MA20034. Probability & Random Processes
Example Sheet Nine

1. In a colony of bacteria, each bacterium lives for one unit of time at the end of which it is replaced by a random number of bacteria chosen as an independent copy of offspring RV L with $\mathbb{P}(L = k) = p_k$ $(k \geq 0)$. Let Z_n be the number of bacteria alive at time n. Suppose that $Z_0 = 1$. For $\theta \in [0, 1]$, define $g(\theta) := \mathbb{E}(\theta^L)$ and $\mu := \mathbb{E}(L)$.
   Let $\pi_n := \mathbb{P}(Z_n = 0) = \mathbb{P}(\text{extinct by time } n)$.

   (a) Show that π_n is increasing and that $\pi_n \uparrow \pi := \mathbb{P}(\text{ultimate extinction})$.
   (b) Show that $\pi_{n+1} = g(\pi_n)$ and $\pi_0 = 0$. Deduce that $\pi = g(\pi)$.
   (c) For the following special cases, calculate $g(\theta)$, μ, π and include a sketch of $g(\theta)$, indicating π_n and π clearly in your pictures:

   (i) $\mathbb{P}(L = 2) = 1/4$, $\mathbb{P}(L = 0) = 3/4$;
   (ii) $\mathbb{P}(L = 2) = 1/2$, $\mathbb{P}(L = 1) = 1/3$, $\mathbb{P}(L = 0) = 1/6$;
   (iii) $\mathbb{P}(L = 2) = 1/2$, $\mathbb{P}(L = 0) = 1/2$;
   (iv) $\mathbb{P}(L = k) = (1/4)(3/4)^k$ for $k = 0, 1, 2, 3, \ldots$.

2. Consider a discrete time branching process $Z = (Z_n)_{n \geq 0}$ with generic offspring RV X. Assume that $\mu := \mathbb{E}(X) < \infty$.
   Define $g(\theta) := \mathbb{E}(\theta^X) = \sum_{i=0}^{\infty} \mathbb{P}(X = k)\theta^k$ and $g_n(\theta) := \mathbb{E}(\theta^{Z_n})$.
   Observe that for $X_1, X_2, \ldots$ IID copies of X,

$$Z_{n+1} = X_1 + X_2 + \cdots + X_{Z_n} = \sum_{i=1}^{Z_n} X_i$$

   where X_i represents the number of replacements for the i^{th} individual alive in the n^{th} generation which is of size Z_n.
   Show that $\psi(k) := \mathbb{E}(\theta^{Z_{n+1}} | Z_n = k) = g(\theta)^k$.
   Deduce that $\mathbb{E}(\theta^{Z_{n+1}} | Z_n) = g(\theta)^{Z_n}$. Using the Tower property of conditional expectations, show that

$$g_{n+1}(\theta) = g_n(g(\theta)).$$

   Hence, deduce that $g_n(\theta) = (g \circ g \circ \cdots \circ g)(\theta)$, the n-fold iteration of g.
   Deduce $g_{n+1}(\theta) = g(g_n(\theta)) = g_n(g(\theta))$.
   Note, $\pi_n := \mathbb{P}(Z_n = 0) = g_n(0)$, hence $\pi_{n+1} = g(\pi_n)$.

3. Consider a discrete time branching process $Z = (Z_n)_{n\geq 0}$ with $Z_0 = 1$ and generic offspring RV X given by $\mathbb{P}(X = k) = pq^k$ for $k = 0, 1, 2, \ldots$, where $q := 1 - p \in (0, 1)$.

   Show that $g(\theta) := \mathbb{E}(\theta^X) = p/(1 - q\theta)$ and $\mu := \mathbb{E}(X) = g'(1) = q/p$.

   Define $g_n(\theta) := \mathbb{E}(\theta^{Z_n})$. Note, $g_0(\theta) = \theta$ and $g_1(\theta) = g(\theta)$.

   Using the relation $g_{n+1}(\theta) = g(g_n(\theta))$, for $p \neq q$, prove by induction that
$$g_n(\theta) = \frac{(p\mu^n - p) + \theta(q - p\mu^n)}{(q\mu^n - p) + \theta(q - q\mu^n)}.$$

   Deduce that, for $p \neq q$,
$$\pi_n := \mathbb{P}(Z_n = 0) = \frac{p(q^n - p^n)}{q^{n+1} - p^{n+1}}.$$

   Hence, determine $\pi := \mathbb{P}(Z_n = 0 \text{ for some } n) = \lim_{n\to\infty} \pi_n$.
   Verify that π is the minimal solution in $[0, 1]$ of $g(\pi) = \pi$.

   [NB: only a few special cases permit g_n in closed form, as above.]

4. ‡ A population of animals consists of two types, A and B. Animals live for one unit of time and give birth to an independent random number of replacements. Suppose that an A type parent either has one type A child with probability 1/2, or one type B child with probability 1/4, or no children with probability 1/4. Suppose that a B type parent either has two type B offspring with probability 1/2, or one type A offspring with probability 1/2.

   (i) Let a, b be the probabilities of extinction starting from a population of just one type A, or just one type B animal, respectively. Calculate a and b, *proving* that both are less than 1.

   (ii)** Suppose additionally that at the time of birth, each animal is independently killed with probability p. Find the critical value p_0 such that the animal population started with one animal has a positive probability of surviving forever if and only if $p < p_0$. ***Answer:*** $p_0 = (2/\sqrt{3}) - 1$.

Note: Questions marked with ‡ are *optional* and * are *harder*.
8/12/2009 `http://people.bath.ac.uk/massch`