Solutions for sheet 3

(1) $y' = \dfrac{x^3 + 2y^3}{xy^2} = \dfrac{x^2}{y^2} + \dfrac{2y}{x}$. This is a function of y/x, so we can use the substitution $u = y/x$.

Then $u' = \dfrac{y'}{x} - \dfrac{y}{x^2} = \dfrac{1}{x}\left(\dfrac{x^2}{y^2} + \dfrac{2y}{x}\right) - \dfrac{y}{x^2} = \dfrac{1}{x}\left(\dfrac{1}{u^2} + 2u\right) - \dfrac{u}{x}$

$= \dfrac{1}{x}\left(\dfrac{1+u^3}{u^2}\right)$

so $\dfrac{u^2}{1+u^3}u' = \dfrac{1}{x}$, so $\displaystyle\int \frac{u^2}{1+u^3}\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx + C$,

so $\displaystyle\int \frac{u^2}{1+u^3}\,du = \int \frac{1}{x}\,dx + C$, so $\displaystyle\frac{1}{3}\int \frac{d}{du}\left(\ln|1+u^3|\right)du = \ln|x| + C$

so $\dfrac{1}{3}\ln|1+u^3| = \ln|x| + C$, so $|1+u^3| = e^{3\ln|x| + 3C} = |x|^3 \cdot e^{3C}$,

so $1 + u^3 = C_1 x^3$, so $u = \sqrt[3]{C_1 x^3 - 1}$.

So $y = x(C_1 x^3 - 1)^{1/3}$.

(2) $y' = xy - xy^3$ is Bernoulli with $n = 3$, so we use the substitution $u = y^{-2}$.

Then $u' = -2\dfrac{y'}{y^3} = -\dfrac{2}{y^3}(xy - xy^3) = -2xu + 2x$

so $u' + 2xu = 2x$

$e^{x^2}u' + 2xe^{x^2}u = 2xe^{x^2}$

$\dfrac{d}{dx}\left(e^{x^2}u\right) = 2xe^{x^2}$

$e^{x^2}u = \displaystyle\int 2xe^{x^2}\,dx + C$

$= e^{x^2} + C$

$u = Ce^{-x^2} + 1$, so $y = \pm\left(Ce^{-x^2}+1\right)^{-1/2}$.

Note also the trivial solution $y=0$.
Try plotting the graph of these solutions for a few values of C!

③ a) $L_1(4) = \frac{d}{dx}(4) = 0$. b) $L_2(4) = 4/x$.
c) $L_1L_2(4) = \frac{d}{dx}(4/x) = -4/x^2$. d) $L_2L_1(4) = \frac{1}{x}\frac{d}{dx}(4) = 0$.
e) $L_1L_2(x^2) = \frac{d}{dx}\left(\frac{x^2}{x}\right) = 1$. f) $L_2L_1(x^2) = \frac{1}{x}\frac{d}{dx}(x^2) = 2$.

④ a) Is linear: $3(c_1y_1+c_2y_2) = 3c_1y_1 + 3c_2y_2 = c_1L(y_1)+c_2L(y_2)$.
b) Is linear: $(c_1y_1+c_2y_2)\sin x = c_1y_1\sin x + c_2y_2\sin x = c_1L(y_1)+c_2L(y_2)$.
c) Not linear: take $c_1=1$, $c_2=1$, $y_1=x$, $y_2=x$ for example.
$\sin(c_1y_1+c_2y_2) = \sin(2x)$
but $c_1L(y_1)+c_2L(y_2) = \sin(x)+\sin(x) = 2\sin x$
and $\sin(2x) \neq 2\sin x$ in general.
d) Is linear: $\frac{d^2}{dx^2}(c_1y_1+c_2y_2) = c_1\frac{d^2}{dx^2}y_1 + c_2\frac{d^2}{dx^2}y_2 = c_1L(y_1)+c_2L(y_2)$
e) Is linear: $x\left(2(c_1y_1+c_2y_2)+x\frac{d}{dx}(c_1y_1+c_2y_2)\right)$
$= c_1x\left(2y_1+x\frac{d}{dx}y_1\right) + c_2x\left(2y_2+x\frac{d}{dx}y_2\right)$
$= c_1L(y_1)+c_2L(y_2)$.

f) Is linear: in fact this is the same operator as in e), by the chain rule.