Math 263 Sheet 8 solutions MR

(1) Try $y_1 = x+2$. Then $y_1' = 1$, $y_1'' = 0$

so $y_1'' + (x+2)y_1' - y_1 = 0 + (x+2) - (x+2) = 0$.

So $y_1 = x+2$ is a solution.

Now try $y_2 = C(x)(x+2)$.

$$y_2' = C'(x)(x+2) + C(x)$$
$$y_2'' = C''(x)(x+2) + 2C'(x)$$

$$y_2'' + (x+2)y_2' - y_2 = C''(x)(x+2) + 2C'(x) + C'(x)(x+2)^2 + C(x)(x+2) - C(x)(x+2)$$

$$= C''(x)(x+2) + C'(x)\left((x+2)^2+2\right)$$

so we need $C''(x) + C'(x)\left((x+2) + \frac{2}{x+2}\right) = 0$

i.e. $$\frac{C''(x)}{C'(x)} = -(x+2) - \frac{2}{x+2}$$

$$\Rightarrow \ln C'(x) = -\int \left(x+2+\frac{2}{x+2}\right)dx$$

$$\Rightarrow \ln C'(x) = -\frac{x^2}{2} - 2x - 2\ln(x+2)$$

$$\Rightarrow C(x) = \int \frac{e^{-\frac{x^2}{2}-2x}}{(x+2)^2}\,dx$$

so the general solution to $y'' + (x+2)y' - y = 0$

is $$y = A(x+2) + B(x+2)\int \frac{e^{-\frac{x^2}{2}-2x}}{(x+2)^2}\,dx.$$

(2) Attempting $y_p = C(x)(x+2)$ as before leads to

$$C''(x)(x+2) + C'(x)\left((x+2)^2+2\right) = \ln x$$

Use the integrating factor

$$\mu = e^{\int (x+2+\frac{2}{x+2})\,dx} = (x+2)^2 e^{\frac{1}{2}x^2+2x} :$$

we get $(x+2)^2 e^{\frac{1}{2}x^2+2x} + \left((x+2)^3 + 2(x+2)\right) e^{\frac{1}{2}x^2+2x} C'(x)$

$$= (x+2) e^{\frac{1}{2}x^2+2x} \ln x$$

$$\Rightarrow \quad \frac{d}{dx}\left(C'(x)(x+2)^2 e^{\frac{1}{2}x^2+2x} \right) = (x+2) e^{\frac{1}{2}x^2+2x} \ln x$$

$$\Rightarrow \quad C'(x) = \frac{e^{-\frac{1}{2}x^2-2x}}{(x+2)^2} \int (x+2) e^{\frac{1}{2}x^2+2x} \ln x \, dx$$

$$\Rightarrow \quad C(x) = \int \left(\frac{e^{-\frac{1}{2}x^2-2x}}{(x+2)^2} \int (x+2) e^{\frac{1}{2}x^2+2x} \ln x \, dx \right) dx .$$

Thus the general solution is

$$y = A(x+2) + B(x+2) \int \frac{e^{-\frac{1}{2}x^2-2x}}{(x+2)^2}\,dx + (x+2) \int \left(\frac{e^{-\frac{1}{2}x^2-2x}}{(x+2)^2} \int (x+2) e^{\frac{1}{2}x^2+2x} \ln x \, dx \right) d$$

③ $$\frac{1}{(s-1)(s^2-4s+8)} = \frac{A}{s-1} + \frac{Bs+C}{s^2-4s+8}$$

$$\Rightarrow 1 = A(s^2-4s+8) + (Bs+C)(s-1)$$

$$\Rightarrow 1 = (A+B)s^2 + (-4A-B+C)s + 8A - C$$

$$\Rightarrow A+B=0, \quad -4A-B+C=0, \quad 8A-C=1$$

$$4A - B = 1$$

$$\Rightarrow 5A = 1 \quad \Rightarrow A = 1/5, \quad B = -1/5, \quad C = 3/5 .$$

So $$\frac{s}{(s-1)(s^2-4s+8)} = \frac{s}{5(s-1)} - \frac{s^2-3s}{5(s^2-4s+8)}$$

$$= \frac{1}{5} + \frac{1}{5(s-1)} - \frac{(s^2-4s+8)+s-8}{5(s^2-4s+8)}$$

$$= \frac{1}{5(s-1)} - \frac{s-8}{5(s^2-4s+8)}$$

$$= \frac{1}{5(s-1)} - \frac{s-8}{5((s-2)^2+4)}$$

$$= \frac{1}{5(s-1)} - \frac{1}{5}\left(\frac{s-2}{(s-2)^2+4}\right) + \frac{3}{5}\left(\frac{2}{(s-2)^2+4}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2-4s+8)}\right\} = \frac{1}{5}e^{t} - \frac{1}{5}e^{2t}\cos 2t + \frac{3}{5}e^{2t}\sin 2t.$$

(4) Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

Then $\mathcal{L}\{y'(t)\}(s) = sY(s) - y(0) = sY(s)$

so applying $\mathcal{L}$ to our ODE we get

$$(s-1)Y(s) = \frac{2}{(s-2)^2+2^2} + \frac{(s-2)}{(s-2)^2+2^2} = \frac{s}{s^2-4s+8}$$

$$\Rightarrow Y(s) = \frac{s}{(s-1)(s^2+4s+8)}$$

$$\Rightarrow y(t) = \frac{1}{5}e^{t} - \frac{1}{5}e^{2t}\cos 2t + \frac{3}{5}e^{2t}\sin 2t.$$