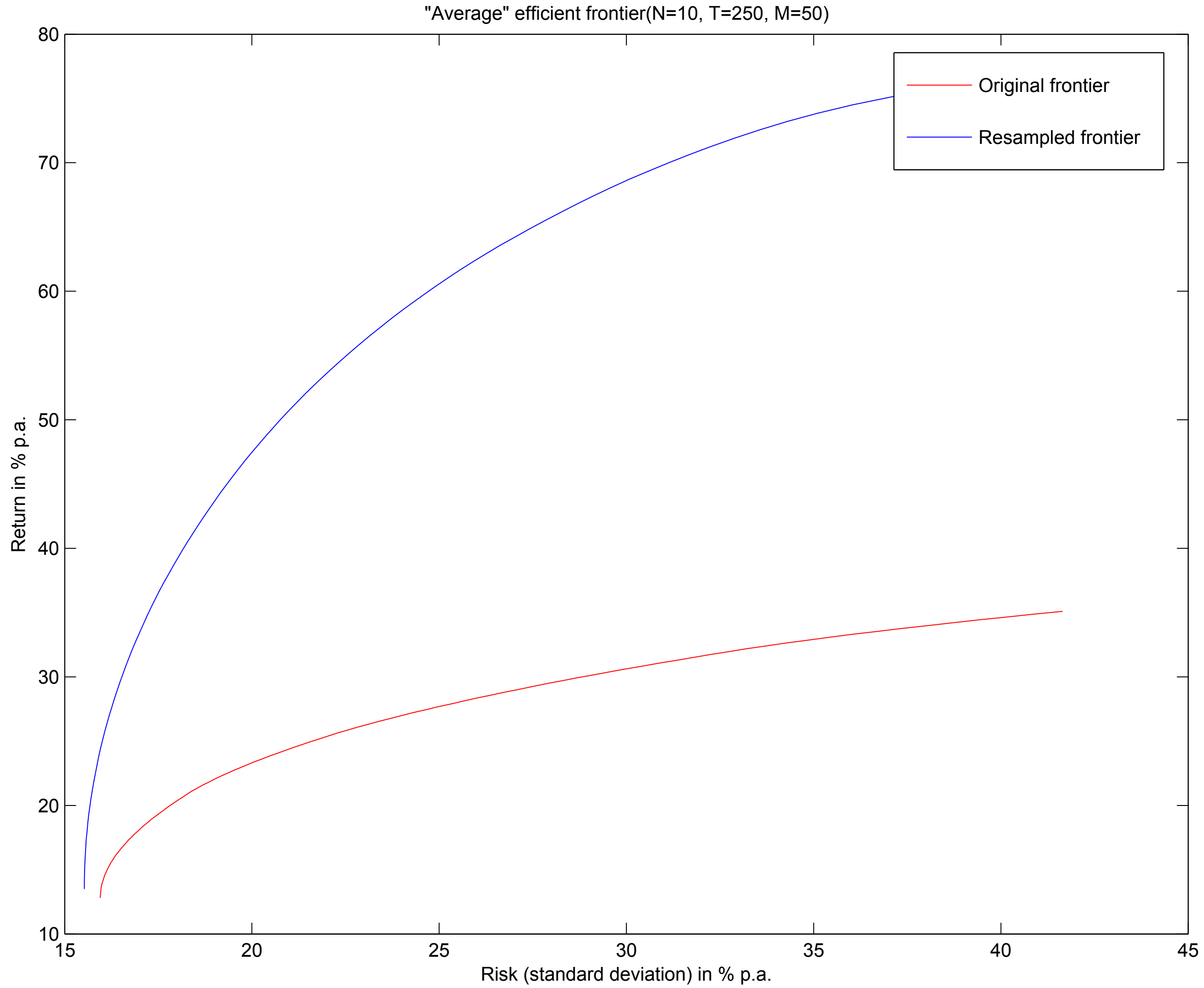
Efficient frontiers are estimates

- Means and covariances are estimates have confidence intervals, so will have efficient frontiers
- Based on the distribution of the estimates, we can obtain a random mean and covariance and calculate an efficient frontier
- We can use Monte-Carlo simulations and get many such efficient frontiers
- We then average these efficient frontiers
- This technique is called resampling

