MA20034. Probability & Random Processes
Example Sheet Five

1. **Simple Random Walk reflected at the origin.**
   Let $W = (W_n)_{n\geq 0}$ be a symmetric Simple Random Walk on $\mathbb{Z}$, $SRW(1/2)$. Define $\hat{W} = (\hat{W}_n)_{n\geq 0}$ by $\hat{W}_n := |W_n|$. Draw a typical path of $\hat{W}$ and describe its motion. Show that, for $y > 0$,
   $$\mathbb{P}_x(\hat{W}_n = y) = \mathbb{P}_x(W_n = y) + \mathbb{P}_x(W_n = -y).$$

2. **Maximum of a Simple Random Walk.**
   Let $W = (W_n)_{n\geq 0}$ be a $SRW(p)$. Define the maximum of the RW up to time n by $M_n := \max_{k\leq n} W_k$. Prove that, for $m \geq w$ and $m \geq 0$,
   $$\mathbb{P}_0(M_n \geq m\,;\, W_n = w) = \binom{n}{u}\, p^v q^{n-v}$$
   where $u := m + \frac{1}{2}(n-w)$ and $v := \frac{1}{2}(w+n)$.
   Deduce that when $p = q = 1/2$ and $w \leq m$, then
   $$\mathbb{P}_0(M_n = m\,;\, W_n = w) = \mathbb{P}_0(W_n = 2m - w) - \mathbb{P}_0(W_n = 2m + 2 - w)$$
   and then that
   $$\mathbb{P}_0(M_n = m) = \mathbb{P}_0(W_n = m) + \mathbb{P}_0(W_n = m + 1).$$

3. **Simple Random Walk killed at the origin.**
   Let $\tilde{W} = (\tilde{W}_n)_{n\geq 0}$ be a $SRW(p)$ *killed* at 0. Show that, for $x > 0$,
   $$\begin{aligned}\mathbb{P}_x(\tilde{W}_n = y) &= \left\{\binom{n}{u} - \binom{n}{v}\right\} p^u q^{n-u} \\ &= \mathbb{P}_x(W_n = y) - \left(\frac{p}{q}\right)^y \mathbb{P}_x(W_n = -y),\end{aligned}$$
   where $u := (n+y-x)/2$, $v := (n+y+x)/2$ and $W = (W_n)_{n\geq 0}$ is a (standard) $SRW(p)$ on $\mathbb{Z}$.
   *[**Hint:** Any given path going from x to y over n steps must have exactly u upward jumps and the same probability of occurring. Use the reflection principle to find the number of such paths that avoid the origin.]*

4. ‡* Consider the 8 possible patterns of three coin tosses: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. Show that if I name *any* of these patterns, then you could name another one of them such that the chance that your pattern appears before mine in a sequence of fair coin tosses is *strictly* greater than 1/2. Try playing this game with your friends - how long until they realise what you are doing?

5. ‡** **Random Walk reflected in a strip.**
Let $\hat{W}$ be a *symmetric SRW reflected at* 0 *and* b, where $b \in \mathbb{N}$. That is, if $1 \leq \hat{W}_n \leq b-1$ the next jump is equally likely to be ± 1 independent of the past; if $\hat{W}_n = b$, then $\hat{W}_{n+1} = b-1$ with probability 1; if $\hat{W}_n = 0$, then $\hat{W}_{n+1} = 1$ with probability 1.
It can be shown that, for $x, y \in \{0, 1, \ldots, b\}$,

$$\mathbb{P}_x(\hat{W}_n = y) = \sum_{n=-\infty}^{+\infty} \{\mathbb{P}_x(W_n = y + 2nb) + \mathbb{P}_x(W_n = -y + 2nb)\}$$

where $W = (W_n)_{n \geq 0}$ is a $SRW(1/2)$ on $\mathbb{Z}$.
Draw the integer line, divide into strips of width b and mark with plus signs '$\oplus$' all the positions of W_n that appear in the sum above.
Observe, although convenient to write as an infinite summation, there will only be a finite number of non-zero terms since $|W_n - W_0| \leq n$.

Can you convince yourself why this result should be true?

[***Hint:*** *Use the reflection principle repeatedly! Each time the path of a RW on* $\mathbb{Z}$ *goes outside the strip* $0, \ldots, b$, *think of reflecting the path back inside. This way, try to map any given path of a RW on* $\mathbb{Z}$ *to a path of the reflected RW. Similarly, from a path of a reflected RW, construct all the possible paths of the RW on* $\mathbb{Z}$ *that could have given rise to it after applying your map. Each time the reflected path hits either boundary you should double the number of possible RW on* $\mathbb{Z}$ *paths that could give rise to it!]*

Note: Questions marked with ‡ are *optional* and * are *harder*.