Solutions to ODEs sheet 2.

① The easiest way is to divide by $y$ - then we get a linear homogeneous ODE. It's also possible to solve it using an integrating factor to make it exact, and the students may try this.

$$y \cot x + 2(1 + 2\operatorname{cosec} x) y' = 0$$

$$\frac{y'}{y} = -\frac{\cot x}{2(1+2\operatorname{cosec} x)}$$

$$\ln|y| = -\frac{1}{2}\int \frac{\cot x}{1+2\operatorname{cosec} x}\,dx + C = -\frac{1}{2}\int \frac{\cos x}{2+\sin x}\,dx + C$$

$$= -\frac{1}{2}\int \frac{d}{dx}\left(\ln|2+\sin x|\right) dx + C$$

$$= -\frac{1}{2}\ln(2+\sin x)$$

$$\Rightarrow y = C_1 e^{-\frac{1}{2}\ln(2+\sin x)} = \frac{C_1}{\sqrt{2+\sin x}}$$

$$y(0) = 1 \Rightarrow 1 = \frac{C_1}{\sqrt{2}} \Rightarrow C_1 = \sqrt{2} \Rightarrow y = \sqrt{\frac{2}{2+\sin x}}.$$

② Now we do need to make it exact.

$$M = y^2 \cot x, \quad N = 2y(1 + 3y\operatorname{cosec} x)$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y\cot x + 6y^2 \cot x \operatorname{cosec} x = 2y\cot x(1+3y\operatorname{cosec} x)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \cot x = p(x).$$

So we can choose $\mu = e^{\int p(x)\,dx} = e^{\ln \sin x} = \sin x$.

This gives us the equation

$$y^2\cos x + 2y(\sin x + 3y)y' = 0 ,$$ which is exact.

$\frac{\partial G}{\partial x} = M = y^2\cos x$ so $G = y^2\sin x + \phi(y)$ for some $\phi$.

Thus $\frac{\partial G}{\partial y} = 2y\sin x + \phi'(y)$, but $\frac{\partial G}{\partial y} = N = 2y\sin x + 6y^2$

so $\phi'(y) = 6y^2$, so $\phi(y) = 2y^3 + C$.

Thus $G = y^2\sin x + 2y^3 + C$, and our equation has solutions $y^2\sin x + 2y^3 = C_1$.

$$y(0) = 1 \Rightarrow C_1 = 2 \Rightarrow \quad y^2\sin x + 2y^3 = 2.$$

(3) Again we make it exact, though this is harder since the integrating factor is a function of $x$ and $y$.

$$M = 1 + 2x^2, \quad N = \frac{e^{-x^2}}{y^2} + \frac{x}{y}.$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2xe^{-x^2}}{y^2} - \frac{1}{y} = 2x\left(\frac{e^{-x^2}}{y^2} + \frac{x}{y}\right) - \frac{2x^2}{y} - \frac{1}{y}$$

$$= 2xN - \frac{1}{y}M.$$

So we can choose $\mu = e^{\int 2x\,dx + \int \frac{1}{y}dy} = ye^{x^2}$.

This gives the ODE $ye^{x^2} + 2yx^2e^{x^2} + \left(\frac{1}{y} + xe^{x^2}\right)y' = 0$.

It's easier to start from $\frac{\partial G}{\partial y} = N = \frac{1}{y} + xe^{x^2}$

$$\Rightarrow G = \ln y + xe^{x^2}y + \phi(x)$$

$\Rightarrow \frac{\partial G}{\partial x} = e^{x^2}y + 2x^2e^{x^2}y + \phi'(x)$. But $\frac{\partial G}{\partial x} = M = e^{x^2}y + 2x^2e^{x^2}y$

so $\phi'(x) = 0$, so $\phi(x) = C$. Thus $G = \ln y + xe^{x^2}y + C$.

So solns are $\ln y + xe^{x^2}y = C_1$. $y(1) = 1 \Rightarrow C_1 = e$

$$\Rightarrow \ln y + xe^{x^2}y = e.$$