Solutions to Exercise sheet 1, Math 263 MR

① $y' = \dfrac{1}{x(e-1-x)}$, $x > e-1$, $y(e) = \dfrac{1}{e-1}$.

Solution: integrate wrt x.

$$y = \int \frac{1}{x(e-1-x)}\,dx + C.$$

Use partial fractions:

$$\frac{1}{x(e-1-x)} = \frac{A}{x} + \frac{B}{e-1-x}; \quad 1 = A(e-1-x) + Bx;$$

$$A = \frac{1}{e-1}, \quad B = \frac{1}{e-1}.$$

So $y = \dfrac{1}{e-1}\displaystyle\int \frac{1}{x}\,dx + \frac{1}{e-1}\int \frac{1}{e-1-x}\,dx + C$

$$= \frac{1}{e-1}\ln|x| - \frac{1}{e-1}\ln|e-1-x| + C.$$

$$= \frac{1}{e-1}\ln x - \frac{1}{e-1}\ln(x-e+1) + C \quad \text{since } x > e-1.$$

$y(e) = \dfrac{1}{e-1} \Rightarrow \dfrac{1}{e-1} = \dfrac{1}{e-1} - 0 + C \Rightarrow C = 0.$

So $\boxed{y = \dfrac{1}{e-1}\ln x - \dfrac{1}{e-1}\ln(x-e+1)}$.

② $y' = y\cot x$, $x \in (0,\pi)$, $y(\frac{\pi}{2}) = 1$.

Solution: divide by y and note that $\dfrac{y'}{y} = \dfrac{d}{dx}(\ln|y|)$.

Thus $\ln|y| = \displaystyle\int \cot x\,dx + C = \int \frac{\cos x}{\sin x}\,dx + C$

$$= \int \frac{d}{dx}(\ln|\sin x|)\,dx + C$$

$$= \ln|\sin x| + C.$$

So $y = \pm e^{c} \sin x$

$= C_1 \sin x$.

$y\left(\frac{\pi}{2}\right) = 1 \Rightarrow 1 = C_1 \times 1 \Rightarrow C_1 = 1$

so $y = \sin x$.

③ $y' = \left(\tan x - \frac{1}{x}\right)y + 2\sec x, \quad x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \quad y(2\pi) = 0.$

Solution: we need to use the integrating factor

$$e^{\int\left(\frac{1}{x} - \tan x\right)dx} = e^{\ln|x| - \ln|\sec x|} = \frac{|x|}{|\sec x|} = |x\cos x|.$$

In fact, since $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, $x\cos x > 0$, so we just use $x\cos x$ rather than $|x\cos x|$.

$$(x\cos x)y' + (x\cos x)\left(\frac{1}{x} - \tan x\right)y = 2(x\cos x)\sec x$$

$$\Rightarrow (x\cos x)y' + (\cos x - x\sin x)y = 2x$$

$$\Rightarrow \frac{d}{dx}\left((x\cos x)y\right) = 2x$$

$$\Rightarrow (x\cos x)y = x^2 + C$$

$$\Rightarrow y = \frac{x^2 + C}{x\cos x}.$$

$$y(2\pi) = 0 \Rightarrow 0 = \frac{4\pi^2 + C}{2\pi} \Rightarrow C = -4\pi^2$$

so $$y = \frac{x^2 + 4\pi^2}{x\cos x}.$$