Math 263 Sheet 7 solutions MR

① a) $-\frac{1}{2}(i-1)(i+1) = -\frac{1}{2}(-1-i+i-1) = -\frac{1}{2}(-2) = 1$.

b) $-\frac{1}{2}(i+1)$, by the above.

② a) i) D^3, ii) $(D+4)^4$, iii) $(D^2+25)^5$ (note that $\cos(-5x) = \cos 5x$)

b) $A(\tan x) = D(\tan x) - 1 - \tan^2 x = \sec^2 x - (1+\tan^2 x) = 0$.
But an annihilator has to be a polynomial in D, and A is not a polynomial in D, so A is not an annihilator.

③ a) $Q(D) = (D-1)^2 + 4 = (D-1-2i)(D-1+2i)$.
General solution to $Q(y_p) = 0$ is

$$y_p = c_1 e^x \sin 2x + c_2 e^x \cos 2x$$
$$y_p' = (c_1 - 2c_2) e^x \sin 2x + (c_2 - c_1) e^x \cos 2x$$
$$y_p'' = (-3c_1 - 4c_2) e^x \sin 2x + (4c_1 - 3c_2) e^x \cos 2x.$$

$\cancel{y_p'' + 3y_p' - 10y_p = (-3c_1 - 4c_2 + 3c_1 - 6c}$

$$y_p'' - 4y_p' + 7y_p = (-3c_1 - 4c_2 - 4c_1 + 8c_2 + 7c_1) e^x \sin 2x$$
$$+ (4c_1 - 3c_2 - 4c_2 + 4c_1 + 7c_2) e^x \sin 2x.$$

so we need $4c_2 = 1$ and $8c_1 = 0$, i.e. $c_1 = 0$, $c_2 = 1/4$.
So $y_p = \frac{1}{4} e^x \cos 2x$.
$P(r) = r^2 - 4r + 7$ which has roots $\frac{4 \pm \sqrt{16-28}}{2} = 2 \pm \sqrt{3}i$
so the general solution is

$$y = Ae^{2x} \sin \sqrt{3}x + Be^{2x} \cos \sqrt{3}x + \frac{1}{4} e^x \cos 2x.$$

b) Start by finding a particular solution to
$$\tilde{y}'' - 4\tilde{y}' + 7\tilde{y} = e^{(1+2i)x}.$$

$\tilde{Q}(D) = (D-1-2i)$. General solution to $\tilde{Q}(D)(y_p) = 0$ is

$\tilde{y}_p = c_1 e^{(1+2i)x}$, $\tilde{y}_p' = c_1(1+2i)e^{(1+2i)x}$, $\tilde{y}_p'' = c_1(1+2i)^2 e^{(1+2i)x}$

so $$\tilde{y}_p'' - 4\tilde{y}_p' + 7\tilde{y}_p = c_1\left((1+2i)^2 - 4(1+2i) + 7\right)e^{(1+2i)x}$$
$$= c_1(1+4i-4-4-8i+7)e^{(1+2i)x}$$
$$= -4c_1 i e^{(1+2i)x}$$

so we need $c_1 = \frac{i}{4}$.

Thus $$\tilde{y}_p = \frac{i}{4}e^{(1+2i)x} = \frac{i}{4}e^{x}\cos 2x + \frac{i}{4}\cdot i e^{x}\sin 2x$$
$$= \frac{i}{4}e^{x}\cos 2x - \frac{1}{4}e^{x}\sin 2x,$$

and $y_p = \operatorname{Im}(\tilde{y}_p) = \frac{1}{4}e^{x}\cos 2x$.

$P(r) = r^2 - 4r + 7$ which has roots $\frac{4 \pm \sqrt{16-28}}{2} = 2 \pm \sqrt{3}i$

so the general solution is
$$y = Ae^{2x}\sin\sqrt{3}x + Be^{2x}\cos\sqrt{3}x + \frac{1}{4}e^{x}\cos 2x.$$

(4) Since we don't know an annihilator for $\ln x$, we use the method of variation of parameters.

$P(D) = D^2 + 3D - 10 = (D+5)(D-2)$ so we can take $y_1 = e^{-5x}$ and $y_2 = e^{2x}$. Then $W(x) = y_1y_2' - y_2y_1' = 7e^{-3x}$.

$$y_p = -y_1(x)\int \frac{b(x)y_2(x)}{W(x)}\,dx + y_2(x)\int \frac{b(x)y_1(x)}{W(x)}\,dx$$
$$= -e^{-5x}\int \frac{e^{2x}\ln x}{7e^{-3x}}\,dx + e^{2x}\int \frac{e^{-5x}\ln x}{7e^{-3x}}\,dx$$
$$= -\frac{1}{7}e^{-5x}\int e^{5x}\ln x\,dx + \frac{1}{7}e^{2x}\int e^{-2x}\ln x\,dx.$$

So the general solution is
$$y = Ae^{-5x} + Be^{2x} - \frac{1}{7}e^{-5x}\int e^{5x}\ln x\,dx + \frac{1}{7}e^{2x}\int e^{-2x}\ln x\,dx.$$