Differential equations

Suppose we're given a differential equation, like

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

We guess that there might be solutions of the form $y = e^{mx}$ for some $m \in \mathbb{C}$. Try it...

$$y = e^{mx}$$
$$\frac{dy}{dx} = me^{mx}$$
$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

so

$$m^2e^{mx} - me^{mx} - 6e^{mx} = 0$$

so

$$m^2 - m - 6 = 0$$

so the equation is satisfied if we have $m = 3$ or $m = -2$. This gives us two (linearly independent, if you know what that means) solutions, and since the equation is second order we deduce that all solutions are of the form

$$y = Ae^{3x} + Be^{-2x}$$

for some A and B. If we have boundary conditions (like "$\frac{dy}{dx} = \frac{d^2y}{dx^2} = 1$ when $x = 0$" or something) then we can hopefully figure out what A and B are.

What if we had a third order differential equation (i.e. involving $\frac{d^3y}{dx^3}$ as well)? Well, we could try the same thing. It'll give us a cubic equation in m, instead of a quadratic. Cubics are a bit harder to solve but in simple cases you might be able to guess one solution and take that out as a factor. Then we hope that we'll get three different solutions to the cubic, call them m_1, m_2 and m_3. And then the general solution will be

$$y = Ae^{m_1x} + Be^{m_2x} + Ce^{m_3x}.$$

Now for the more complicated bit — not necessary for Sheet 3.
Why did we guess solutions of the form $y = e^{mx}$?
Well, because we were told to in lectures. (In fact, we might just have been told to skip straight to the $m^2 - m - 6 = 0$ stage.) But how did the lecturer know we should do that? Maybe because she were told to when she were a lad. But also because she knows solutions to differential equations often look like that. They don't always — sometimes you will get a differential equation with solutions that look different. For example, you might find that the quadratic you have to solve gives you a repeated root, so you only get one solution of the usual form. Then you have to look for other solutions. One thing to try is solutions of the form xe^{mx}. That sometimes works. Again, sometimes it doesn't and you have to be cleverer.

Difference equations

Suppose we're given a difference equation, like

$$y_{k+2} - y_{k+1} - 6y_k = 0.$$

We guess that there might be solutions of the form $y_k = m^k$ for some $m \in \mathbb{C}$. Try it...

$$\begin{aligned} y_k &= m^k \\ y_{k+1} &= m^{k+1} \\ y_{k+2} &= m^{k+2} \end{aligned}$$

so

$$m^{k+2} - m^{k+1} - 6m^k = 0$$

so (dividing by m^k... what are we assuming here?)

$$m^2 - m - 6 = 0$$

so the equation is satisfied if we have $m = 3$ or $m = -2$. This gives us two (linearly independent, if you know what that means) solutions, and since the equation is second order we deduce that all solutions are of the form

$$y_k = A3^k + B(-2)^k$$

for some A and B. If we have boundary conditions (like "$y_0 = 4, y_1 = 5$" or something) then we can hopefully figure out what A and B are.

What if we had a third order difference equation (i.e. involving y_{k+3} as well)? Well, we could try the same thing. It'll give us a cubic equation in m, instead of a quadratic. Cubics are a bit harder to solve but in simple cases you might be able to guess one solution and take that out as a factor. Then we hope that we'll get three different solutions to the cubic, call them m_1, m_2 and m_3. And then the general solution will be

$$y_k = Am_1^k + Bm_2^k + Cm_3^k.$$

Now for the more complicated bit — not necessary for Sheet 3.
Why did we guess solutions of the form $y_k = m^k$?
Well, because we were told to in the example sheet. But how did the lecturer know we should do that? Because he knows solutions to difference equations often look like that. They don't always — sometimes you will get a difference equation with solutions that look different. For example, you might find that the quadratic you have to solve gives you a repeated root, so you only get one solution of the usual form. Then you have to look for other solutions. One thing to try is solutions of the form $y_k = km^k$. That sometimes works. Again, sometimes it doesn't and you have to be cleverer.