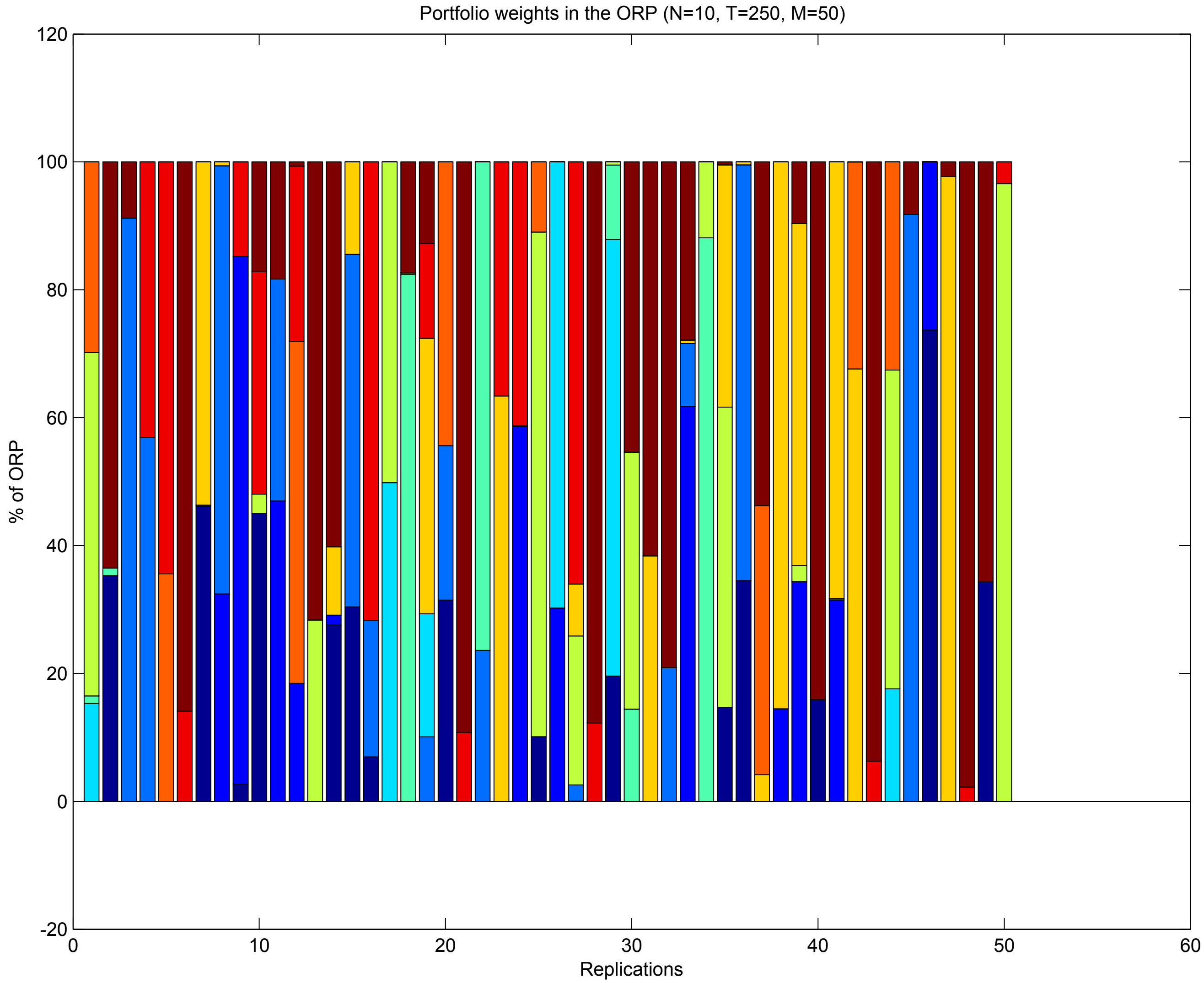
Resampled efficient frontiers



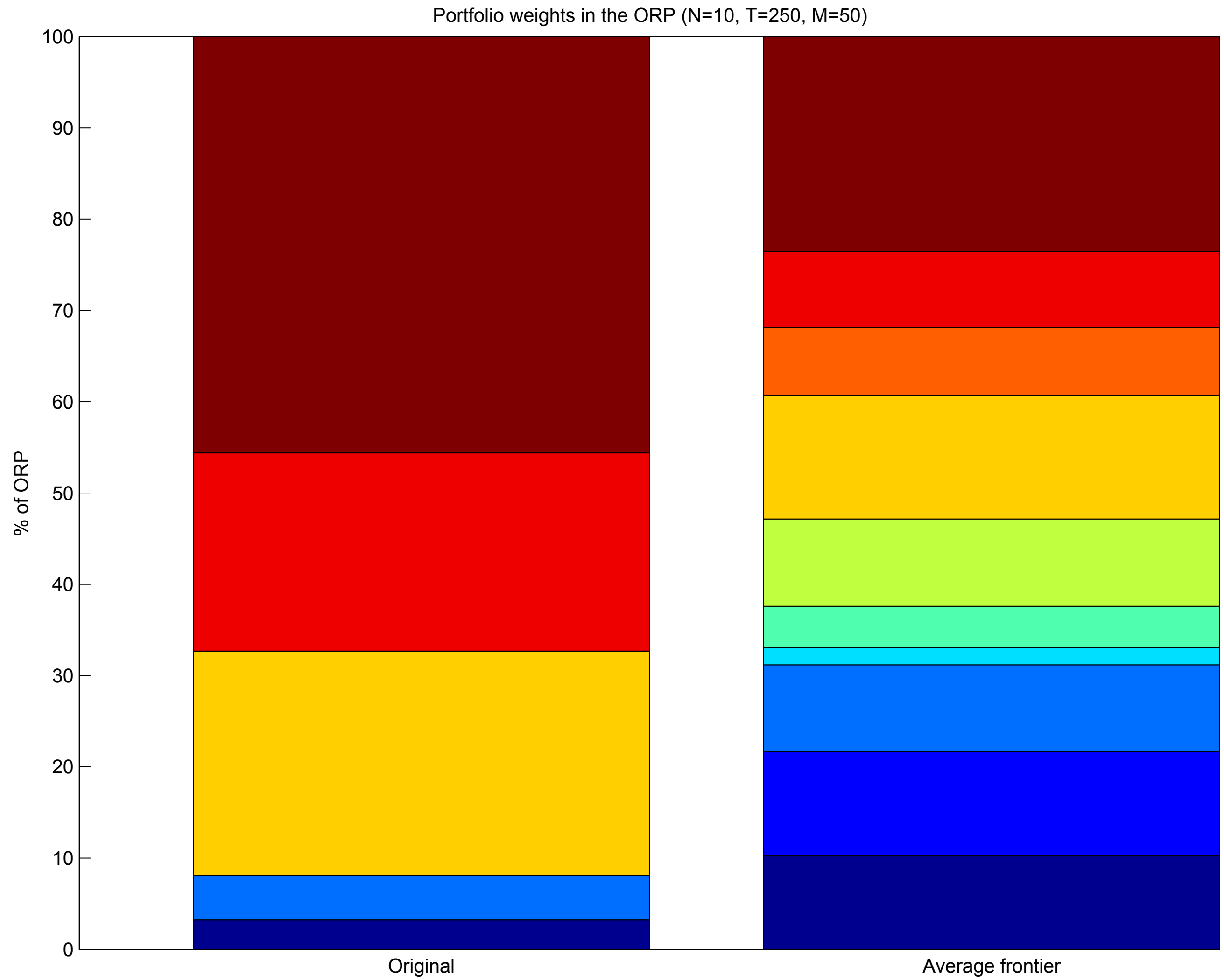
Average frontier



ORP weights in different frontiers



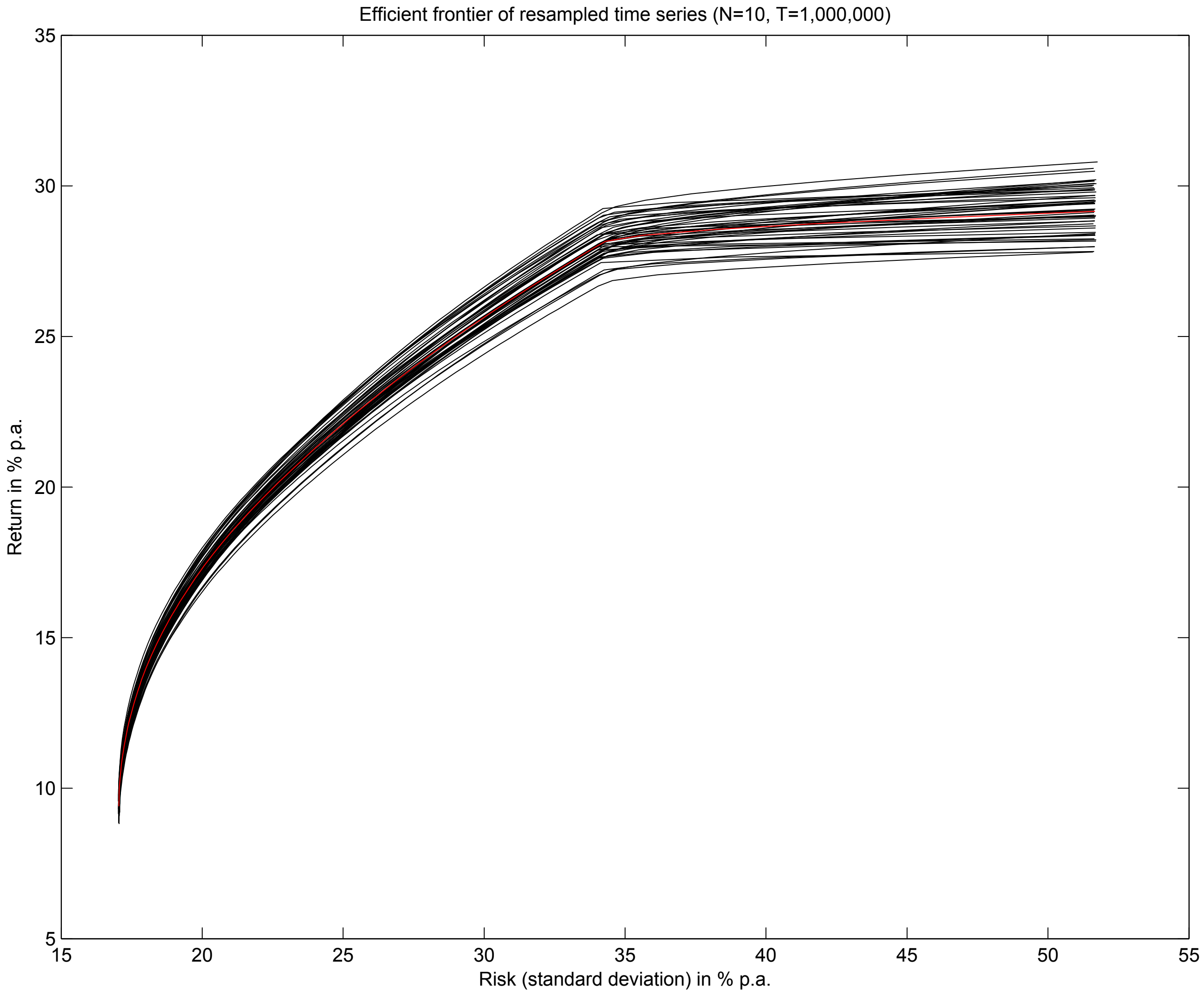
ORP of resampled frontier



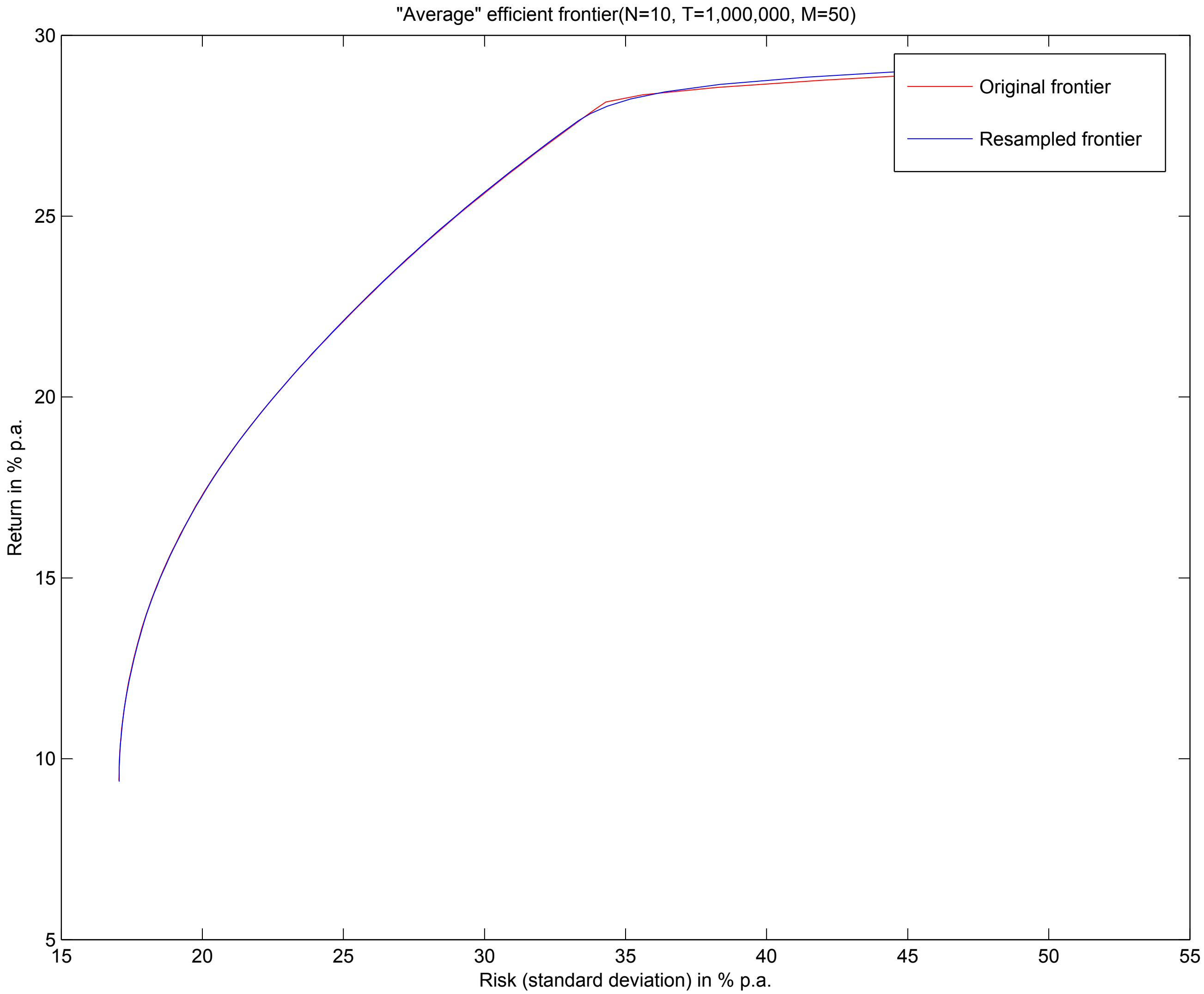
Increasing sample sizes and stocks considered

