MA20034. Probability & Random Processes
Example Sheet Seven

1. Suppose $p \geq 1$. Recall that $X \in \mathcal{L}^p$ means that $\mathbb{E}(|X|^p) < \infty$. Show that $X \in \mathcal{L}^p$ implies that $X \in \mathcal{L}^q$ for all $q \in [1, p]$.
   Hint: *Observe that* $y^q \leq 1 + y^p$ *for* $y \geq 0$.

2. Use Markov's inequality to show that $X_n \to X$ in $\mathcal{L}^1$ implies that $X_n \to X$ in probability.

3. Suppose $Y_1, Y_2, \dots$ is a sequence of RVs with $\mathbb{P}(Y_n = 0) = n^{-2}$ and $\mathbb{P}(Y_n = n^3) = 1 - n^{-2}$. Show that $Y_n \to 0$ in probability but it is not the case that $Y_n \to 0$ in $\mathcal{L}^1$. Does $Y_n \to 0$ almost surely here?
   In general, does $Z_n \to Z$ almost surely imply that $Z_n \to Z$ in $\mathcal{L}^1$?

4. Let $\mu > 0$, $\alpha \in (0, 1)$ and $\lambda_n := \mu(1 - \alpha)\alpha^n$. Suppose that $X_0, X_1, \dots$ is a sequence of independent RVs where $X_n \sim Po(\lambda_n)$.
   (a) Find the PGF of X_n, $g_n(s) := \mathbb{E}(s^{X_n})$.
   (b) Define $Y_n := \sum_{i=0}^{n} X_i$. Find the PGF of Y_n, $h_n(s) := \mathbb{E}(s^{Y_n})$. Deduce the distribution of Y_n.
   (c) Suppose Z is a Poisson RV of parameter μ. Stating any result you appeal to, use PGFs to show that Y_n converges in distribution to Z, that is, for each $k \in \mathbb{Z}^+$, $\mathbb{P}(Y_n = k) \to \mathbb{P}(Z = k)$ as $n \to \infty$.
   (d) Define $Y := \sum_{i=1}^{\infty} X_i$. Use monotone convergence (or continuity of $\mathbb{P}$) to show that $\mathbb{P}(Y \leq k) = \lim_{n\to\infty} \mathbb{P}(Y_n \leq k)$, and thus $Y \sim Po(\mu)$. Show that (i) $Y_n \to Y$ in probability, (ii) $Y_n \to Y$ in $\mathcal{L}^1$, and (iii) $Y_n \to Y$ almost surely.
   Hint: *For (i) and (ii), observe that* $Y - Y_n \sim Po(\theta_n)$ *where* $\theta_n := \sum_{i=n+1}^{\infty} \lambda_n = \dots$ *For (iii), use Borel-Cantelli to show only finitely many of the* X_i*'s are non-zero.*

5. **Jensen's inequality.** Suppose that $c : \mathbb{R} \to \mathbb{R}$ is a *convex* function, so that for any $a \in \mathbb{R}$, there exists a constant $\lambda \in \mathbb{R}$ such that

$$\forall a \in \mathbb{R},\ \exists \lambda \in \mathbb{R}, \text{ such that } c(x) \geq c(a) + \lambda(x - a), \quad \forall x \in \mathbb{R}.$$

   Draw a picture to visualise this property. Note that $x^2, e^{\theta x}, |x|$ are all convex functions.
   Show that for any $\mathbb{R}$-valued RV X with $\mathbb{E}(|X|) < \infty$ and $\mathbb{E}(|c(X)|) < \infty$ for some convex function c, we have Jensen's inequality:

$$\mathbb{E}(c(X)) \geq c(\mathbb{E}(X)).$$

6. ‡ **Cauchy-Schwarz inequality.** Suppose $X, Y \in \mathcal{L}^2$. Prove the Cauchy-Schwarz inequality that

$$\{\mathbb{E}(XY)\}^2 \leq \mathbb{E}(X^2)E(Y^2).$$

Hint: *Set* $Z := \lambda X - Y$ *and note that* $f(\lambda) := \mathbb{E}(Z^2) \geq 0$ *is a quadratic in* λ*, so can have at most one real solution. Now think of what that means in the quadratic formula.*

7. ‡ Find a sequence of RVs $X_1, X_2, \ldots$ taking values in $\{0, 1\}$ such that $X_n \to 0$ in $\mathcal{L}^1$ but it is *not* the case that $X_n \to 0$ almost surely.

8. ‡ Let $p > q \geq 1$. Suppose $(X_i)_{i \geq 1}$ is a sequence of real valued RVs with $\mathbb{P}(X_n = 0) = 1 - n^{-(p+q)/2}$ and $\mathbb{P}(X_n = n) = n^{-(p+q)/2}$.
(a) Show that $X_n \to 0$ in probability.
Recall, $Z_n \to Z$ in $\mathcal{L}^p$ if $\mathbb{E}(|Z_n - Z|^p) \to 0$ as $n \to \infty$.
(b) Calculate $\mathbb{E}(|X_n|^q)$ and hence show that $X_n \to 0$ in $\mathcal{L}^q$.
(c) Demonstrate that, in general, $X_n \to X$ in $\mathcal{L}^q$ for some $q \geq 1$ does *not* imply that $X_n \to X$ in $\mathcal{L}^p$ for $p > q$.

Note: Questions marked with ‡ are *optional* and * are *harder*.
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