Math 263 - Solutions to sheet 5 MR

① $P(r) = 2r^2 + 4r + 8$ has roots $\dfrac{-4 \pm \sqrt{16 - 4 \times 2 \times 8}}{4}$

$$= -1 \pm \tfrac{1}{4}\sqrt{48}\, i$$

$$= -1 \pm \sqrt{3}\, i.$$

Thus the general solution to $2y'' + 4y' + 8y = 0$ is

$$y = Ae^{-t}\sin(\sqrt{3}t) + Be^{-t}\cos(\sqrt{3}t).$$

$y(0) = -1 \Rightarrow -1 = A \times 1 \times 0 + B \times 1 \times 1$

so $B = -1$.

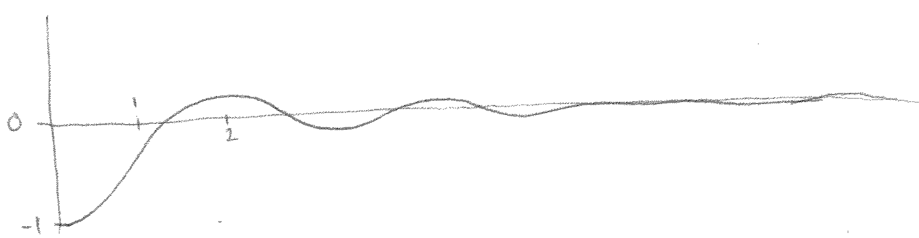
$y'(t) = -Ae^{-t}\sin(\sqrt{3}t) + \sqrt{3}Ae^{-t}\cos(\sqrt{3}t) - Be^{-t}\cos(\sqrt{3}t) - \sqrt{3}Be^{-t}\sin(\sqrt{3}t$

$y'(0) = 0 \Rightarrow -A \times 1 \times 0 + \sqrt{3}A \times 1 \times 1 - B \times 1 \times 1 - \sqrt{3}B \times 1 \times 0 = 0$

$\Rightarrow \sqrt{3}A - B = 0$

$\Rightarrow A = \dfrac{-1}{\sqrt{3}}.$

So $y(t) = -\dfrac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t) - e^{-t}\cos(\sqrt{3}t).$



② Assume that $(D-a)^k = e^{ax}D^k e^{-ax}$. Then

$$(D-a)^{k+1} = (D-a)(D-a)^k = (D-a)(e^{ax}D^k e^{-ax})$$

$$= D(e^{ax}D^k e^{-ax}) - ae^{ax}D^k e^{-ax}$$

product rule

$$= ae^{ax}D^k e^{-ax} + e^{ax}D(D^k e^{-ax}) - ae^{ax}D^k e^{-ax}$$

$$= e^{ax}D^{k+1}e^{-ax}.$$

□

(3) $P(r) = r^5 + r^4 + 3r^3 + 3r^2$.

Clearly $r=0$ is a root of multiplicity 2:

$$P(r) = r^2(r^3 + r^2 + 3r + 3)$$

Now notice that -1 is a root, so we can take out $(r+1)$ as a factor:

$$P(r) = r^2(r+1)(r^2+3)$$

so $P(r)$ has roots $0, 0, -1, \sqrt{3}i, -\sqrt{3}i$

so $y^{(5)} + y^{(4)} + 3y^{(3)} + 3y'' = 0$ has solutions

$$y_1 = 1,\ y_2 = x,\ y_3 = e^{-x},\ y_4 = \sin(\sqrt{3}x),\ y_5 = \cos(\sqrt{3}x).$$

So the general solution is

$$y = C_1 + C_2 x + C_3 e^{-x} + C_4 \sin(\sqrt{3}x) + C_5 \cos(\sqrt{3}x).$$

(4)
$$y_1 = x\sin x$$
$$y_1' = \sin x + x\cos x$$
$$y_1'' = 2\cos x - x\sin x$$
$$y_1'' - \frac{2}{x}y_1' + \left(1+\frac{2}{x^2}\right)y_1$$
$$= 2\cos x - x\sin x - \frac{2}{x}\sin x - 2\cos x + x\sin x + \frac{2}{x}\sin x$$
$$= 0.$$

$$y_2 = x\cos x$$
$$y_2' = \cos x - x\sin x$$
$$y_2'' = -2\sin x - x\cos x$$
$$y_2'' - \frac{2}{x}y_2' + \left(1+\frac{2}{x^2}\right)y_2$$
$$= -2\sin x - x\cos x - \frac{2}{x}\cos x + 2\sin x + x\cos x + \frac{2}{x}\cos x$$
$$= 0.$$

So y_1 and y_2 are solns. $W(x) = \begin{vmatrix} x\sin x & x\cos x \\ \sin x + x\cos x & \cos x - x\sin x \end{vmatrix}$

$W(\pi/2) = \begin{vmatrix} \pi/2 & 0 \\ 1 & -\pi/2 \end{vmatrix} = -\frac{\pi^2}{4}$. Our theorem said if y_1, y_2 are solns to the same homogeneous linear ODE then W never changes sign. So there does not exist $x>0$ such that $W(x) \geq 0$.

But $W(0) = 0$?!? This highlights the fact that we have implicitly assumed our ODE is continuous. $\frac{1}{x}$ and $1+\frac{2}{x^2}$ are not cts at $x=0$.