- Increasing the sample number (number of repetitions) does not reduce the bias
- Increasing the number of stocks does not reduce the bias

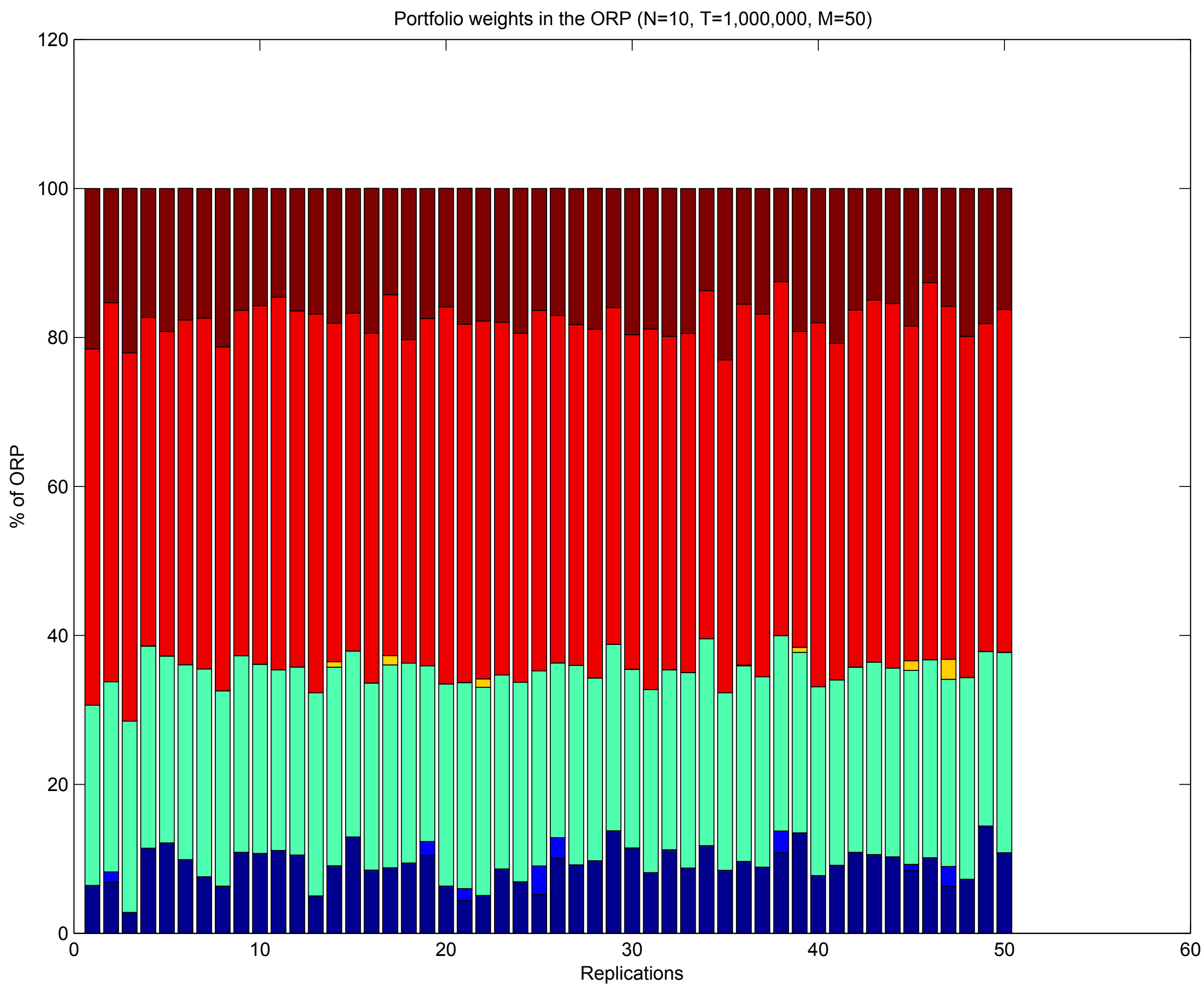
Increasing the length of the time series



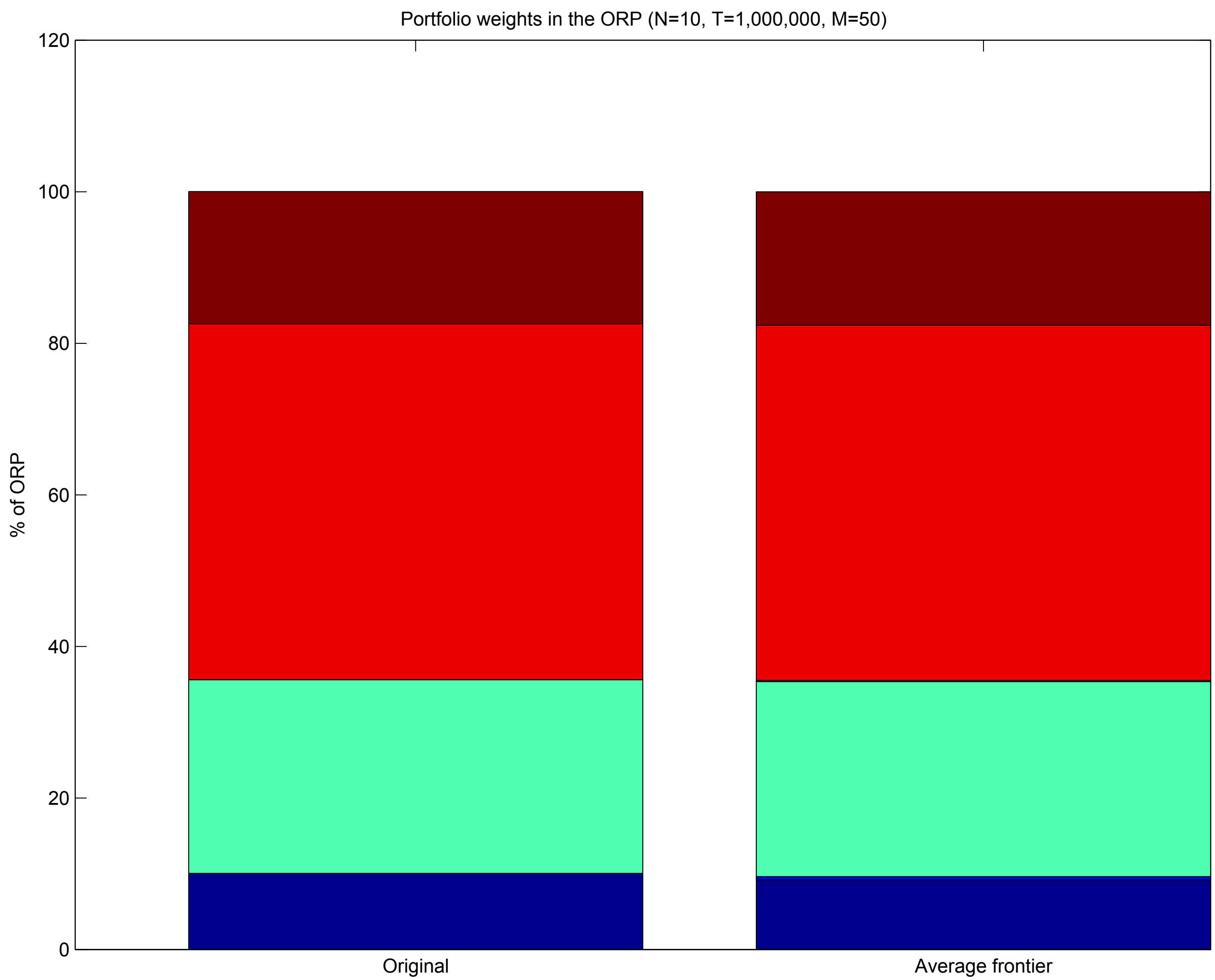
Average efficient frontiers



ORP Composition



Average ORP


Portfolio weights in the ORP (N=10, T=1,000,000, M=50)
% of ORP
120
100
80
60
40
20
0
Original
Average frontier

Outlook

1. So far we only considered the investment decision
2. Consumers, however, also have to make consumption decisions
3. the coming lectures will include these elements into the portfolio decisions

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Lecture 4 - Consumption and portfolio choice

Structure of this lecture

1. Power utility of consumption
2. Epstein-Zin utility function
3. The intertemporal budget constraint
4. Optimal portfolio choice
5. Outlook

Importance of consumption

- The aim of investment is usually to increase future consumption
- Objective of investors is therefore not to maximize expected utility of final wealth but expected utility of future consumption
- We assuming investors to have a time-separable power utility function

