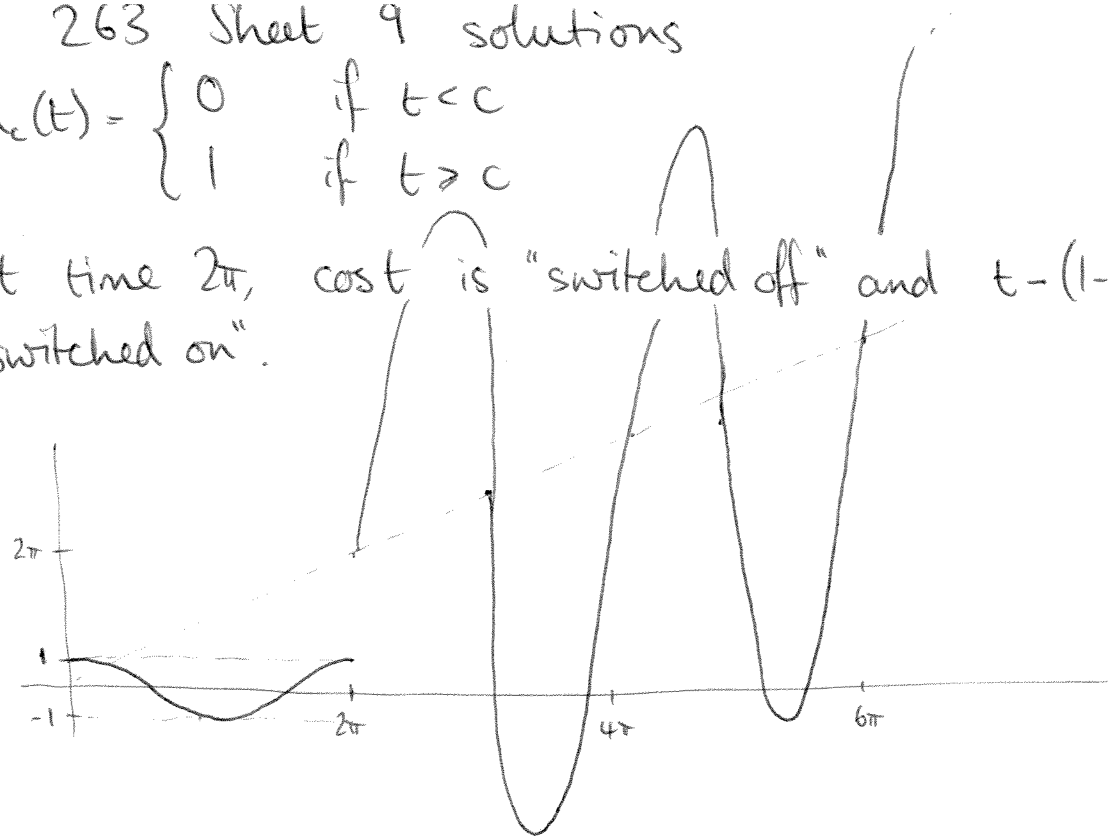
Math 263 Sheet 9 solutions MR

① $u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$

② At time 2π, $\cos t$ is "switched off" and $t-(1-4\pi^2)\sin t$ is "switched on".



③ Note that $\cos t = \cos(t-2\pi)$ and $\sin t = \sin(t-2\pi)$.

$$\mathcal{L}\{(1-u_{2\pi}(t))\cos t\} = \mathcal{L}\{\cos t - u_{2\pi}(t)\cos(t-2\pi)\}$$

$$= \frac{s}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

and $\mathcal{L}\{u_{2\pi}(t)(t-(1-4\pi^2)\sin t)\}$

$$= \mathcal{L}\{u_{2\pi}(t)(t-2\pi) + 2\pi u_{2\pi}(t) - (1-4\pi^2)u_{2\pi}(t)\sin(t-2\pi)\}$$

$$= \frac{e^{-2\pi s}}{s^2} + \frac{2\pi e^{-2\pi s}}{s} - \frac{(1-4\pi^2)e^{-2\pi s}}{s^2+1}.$$

Thus $\mathcal{L}\{(1-u_{2\pi}(t))\cos t + u_{2\pi}(t)(t-(1-4\pi^2)\sin t)\}$

$$= \frac{e^{-2\pi s}(s^2+1) + 2\pi e^{-2\pi s}s(s^2+1) - (1-4\pi^2)s^2e^{-2\pi s} + 2\pi s^3 - 2\pi e^{-2\pi s}s}{s^2(s^2+1)}$$

$$= \frac{4\pi e^{-2\pi s}s^2 + e^{-2\pi s} + 2\pi e^{-2\pi s}s + 2\pi s^3}{s^2(s^2+1)}$$

$$= \frac{1}{s^2+1}\left(4\pi e^{-2\pi s} + \frac{e^{-2\pi s}}{s^2} + \frac{2\pi e^{-2\pi s}}{s} + 2\pi s\right).$$

(4) Let $Y(s) = \mathcal{L}\{y(t)\}(s)$. Then

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2\pi$$

and

$$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - 2\pi s.$$

$$\mathcal{L}\{\delta(t-2\pi)\}(s) = \int_0^\infty \delta(t-2\pi) e^{-st}\, ds = e^{-2\pi s}$$

$$\mathcal{L}\{t u_{2\pi}(t)\}(s) = \cancel{e^{2\pi s}}\, \mathcal{L}\{u_{2\pi}(t)(t-2\pi) + 2\pi u_{2\pi}(t)\}(s)$$

$$= \frac{e^{-2\pi s}}{s^2} + \frac{2\pi e^{-2\pi s}}{s}.$$

So applying $\mathcal{L}$ to both sides of $y'' + y = 4\pi\delta(t-2\pi) + t u_{2\pi}(t)$,

we get

$$s^2 Y(s) - 2\pi s + Y(s) = 4\pi e^{-2\pi s} + \frac{e^{-2\pi s}}{s^2} + \frac{2\pi e^{-2\pi s}}{s}$$

so

$$Y(s) = \frac{1}{s^2+1}\left(4\pi e^{-2\pi s} + \frac{e^{-2\pi s}}{s^2} + \frac{2\pi e^{-2\pi s}}{s} + 2\pi s\right).$$

From q3, we see that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = (1 - u_{2\pi}(t))\cos t + u_{2\pi}(t)\left(t - (1-4\pi^2)\sin t\right).$$