Math 263 Sheet 10 solutions MR.

① a) $x^3 = \sum_{n=0}^{\infty} a_n (x-2)^n$. Clearly ~~we~~ we need $a_4 = a_5 = \dots = 0$, since there are no terms $x^4, x^5, \dots$ on the left-hand side. Thus we must choose $a_3 = 1$ to make the coefficient of x^3 agree. Then $a_3(x-2)^3 = x^3 - 6x^2 + 12x - 8$, so to make the coeffs of x^2 agree we need $a_2 = 6$.

Then $a_3(x-2)^3 + a_2(x-2)^2 = x^3 - 6x^2 + 12x - 8 + 6x^2 - 24x + 24$

$$= x^3 - 12x + 16,$$

so to make the coeffs of x agree we need $a_1 = 12$.

Then $a_3(x-2)^3 + a_2(x-2)^2 + a_1(x-2)$

$$= x^3 - 12x + 16 \quad + 12x - 24$$

$$= x^3 - 8$$

so to make the constants agree we need $a_0 = 8$.

So $x^3 = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$.

b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

c) $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$

(You can work these out by hand if needed, but you can also just memorise them as they are very standard!)

② $y'' - x^3 y = 0$, $y(0) = 1$, $y'(0) = 0$.

Substitute $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$y(0) = 1 \Rightarrow a_0 = 1$

$y'(0) = 0 \Rightarrow a_1 = 0$.

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - x^3\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=3}^{\infty}a_{n-3}x^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 x + 12a_4 x^2 + \sum_{n=3}^{\infty}\left((n+1)(n+2)a_{n+2} - a_{n-3}\right)x^n = 0$$

$$\Rightarrow a_2 = a_3 = a_4 = 0$$

and for $n \geq 3$, $(n+1)(n+2)a_{n+2} = a_{n-3}$

i.e. for $n \geq 0$, $(n+4)(n+5)a_{n+5} = a_n$

i.e. $$a_{n+5} = \frac{a_n}{(n+4)(n+5)}.$$

In summary we have $a_0 = 1$, $a_1 = a_2 = a_3 = a_4 = 0$, $a_5 = \frac{1}{20}$, $a_6 = ($

(3) Here 0 is a regular singular point.

$xp(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ so $p_0 = 0$, and

$x^2 q(x) = \frac{14}{89}\cos x = \frac{14}{89}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$ so $q_0 = \frac{14}{89}$.

Thus we get the indicial equation

$$F(r) = r(r-1) + \frac{14}{89} = 0$$

i.e. $$r^2 - r + \frac{14}{89} = 0$$

i.e. $$\left(r - \frac{2}{9}\right)\left(r - \frac{7}{9}\right) = 0$$

so we have roots $r_1 = \frac{7}{9}$, $r_2 = \frac{2}{9}$. ~~Then~~ $r_1 - r_2$ is not an integer, so we have two LI solutions

$$y_1 = x^{7/9}\sum_{n=0}^{\infty}a_n(r_1)x^n \quad \text{and} \quad y_2 = x^{2/9}\sum_{n=0}^{\infty}a_n(r_2)x^n.$$