Objective function

- $E_t\left[\sum_{i=0}^{+\infty}\rho^i U(C_{t+i})\right] = E_t\left[\sum_{i=0}^{+\infty}\rho^i \frac{C_{t+i}^{1-\gamma}}{1-\gamma}\right] \rightarrow \max_{\{C_{t+i}\}}$
- $W_{t+1} = (W_t - C_t)(1 + R_{t+1})$

First order condition

- Using dynamic programming we can get the first order condition as
- $U'(C_t) = E_t \left[\rho U'(C_{t+1})(1 + R_{t+1}) \right]$
- Marginal consumption today has to equal the expected marginal consumption tomorrow

Stochastic discount factor

- Using the power utility function $U'(C_t) = C_t^{-\gamma}$ we get
- $1 = E_t\left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1 + R_{t+1}\right]$
- $\frac{1}{1+R_f} = E_t\left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$
- $\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ is the stochastic discount factor

The problem with power utility functions

- Risk aversion determines the elasticity of intertemporal substitution
- $\psi = \frac{1}{\gamma}$
- Using the Epstein-Zin utility function eliminates this link
- $U_t = \left((1-\rho) C_t^{\frac{1-\gamma}{\theta}} + \rho E_t \left[U_{t+1}^{1-\gamma} \right]^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\gamma}}$
- $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$

Interpretation of the utility function

- Utility is a weighted average of the present consumption and future expected utility, i.e. future consumption
- The weights are determined by the time preferences (ρ) and by the risk aversion to uncertainty in future consumption (γ)

First order condition

- The budget constraint is $W_{t+1} = (W_t - C_t)(1 + R_p)$
- $1 = E_t \left[\rho^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} (1 + R_p)^{-\theta} \right]$

Approximation of the condition

- Taking logs we get
- $E_t\left[\ln\frac{C_{t+1}}{C_t}\right]=\psi\ln\rho+\psi E_t[R_p]+\frac{\theta}{2\psi}Var\left[\ln\frac{C_{t+1}}{C_t}-\psi R_P\right]$
- The last term arises from approximating $\ln E_t[x]$ by $E_t[\ln x]$ as $E[\ln x]=\mu\Leftrightarrow\ln E[x]=\mu-\frac{1}{2}\sigma^2$ for normal distributions

Consumption growth

Consumption grows over times at a higher rate if

- expected return are higher: more wealth is accumulated
- the investment risk is higher: precautionary savings initially let us start from a lower base

Risk premium

- This result can be used to derive the risk premium of an asset
- $\mu - r + \frac{1}{\sigma^2} = \theta \frac{Cov[r_i, \Delta c_{t+1}]}{\psi} + (1-\theta) Cov[r_i, r_p]$
- The risk premium depends on the covariance of the asset with consumption growth and the market portfolio

Rewriting the budget constraint

- $W_{t+1} = (W_t - C_t)(1 + R_p)$
- $\frac{W_{t=1}}{W_t} = (1 + R_p)\left(1 - \frac{C_t}{W_t}\right)$
- $\Delta \ln W_{t+1} = R_p + \ln\left(1 - e^{\ln C_t - \ln W_t}\right)$

Taylor approximation

- The second term can be approximated as
- $\ln\left(1 - e^{\ln C_t - \ln W_t}\right) = \ln\delta + (1-\delta)\ln\frac{1-\delta}{\delta} + \underbrace{\left(1 - \frac{1}{\delta}(\ln C_t - \ln W_t\right)}_{k}$
- $\delta = 1 - e^{E_t[\ln C_t - \ln W_t]}$ is the fraction of reinvested wealth
- Inserting gets us finally $\Delta \ln W_{t+1} = R_p + k + \left(1 - \frac{1}{\rho}\right)(\ln C_t - \ln W_t)$

Rewriting the wealth growth

- $\Delta \ln W_{t+1} \equiv \Delta \ln C_{t+1} + (\ln C_t - \ln W_t) - (\ln C_{t+1} - \ln W_{t+1})$
- is a difference equation on $\ln C_t - \ln W_t$ which we can solve forward as
- $\ln C_t - \ln W_t = \sum_{i+1}^{+\infty} \delta^i (R_{p,t+i} - \Delta \ln C_{t+i}) + \frac{\delta k}{1-\delta}$

Unexpected changes in wealth

- We can now take expectations and make a number of transformations
- In the end we get
- $\Delta \ln C_{t+1} - E_t[\Delta \ln C_{t+1}] =$
  $(E_{t+1} E_t) \left[\sum_{j=0}^{+\infty} \delta^j R_{p,t+1+j} - \sum_{j=0}^{+\infty} \delta^j \Delta \ln C_{C+1+j}\right]$

Interpretation of results

An unexpected change in consumption is the result of a

- change in expected returns
- change in future consumption growth

Bringing the utility function back in

- So far we have only manipulated the budget constraint
- Epstein-Zin utility was
  $E_t\left[\ln\frac{C_{t+1}}{C_t}\right] = \psi\ln\rho + \psi E_t[R_p] + \frac{\theta}{2\psi}Var\left[\ln\frac{C_{t+1}}{C_t} - \psi R_P\right]$
- If the variance remains constant we can write this as
- $E_t\left[\ln\frac{C_{t+1}}{C_t}\right] = \varphi + \psi E_t[R_p]$

Consumption ratio

- Inserting this into the budget constraint we get
- $\ln C_t - \ln W_t = \frac{\delta(k-\varphi)}{1-\delta} + (1-\psi)E_t\left[\sum_{j=1}^{+\infty} \delta^i R_{p,t+j}\right]$

Income and substitution effect

- If portfolio returns are high, consumption can be *higher* (income effect)
- This arises from the increased income the high returns generate
- If portfolio returns are high, consumption can be *lower* (substitution effect)
- This arises from the incentive to delay consumption in order to benefit from the current high returns and increase consumption later
- if $\psi < 1$ the income effect dominates, else the substitution effect

Changes in consumption

- Inserting the above relationship we get
- $\Delta \ln C_{t+1} - E_t[\Delta \ln C_{t+1}] =$
  $R_{p,t+1} - E_t[R_{P,t+1}] + (1-\psi)(E_{t+1} - E_t)\left[\sum_{j=0}^{+\infty} \delta^i R_{P,t+1+j}\right]$

Interpretation

Consumption changes if

- the portfolio return is unexpectedly high (or low)
- expectations about future portfolio returns are revised

If $\psi < 1$ consumption increases with returns, i.e. the income effect dominates

Asset pricing

- Using the results on asset pricing we get
- $\mu - r + \frac{1}{\sigma^2} =$
  $\gamma Cov[R, R_p] + (1-\gamma) Cov\left[R, (E_{t+1} - E_t)\left[\sum_{j=0}^{+\infty} \delta^i R_{P,t+1+j}\right]\right]$

Assumptions

- Variances and risk premia are constant
- If the risk premium is constant, any changes in the returns are due to changes in the risk free rates
- $(E_{t+1} - E_t)\left[\delta^i R_{P,t+1+j}\right] = (E_{t+1} - E_t)\left[\delta^i R_{f,t+1+j}\right]$

Single risky asset

- For simplicity assume we only have a single risky asset with return R
- If we invest a fraction α into this risky asset and the remainder into the risk free asset we get
- $Cov_t[R, R_p] = \alpha\sigma^2$

Optimal portfolio

- Based on these results we can now derive the optimal portfolio using the asset pricing relationship and first order conditions
- $\alpha = \frac{1}{\gamma}\frac{\mu - r_r + \frac{1}{2}\sigma^2}{\sigma^2} + \left(1 - \frac{1}{\gamma}\right)\frac{Cov\left[R, -(E_{t+1} - E_t)\left[\delta^i R_{f,t+1+j}\right]\right]}{\sigma^2}$

Interpreting the optimal portfolio

The weight of the risky asset depends on

- the risk premium relative to its risk
- his risk aversion
- the covariance of the asset with changes in the risk-free rate

The problem of the risk-free rate

- Short-term treasury bills are not risk-free for long-term investors as interest changes on roll-over dates
- The intermediate value of long-term bonds also fluctuates as the interest rate changes
- Hence no really risk-free asset exists

Full evaluation of the model

- To solve the above model properly we need a model of the term structure
- This allows us to model the covariance term for the optimal portfolio holdings

General properties

For a realistic term structure model we find the following properties

- The more risk averse investors are the less equity they hold
- Long-term bonds serve as a hedging against consumption risk arising from falling interest rates
- Short-term bonds are generally not held much, mostly in times of high inflation as a hedge against this risk

Outlook

- Including the consumption decision gives rise to many complications in deriving the optimal portfolio
- Detailed solutions maybe of limited value for practical applications
- Nevertheless important general principles can be established
- The coming lecture will include labor income which is more realistic for most investors

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Lecture 5 - Portfolio choice with labor income

Structure of this lecture

Financing consumption

- Investors usually do not only rely on their wealth to finance consumption
- in most cases they also have labor income
- additional complication arise from retirement

Present value of future labor income

- If labor income is known over the life time and fixed then
- $H = \sum i = 0^T \rho^j L_j = \frac{1-\rho^T}{1-\rho} L$
- discount rate ρ is risk free

Total wealth

- The total wealth of the investor is
- $\widehat{W} = W + H$
- Wealth consists of the current wealth plus future labor income

Fraction invested into the stock market

- From before we know that $\alpha = \frac{\mu - r}{z\sigma^2}$
- Including future labor income the fraction of the current wealth is
- $\widehat{\alpha} = \alpha \frac{\widehat{W}}{W} > \alpha$

Retirement

- As the investor approaches retirement, T reduces and so does H
- Implication is that less and less is put into the stock market as people become older
- This does not arise from a changed time horizon or changed preferences but changed circumstances

Uncertain labor income

- $L \sim N(\mu_L, \sigma_L^2)$
- $Cov[L, R] = \sigma_{Lr}$

Objective function

- Investors maximize expected utility of future consumption
- $\max_{\alpha} E\left[\rho \frac{C^{1-\gamma}}{1-\gamma}\right]$
- $C = W(1 + R_p) + L$
- $R_p = \alpha(R - r) + r$

Budget constraint

- $\ln C = \theta + \varphi(\ln W + R) + (1 - \varphi) \ln L$
- $\varphi \approx \frac{(1+R_p)\frac{W}{L}}{1+(1+R_p)\frac{W}{L}}$
- φ is the elasticity of consumption with respect to wealth
- θ is a constant

Determinants of consumption

Consumption increases with

- increasing wealth
- increasing return in the stock market
- increasing labor income

φ determines how strong these relationships are.

Optimal portfolio

- Conduction the maximization we get
- $\alpha = \frac{1}{\varphi}\frac{\mu - r}{\gamma\sigma^2} + \left(1 - \frac{1}{\rho}\right)\frac{\sigma L_r}{\sigma^2}$

Properties of the optimal portfolio

- Optimal portfolio depends on two components
- The *first* component is identical to the myopic portfolio allocation and accounts for the risk and returns of the investment
- The *second* component is negative as $1 < \varphi < 1$ and hedges against the risk of low labor income
- If labor income and stock market returns are correlated, stock market holdings are reduced as labor income and stock market income are substitutes

Differences to single-period model

- In addition to the optimal amount to invest into the stock market (in each period) we also need to determine the optimal consumption (in each period)
- $E_t\left[\sum_{i=0}^{+\infty} \rho^i \frac{C_{t+i}^{1-\gamma}}{1-\gamma}\right] \rightarrow \max_{\{\alpha_t, C_t\}}$
- Budget constraint: $W_{t+1} = (W_t + L_t - C_t)(1 + R_{P,t+1})$

Retirement and death

- After retirement investors receive $L_t = 0$
- The timing of retirement is uncertain and happens with probability π^r in each time period
- The investor might die with probability π^d in each time period

Investment after retirement

- After retirement investment is identical to myopic investors
- $\alpha_r = \frac{\mu - r}{\gamma\sigma^2}$
- The probability of death does not enter the equation as it only affects the time horizon, which we saw to be irrelevant

Investment prior to retirement

- The result turns out to be
- $\alpha_L = \theta_1 \frac{\mu - r}{\gamma \sigma^2} + \theta_2 \frac{\sigma_{Lr}}{\sigma^2}$
- θ_1 and θ_2 are constant that include π_r

Comparing results

- Result of single period and multiple periods are essentially identical
- Reductions in current wealth can be compensated for by saving less, thus reducing consumption by less

Labor income risk

- Higher labor income risk reduces the amount held in shares
- Higher labor income risk reduces consumption, i.e. the total wealth increases
- Saving increases to reduce exposure to labor income risk

Life cycle investments

- Young investors have a high future labor income which is low risk, so most wealth is invested into shares
- wealth increases relative to income
- As wealth increases the higher exposure to stock market risk leads to reduced share invested into the stock market
- After retirement, cash holding increase even more and wealth reduces due to consumption

Outlook

- We so far only considered long-term investment strategies (strategic asset allocation)
- Based on information, investors will deviate from these long-term strategies (tactical asset allocation)
- The coming lectures will explore the optimal portfolios with such considerations

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Lecture 6 - Tactical asset allocation

Structure of this lecture

1. The separation theorem
2. The optimal tactical bet
3. Outlook

Strategic vs. tactical asset allocation

- Strategic asset allocation is concerned about the long-term investment into several asset classes
- Tactical asset allocation is concerned about short-term, i.e. one time period lasting, deviations from this long-term allocation
- Tactical asset allocation is usually based on information about future performance of the assets

Optimal portfolio

- $\omega'\mu - \frac{1}{2}z\omega'\Sigma\omega \rightarrow \max_\omega$
- $\omega'\iota = 1$
- $L = \omega'\mu - \frac{1}{2}z\omega'\Sigma\omega - \lambda(\omega'\iota - 1) \rightarrow \max_\omega$

First order conditions

- $\frac{\partial L}{\partial \omega} = \mu - z\Sigma\omega - \lambda\iota = 0$
- $\frac{\partial L}{\partial \lambda} = 1 - \omega'\iota = 0$
- $\omega = \frac{1}{z}\Sigma^{-1}(\mu - \lambda\iota)$
- $\omega'\iota = \left(\frac{1}{z}\Sigma^{-1}(\mu - \lambda\iota)\right)'\iota = 1$
- $\lambda = \frac{\iota'\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\iota} - \frac{z}{\iota'\Sigma^{-1}\iota}$

Optimal portfolio

- $\omega = \left(1 - \frac{\iota'\Sigma^{-1}\mu}{z}\right)\frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota} + \frac{\iota'\Sigma^{-1}\mu}{z}\frac{\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\iota}$
- This is a combination of two portfolios, $\frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota}$ and $\frac{\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\iota}$ with their weights

Determining the MRP

- $\sigma_P^2 = \omega' \Sigma \omega \rightarrow \min_\omega$
- $\omega' \iota = 1$
- $\frac{\partial L}{\partial \omega} = \Sigma \omega - \lambda \iota = 0$
- $\omega = \lambda \Sigma^{-1} \iota$
- $\omega' \iota = \lambda \iota' \Sigma^{-1} \iota = 1$
- $\lambda = \frac{1}{\iota' \Sigma^{-1} \iota}$
- $\omega_{MRP} = \frac{\Sigma^{-1} \iota}{\iota' \Sigma^{-1} \iota}$

Re-assessing the optimal portfolio

- $\omega = \left(1 - \frac{\iota'\Sigma^{-1}\mu}{z}\right)\omega_{MRP} + \frac{\iota'\Sigma^{-1}\mu}{z}\frac{\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\iota}$
- $\quad = \omega_{MRP} - \frac{\iota'\Sigma^{-1}\mu}{z}\frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota} + \frac{\Sigma^{-1}\mu}{z}$
- $\quad = \omega_{MRP} + \frac{\Sigma^{-1}}{z}\left(\mu - \iota\frac{\iota'\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\iota}\right)$

Non-equilibrium returns

- Strategic asset allocation is made using the long-term expected returns, i.e. equilibrium expected returns $\widehat{\mu}$
- Short-term investors might have a different opinion, μ

Rewriting the optimal portfolio

- $\omega = \omega_{MRP} + \frac{\Sigma^{-1}}{z}\left((\mu - \widehat{\mu}) + \widehat{\mu}\iota\frac{\iota'\Sigma^{-1}((\mu-\widehat{\mu})+\widehat{\mu}}{\iota'\Sigma^{-1}\iota}\right)$
- $\quad = \ldots$
- $\quad = \omega_{MRP} + \frac{1}{z}\frac{\Sigma^{-1}}{\iota'\Sigma^{-1}\iota}\left(\widehat{\mu}\iota' - \iota\widehat{\mu}'\right)\Sigma^{-1}\iota +$
  $\frac{1}{z}\frac{\Sigma^{-1}}{\iota'\Sigma^{-1}\iota}\left((\mu - \widehat{\mu})\iota' - \iota(\mu - \widehat{(\mu)})'\right)\Sigma^{-1}\iota$
- $\quad = \omega_{MRP} + \omega_S + \omega_T$

Strategic bet

- $\omega'_S\iota = \omega'_T\iota = 0$
- The optimal portfolio consists of 3 elements, the MRP, the long-term strategic portfolio and the tactical portfolio
- $\widehat{\mu}\iota' - \iota\widehat{\mu}' = [\widehat{\mu}_i - \widehat{\mu}_j]$
- The strategic investments are determined by the excess returns of the assets against each other, weighted by the risk and risk aversion
- The strategic bet adjusts the MRP to take into account the higher equilibrium returns of some assets

Tactical bet

- $(\mu - \widehat{\mu})\iota' - \iota(\mu - \widehat{\mu})' = [(\mu_i - \mu_j) - (\widehat{\mu}_i - \widehat{\mu}_j)]$
- This denotes the deviation of excess returns from their equilibrium values
- As they should adjust quickly, we interpret them as tactical

Portfolio decomposition

- We can thus distinguish 3 elements of the portfolio, the MRP, strategic bet and tactical bet
- The composition of these portfolios are independent of preferences
- In standard portfolio theory $\mu = \widehat{\mu}$ and the tactical bet does not exist
- The total portfolio is efficient

The tactical bet

- Assume strategic and tactical bets are determined separately
- $\mu_\theta = \theta' \mu$
- $\sigma_\theta^2 = \theta' \Sigma \theta$
- $\mu_\theta - \frac{1}{2} z_T \sigma_\theta^2$
- $\theta' \iota = 0$

Optimal bet

- $\frac{\partial L}{\partial \theta} = \mu - z_T \Sigma \theta - \lambda \iota = 0$
- $\theta = \frac{1}{z_T} \Sigma^{-1} (\lambda \iota - \mu)$
- $\theta' \iota = \frac{1}{z_T} (\lambda \iota' - \mu') \Sigma^{-1} \iota = 0$
- $\lambda = \frac{\mu' \Sigma^{-1} \iota}{\iota' \Sigma^{-1} \iota}$
- $\theta = \frac{1}{z_T} \Sigma^{-1} \frac{\mu' \iota - \iota \mu'}{\iota' \Sigma^{-1} \iota} \Sigma^{-1} \iota$

Evaluation of the optimal bet

- The optimal tactical bet is identical to the one we had before
- We can allow for a different risk aversion for tactical bets
- The reason is that the tactical bet was determined independently of the properties of the strategic portfolio

Outlook

- We have established the principle of a tactical bet
- The coming lectures will explore the optimal tactical bets in more detail

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Lecture 7 - Performance measurement

Structure of this lecture

1. Jensen's α
2. Optimal tactical bets
3. Optimal bias
4. Outlook

Performance evaluation

- The performance of an investment needs to be evaluated relative to a benchmark
- The risk of the investment needs to be taken into account
- A standard benchmark to use would be the CAPM
- $\widehat{\mu} = r + \beta(\mu_M - r)$

Comparing actual performance with the benchmark

- Using the benchmark from the CAPM we can define the performance as
- $\alpha = R_p - \widehat{\mu}$
- This performance measure is calledd Jensen's α

Measuring the performance of tactical asset allocation

- The natural benchmark for tactical asset allocation would be the strategic portfolio
- $\alpha = R_T - R_S$
- $R_S = \omega R + (1 - \omega) r$
- $R_T = (\omega + \theta) R + (1 - (\omega + \theta)) r$
- $\alpha = \theta(R - r)$

Tracking error

- The risk profiles of the tactical bet can be very different from that of the strategic asset allocation
- The α as defined above does not include the risks of the strategies
- The risk can be measures by the tracking error
- $\vartheta = Var[\alpha]^{\frac{1}{2}}$

Information ratio

- Combining the excess return and the risk, we get the information ratio
- $\kappa = \frac{\alpha}{\vartheta}$
- The information ratio can compare the performance of two investors
- It is very similar to the Sharpe ratio

Signals

- In order to make tactical bets we need information about the future risk premium
- Such information is called a signal ψ
- $R - r = \beta_0 + \beta_1 \psi + \varepsilon$
- $\widehat{\mu} - r = \beta_0 + \beta_1 \widehat{\mu}_\psi$

Regression results

- As this equation is a basic linear regression we get
- $\beta_1 = \frac{\sigma_{R\psi}}{\sigma_{\psi}^2} = \rho_{R\psi}\frac{\sigma_R}{\sigma_\psi}$

Tactical bet

- From the previous models we had the tactical bet as
- $\theta = \frac{1}{\gamma}\Sigma^{-1}\frac{(\mu-\widehat{\mu})\iota'-\iota(\mu-\widehat{\mu})'}{\iota'\Sigma^{(}-1)\iota\Sigma^{(}-1)\iota}$
- If we have only two assets this becomes
- $\theta = \frac{1}{\gamma}\frac{E[R-r]-(\widehat{\mu}-r)}{Var[R-r]}$

Aggressiveness factor

- Define $\widehat{F} = \frac{1}{\gamma} \frac{1}{Var[R-r]}$ as the aggressiveness factor
- $\theta = \widehat{F}\,(E[R-r] - (\widehat{\mu} - r))$

Using the signal

- Inserting from the signal we get
- $\theta = \widehat{F}\left((\beta_0 + \beta_1\psi) - (\beta_0 + \beta_1\widehat{\mu}_\psi)\right)$
- $\quad = \widehat{F}\beta_1(\psi - \widehat{\mu}_\psi)$
- $\quad = F(\psi - \widehat{\mu}_\psi)$
- F is the aggressiveness factor directly applied to the signal

Systematic bias

- Assume we misinterpret the signal systematically:
- $\widehat{\mu} = \beta_0 + \beta_1\psi + \delta + \varepsilon$
- $\theta = \widehat{F}\left(\left(\beta_0 + \beta_1\psi + \delta\right) - \left(\beta_0 + \beta_1\widehat{\mu}_\psi\right)\right)$
- $\quad = F\left(\psi - \widehat{\mu}_\psi + \frac{\delta}{\beta_1}\right)$

Performance of the tactical bet

- $\alpha = \theta(R - r)$
- $\quad = F\left(\psi - \widehat{\mu}_{\psi} + \frac{\delta}{\beta_1}\right)(R - r)$

Average bet

- Over time it obviously is $E[\psi] = \widehat{\mu}_\psi$
- $E[\theta] = F\frac{\delta}{\beta_1} = F\delta\frac{\sigma_\psi^2}{\sigma_{R\psi}}$
- $Var[\theta] = \frac{F^2}{\sigma_\psi^2}$

Average performance

- $E[\alpha] = F\ E\left[\psi(R-r) - \widehat{\mu}_\psi(R-r) + \frac{\delta}{\beta_1}(R-r)\right]$
- $\quad = F\left(\underbrace{E[\psi(R-r)] - \widehat{\mu_\psi}(\mu-r)}_{\sigma_{R\psi}} + \frac{\delta}{\beta_1}(\mu-r)\right)$
- $\quad = F\sigma_{R\psi} + F(\mu-r)\frac{\sigma_\psi^2}{\sigma_{R\psi}}\delta$
- $\quad = \alpha_V + \alpha_b$

Volatility capture

- As the signal provides valuable information and $\sigma_{R\psi}$ measures the value of this information, a high covariance increases the performance
- α_V is called the volatility capture
- The investor can reduce the volatility of his investments through use of the signal

Bias and performance

- α_b is the performance arising from the bias δ
- If $\delta > 0$, i.e. we have a bias in favor of shares, the performance on average increases
- A higher exposure to the stock market causes a higher return, so the result is not surprising
- The full performance needs to be evaluated by including the tracking error

Tracking error

- $\theta = F\sigma_\psi \left((1+\rho^2_{R\psi})\sigma^2_R + (\mu - r)^2 + \frac{\delta^2}{\rho^2_{R\psi}} + 2(\mu - r)\delta \right)^{\frac{1}{2}}$
- If $\delta > 0$ the tracking error is larger for a biased strategy
- If $\delta < -2(\mu - r)\rho^2_{R\psi} (< 0)$ the tracking error is smaller for a biased strategy

Maximizing the information ratio

- We can determine the bias that maximizes the information ratio
- It turns out that
- $\delta^* = \frac{\left(\frac{\sigma_R}{\mu - r}\right)^2 + 1}{\left(\frac{\sigma_R}{\mu - r}\right)^2 - 1}(\mu - r) \approx \mu - r$
- As in reality we usually find $\sigma_R > \mu - r > 0$ we have $\delta^* > 0$

Size of the bias

- Assume simply that $\sigma_R = 0.25$, $\mu - r = 0.05$, then $\delta^* = 0.054$
- With $\gamma = 4$, $\rho_{R\psi} = 0.2$, $\sigma_\psi = 0.5$ we have
- $\widehat{F} = 4$, $\beta_1 = 0.1$, $F = 0.4$
- $\theta_{bias} = F\frac{\delta}{\beta_1} = 0.216$

Substantial bias

- The bias in terms of the asset allocation can be substantial
- If $\rho_{R\psi} > 0$ the performance will have a positive skew, making the performance even more favorable

Origins of the bias

- The bias turns out to be optimal because the tracking error increases initially less than the α
- Only once the bias increases too much does the tracking error reduce the information ratio again

Outlook

The coming lecture will explore in more detail how to explore information, especially multiple signals

ES50106 - Financial investment management

Andreas Krause

Lecture 8 - Tactical bets with multiple signals

Structure of this lecture

1. Optimal aggressiveness
2. Optimal number of signals
3. Outlook

Multiple sources of information

- In most cases investors receive multiple pieces of information
- Each piece of information might be valuable to a number of assets
- About each asset multiple pieces of information can be received
- The problem is which weight to give each piece of information in the decision-making process

Signals

- A signal is some information about the deviation of the return from its long-term average
- $\Psi = \psi - \mu_\psi$
- From before we know that $\theta = F'\Psi$
- F is the aggressiveness with which the investor follows the signal

Properties

- $\alpha_i = F_i \widehat{\alpha}_i$
- $\vartheta_i = F_i \widehat{\vartheta}_i$
- $\widehat{}$ denotes the value for $F_i = 1$.
- $Var[\alpha] = F' Q F$
- Q is the covariance matrix of the αs when trading the signals ($F = \iota$)

Achieving a target α

- Suppose we want to achieve a target performance α
- This should be done by minimizing the tracking error
- $\frac{1}{2}F'QF \to \min_F$
- $F'\widehat{\alpha} = \alpha$

First order conditions

- $\frac{\partial L}{\partial F} = QF - \lambda\widehat{\alpha} = 0$
- $F = \lambda Q^{-1}\widehat{\alpha}$
- $F'\widehat{\alpha} = \lambda\widehat{\alpha}' Q^{-1}\widehat{\alpha} = \alpha$
- $\lambda = \frac{\alpha}{\widehat{\alpha}' Q^{-1}\widehat{\alpha}}$
- $F = \frac{Q^{-1}\widehat{\alpha}}{\widehat{\alpha}' Q^{-1}\widehat{\alpha}}\alpha$

Properties

- Tracking error: $\vartheta = (F'QF)^{\frac{1}{2}} = \frac{\alpha}{(\widehat{\alpha}'Q^{-1}\widehat{\alpha})^{\frac{1}{2}}}$
- Information ratio: $\kappa = \frac{\alpha}{\vartheta} = (\widehat{\alpha}'Q^{-1}\widehat{\alpha})^{\frac{1}{2}}$

Correlation of target αs

- $Cov[\alpha_1, \alpha_2] = F_1' Q F_2 = \frac{\alpha_1 \alpha_2}{\widehat{\alpha}' Q^{-1} \widehat{\alpha}}$
- $Corr[\alpha_1, \alpha_2] = 1$
- The resulting αs are perfectly correlated
- In the (α, θ)-plane the efficient frontier is a straight line with slope κ

Optimal target α

- The optimal target α is the one that maximizes the information ratio
- Assume investors have a power utility function: $U = \frac{\alpha^{1-\gamma}}{1-\gamma}$
- The optimum is where the utility function is tangential to the efficient frontier
- $U' = \alpha^{-\gamma} = \kappa$
- $\alpha = \kappa^{-\frac{1}{\gamma}} = (\widehat{\alpha}' Q^{-1} \widehat{\alpha})^{-\frac{1}{2\gamma}}$

Diversification of signals

- Similar to normal portfolio theory, investors seek to find the optimal portfolio
- Seeking a tradeoff between α (return) and tracking error (risk)
- The optimal portfolio gives the best combination of aggressiveness to the signals (F)

1. Optimal aggressiveness
2. Optimal number of signals
3. Outlook

Following signals is costly

- Following a signals will impose costs on investors, e.g. time, subscription costs, information processing
- We would need to find the optimal balance between costs and benefits

A simple model

- Assume we follow N signals that have a common standard deviation $\overline{\sigma}$ and covariance $\overline{\sigma}_{ij}$
- Assume further we give each signal the same weight, $\frac{1}{N}$
- $\sigma^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{1}{N}\frac{1}{N}\sigma_{ij} = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 + \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\sigma_{ij}$
- $\quad = \frac{1}{N}\overline{\sigma}^2 + \overline{\sigma}_{ij} \rightarrow_{N\rightarrow+\infty} \overline{\sigma}_{ij}$

Information cost

- As N increases, the risk reduces to the covariance risk of the portfolio, the systematic risk
- The reduction in the risk by introducing more and more assets is slowing
- Assume each additional signal costs c
- The optimal number of signals would be where marginal benefits and marginal costs are equal

Optimal number of signals

- Assume that the number of assets does not affect the mean of the signals
- $L = \mu - \frac{1}{2} z \sigma^2$
- $\frac{\partial L}{\partial N} = -\frac{1}{2} z \left(-\frac{1}{N^2} \overline{\sigma}^2\right) = \frac{z\overline{\sigma}^2}{2N^2} = c$
- $N = \sqrt{\frac{z\overline{\sigma}^2}{2c}}$

Properties

- The more risk averse an investor is the more signals he will evaluate
- The more risky an investment is the more signals an investor will evaluate
- The higher the costs of evaluating signals the less signal an investor will evaluate
- If we use the information ratio, a similar result can be obtained

1. Optimal aggressiveness
2. Optimal number of signals
3. Outlook

Outlook

The coming lecture will finally explore a model of how to use multiple signals when trading multiple assets.

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Lecture 9 - The Black-Litterman approach

Structure of this lecture

1. Views on returns
2. Determining the optimal tactical bet
3. Outlook

Structure of this lecture

1. Views on returns
2. Determining the optimal tactical bet
3. Outlook

Assessing information

- It is most common to assess information via the relative impact on the assets
- Thus signals suggest that asset A will outperform asset B by x%
- We can use such information to determine the expected returns and then derive the tactical bet

Market returns

- In a market the expected returns will vary around their long-term averages
- $\mu \sim N(\widehat{\mu}, \Sigma)$

Views

- A view is the opinion of an investor, based on his information
- Views are formed relative to other assets
- $v = P\mu + \varepsilon$
- $\varepsilon \sim N(0, \Omega)$
- $P_{ij} \in \{-1, 0, 1\}$
- If we know the differences between assets 1 and 3 and between assets 2 and 3 it is $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

Obtaining expected returns

- If we know the views we can determine the expected returns that deviate least from the long-run equilibrium
- $(\mu - \widehat{\mu})'\Sigma(\mu - \widehat{\mu}) \rightarrow \min_{\mu}$
- $v = P\mu$

First order conditions

- Let us assume that investor are certain about their views, i.e. $\Omega = 0$
- $\frac{\partial L}{\partial \mu} = 2\Sigma(\mu - \widehat{\mu})0\lambda P = 0$
- $\mu = \widehat{\mu} + \frac{1}{2}\lambda\Sigma^{-1}P'\iota$
- $P\mu = P\widehat{\mu} + \frac{1}{2}\lambda P\Sigma^{-1}P' = v$
- $\lambda = 2(v - P\widehat{\mu})(P\Sigma^{-1}P)^{-1}$

Expected returns

- $\mu = \widehat{\mu} + \Sigma^{-1}P'(P\Sigma^{-1}P')^{-1}(v - P\widehat{\mu})$
- We thus observe that the equilibrium returns are adjusted by the information used to form the views

Including uncertainty in views

- If $\Omega \neq 0$, i.e. views are uncertain, we need to use Bayesian regression and obtain
- $E[\mu] = \left((\Sigma\Omega)^{-1} + P'P\right)^{-1} \left(\Sigma^{-1}\Omega\widehat{\mu} + P'v\right)$
- even if no explicit views on assets are held, the expected returns will be affected
- This arises from the fact that assets are correlated and this correlation will affect also assets on which no information is directly available

- Dividing the expression by Ω^{-1} and expanding the final term we get
- $E[\mu] = \left(\Sigma^{-1} + P'\Omega^{-1}P\right)^{-1} \left(\Sigma^{-1}\widehat{\mu} + (P'\Omega^{-1}P)(P'P)^{-1}P'v\right)$
- As $v = P\mu + \varepsilon$ we can get the least square estimate of μ as
- $\widehat{\widehat{\mu}} = (P'P)^{-1}Pv$
- $E[\mu] = \left(\Sigma^{-1} + P'\Omega^{-1}P\right)^{-1} \left(\Sigma^{-1}\widehat{\mu} + P'\Omega^{-1}P\widehat{\widehat{\mu}}\right)$

Interpretation of expected returns

- The expected returns are a weighted average of the equilibrium returns ($\widehat{\mu}$) and the views of investors ($\widehat{\widehat{\mu}}$)
- The more sure investors are about their views, i.e. the smaller Ω is, the larger the weight on their views

1. Views on returns
2. Determining the optimal tactical bet
3. Outlook

The resulting tactical bet

- From previous analysis we know that in equilibrium
  $\omega_S = \omega_{MRP} + \frac{1}{\gamma}\Sigma^{-1}\left(\widehat{\mu} - \iota\frac{\iota'\Sigma^{-1}\widehat{\mu}}{\iota'\Sigma^{-1}\iota}\right)$
- When holding a different opinion it is
  $\omega_{BL} = \omega_{MRP} + \frac{1}{\gamma}\Sigma^{-1}\left(\mu - \iota\frac{\iota'\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\iota}\right)$
- The tactical bet will then be the difference between these two allocations:
- $\theta = \omega_{BL} - \omega_S = \frac{1}{\gamma}\Sigma^{-1}\left((\mu - \widehat{\mu}) - \iota\frac{\iota'\Sigma^{-1}(\mu - \widehat{\mu})}{\iota'\Sigma^{-1}\iota}\right)$

Including views of investors

- Inserting from the views of investors will provide a solution to the optimal tactical bet
- We take into account the impact views on individual stocks have on all stocks via their correlation structure
- Uncertainty about their views is also taken into account
- The Black-Litterman model has been developed in the early 1990s at GoldmanSachs and in a modified version still used today

Outlook

- With this lecture we have covered the most common methods for tactical asset allocation
- The final lecture will look at aspects of risk management
- This allows us to evaluate techniques financial institutions commonly use to manage their risk exposure explicitly
- These risk management aspects serve commonly as a restriction on tactical bets

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Lecture 10 - Risk Management

Structure of this lecture

1. Value at Risk
2. Constant Proportion Portfolio Insurance

1. Value at Risk
2. Constant Proportion Portfolio Insurance

Common definition of risk

- Risk is the possibility of a loss
- Volatility does not only capture losses but also gains

The VaR measure

- In reaction to huge losses in financial scandals JP Morgan developed an alternative risk measure which focuses only on losses
- The *Value-at-Risk* says that "The loss will not exceed the VaR with a probability of c over the next T time periods."

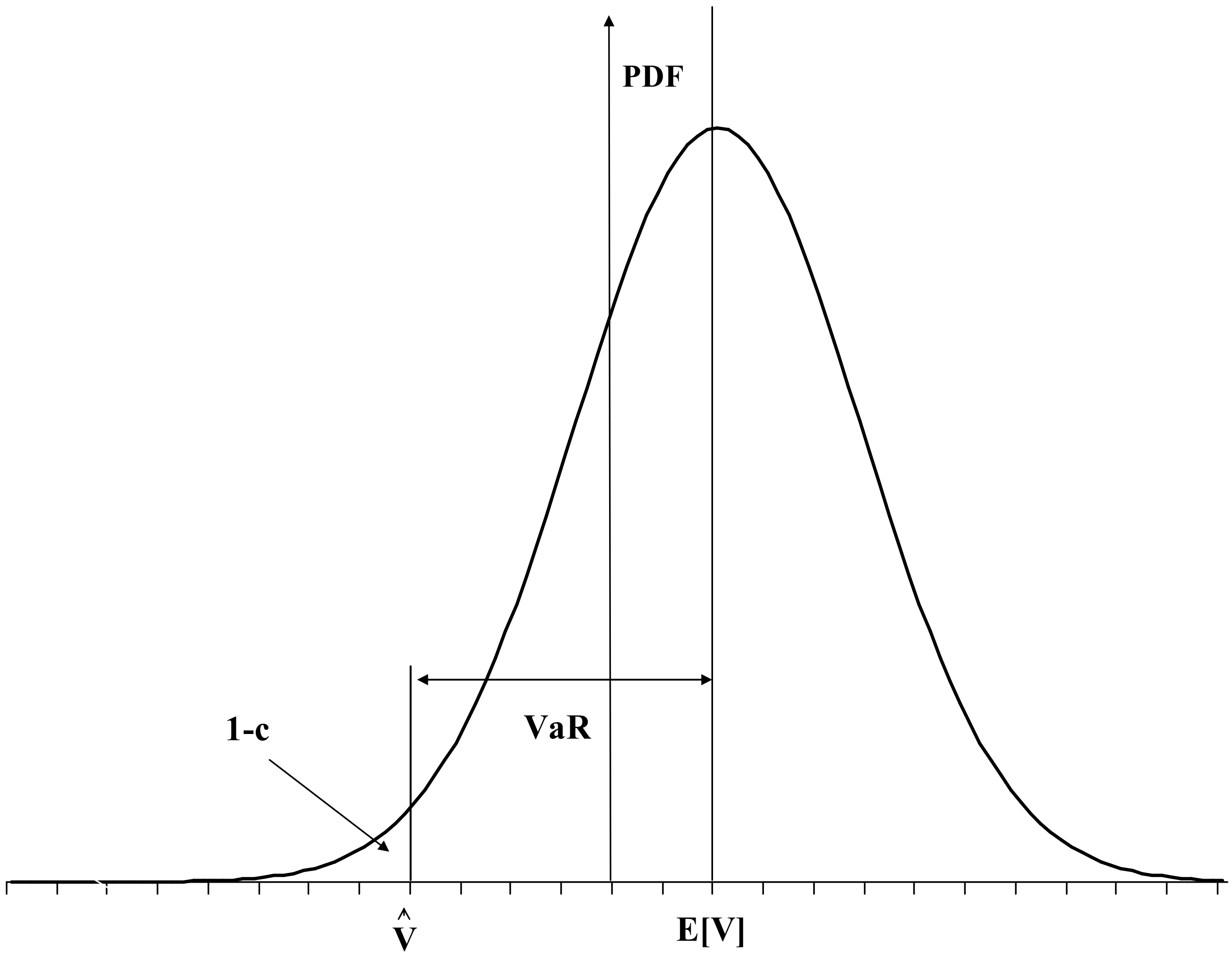
Interpreting VaR

- VaR can be interpreted as the *reasonable* amount that can be lost within a given time horizon
- What is reasonable will depend on the risk aversion of the user
- The more risk averse the user is, the smaller the probability that the loss will be exceeded

Defining losses

- A loss needs to be defined relative to a benchmark
- Common benchmarks are the zero return (absolute loss) and the expected outcome (relative loss)
- For simplicity we focus on the relative loss here

Illustration of VaR



Formal definition of VaR

- Select $\widehat{V}$ such that $Prob(V < \widehat{V}) = 1 - c$
- $VaR = E[V] - \widehat{V}$
- VaR is the estimation of the c-quantile $(\widehat{V})$

Choice of confidence level

- The choice of confidence level will depend on the risk aversion
- The more risk averse an investor is, the higher the confidence level chosen

Normal distribution

- If the distribution of returns is normal we have
- $\widehat{R} = \mu - \alpha\sigma$
- $VaR = \alpha\sigma W_0$

Portfolio with normal distributions

- $\sigma_P^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \omega_i \omega_j \sigma_{ij}$
- Marginal impact of changing the weight of asset i:
- $\frac{\partial \sigma_P^2}{\partial \omega_i} = 2\omega_i \sigma_i^2 + 2 \sum_{j=1, j \neq i}^{N} \omega_j \sigma_{ij} = 2Cov \left[R_i, \omega_i R_i + \sum_{j=1, j \neq i}^{N} \omega_j R_j \right]$
- $\quad = 2Cov[R_i, R_P] \equiv 2\sigma_{iP}$

Marginal impact

- $\frac{\partial \sigma_P^2}{\partial \omega_i} = 2\sigma_P \frac{\partial \sigma_P}{\partial \omega_i}$
- $\frac{\partial \sigma_P}{\partial \omega_i} = \frac{\sigma_{iP}}{\sigma_P} = \beta_i \sigma_P$
- $\beta_i = \frac{\sigma_{iP}}{\sigma_P^2}$

Marginal VaR

- $VaR = \alpha \sigma_P W_0$
- $\frac{\partial \frac{VaR}{W_0}}{\partial \omega_i} = \alpha \frac{\partial \sigma_P}{\partial \omega_i} = \alpha \sigma_P \beta_i = \beta_i \frac{VaR}{W_0} \equiv \partial VaR_i$ If we change the weight of asset i by a small amount, the (relative) VaR changes by ∂VaR_i

Linear approximation

- If we change the weight more than marginally, we can use a linear approximation
- $\Delta VaR_i = W_0 \Delta\omega_i \partial VaR_i = \beta_i \Delta\omega_i VaR$

Reducing portfolio risk

- The change in the Portfolio VaR is equal to the changes for the individual components (assets)
- $\Delta VaR = \sum_{i=1}^{N} \Delta VaR_i = VaR \sum_{i=1}^{N} \beta_i \Delta\omega_i$
- If we only rearrange the weights, the total changes in the weights must be zero: $\sum_{i=1}^{N} \Delta\omega_i = 0$

Changing portfolio risk

- If we want to reduce portfolio risk, we reduce the weight of those assets with high β_i and increase those with low β_i
- For $N = 2$ the solution is unique, but for $N > 2$ many solutions exist.
- If we wanted to we could specify that the solution should involve the minimum change in weights as to preserve the original portfolio as much as possible, i.e. choose the solution which minimizes $\sum_{i=1}^{N}(\Delta\omega_i)^2$

Least distorting portfolio

- If we wanted to find a solution which is least distorting to the portfolio we would minimize the change in weights
- $\Delta\omega'\Delta\omega \rightarrow \min_{\Delta\omega}$
- $\Delta VaR = VaR\beta'\Delta\omega$
- $\Delta\omega'\iota = 0$

Resulting portfolio changes

- Conducting the optimization we finally get
- $\Delta\omega = \frac{\Delta VaR}{VaR\beta'\beta}\frac{1}{2\Delta VaR+\iota'\iota}\left(\iota'\iota\beta + \beta'\iota\iota\right)$

Aim of portfolio insurance

- VaR is primarily a risk measure and does not prevent losses
- In many cases investors want to ensure a ceratin minimum value of their investment at the end of their time horizon
- This can be driven by regulatory requirements or the need to meet certain obligation
- In principle derivatives can be used to achieve this aim

Limits in the use of derivatives

- Derivatives do not exist for less commonly traded stocks
- Using derivatives for portfolios with many assets can be very costly
- Alternative ways need to be found

Buy and hold

- An investor needs to ensure that $V_T \geq \alpha V_0$
- If the investor invests into the risk free asset this is ensured: $F_t = \alpha V_0 e^{-r\Delta t}$
- The remainder is invested into the risk-free asset

Additional assumption

- Assume we know that an asset cannot loose more than a certain amount before it can be sold
- This amount that can be lost might be the 99% VaR or similar considerations

The strategy

- Define $C_t = V_t - F_t$
- Invest mC_t into the risky asset and F_t into the risk free asset
- F_t is the floor
- C_t is the cushion

Dynamics of the investment

- z is the maximum loss possible before it can be sold (in %)
- The investment into the asset is X_t
- Worst case scenario: $V_{t+1} = (V_t - X_t z) + r(V_t - X_t)$

Evolution of the cushion

- As $V_t = C_t + F_t$ and $X_t = mC_t$ we get
- $C_{t+1} + F_{t+1} = \underbrace{C_t + F_t}_{V_t} - \underbrace{mC_t}_{X_t} z + \underbrace{((1-m)C_t + F_t)}_{V_t - X_t} r$
- $\quad = C_t(1 - mz + (1-m)r) + F_t(1+r)$
- $F_{t+1} \approx F_t(1+r)$
- $C_{t+1} = C_t(1 - mz + (1-m)r)$

Optimal risky investment

- The value will be guaranteed as long as the cushion remains positive
- $1 - mz + (1 - m)r \geq 0$
- $m \leq \frac{1+r}{z-r} \approx \frac{1}{z}$

CPPI

- As the investment consists of a fixed proportion m invested into the risky asset, it is called the Constant Proportion Portfolio Insurance (CPPI)
- In CPPI the portfolio composition remains unchanged, only the amount of risk free assets is changed
- This is similar to buying derivatives for each asset held