MA20034. Probability & random processes
Example Sheet Three

1. **Primes and the Riemann-Zeta function.** Recall that a prime number of $\mathbb{N}$ is one of $2, 3, 5, 7, \ldots,$ but note that 1 is *not* considered prime. Let $s > 1$ and suppose a RV X has

$$\mathbb{P}(X = n) = \frac{n^{-s}}{\zeta(s)} \qquad (n \in \mathbb{N})$$

where ζ is the famous *Riemann-Zeta function* defined by $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, where the series converges for $s > 1$.
Let E_m be the event that X is divisible by m.
(a) Prove that $\mathbb{P}(E_m) = m^{-s}$ for $m \in \mathbb{N}$.
(b) Prove that the events $(E_p : p \text{ prime})$ are independent.
[**Hint:** If p_1 and p_2 are distinct primes, then a number is divisible by both p_1 and p_2 if and only if it is divisible by $p_1 p_2$; and similarly for more than two distinct primes.]
(c) By considering $\cap_{p \text{ prime}} E_p^c$, prove Euler's formula that

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right).$$

2. Consider a RW W on $\mathbb{Z}$, $W_n := a + X_1 + \cdots + X_n$, where $X_1, X_2, \ldots$ are IID with $\mathbb{P}(X_n = -1) = \frac{2}{7}, \mathbb{P}(X_n = 1) = \frac{2}{7}, \mathbb{P}(X_n = 2) = \frac{3}{7}$.
(a) Define $x_k := \mathbb{P}_k(\text{hit } 0) := \mathbb{P}(\text{hit } 0 | W_0 = k)$. By splitting the event of hitting 0 over the first move taken by the random walk, show that

$$x_1 = \frac{2}{7} + \frac{2}{7} x_2 + \frac{3}{7} x_3.$$

Noting that the RW can only move down 1 unit at any time, explain why $x_k = x_1^k$ for $k \geq 1$.
Calculate $\mathbb{E}(X_n)$ and explain why the Random Walk will drift off to $+\infty$, hence deduce that $x_1 < 1$. Show that $\mathbb{P}_k(\text{hit } 0) = 3^{-k}$ for $k \geq 1$.
(b) For $k \geq -1$, define $y_k := \mathbb{P}_0(\text{hit } k) = \mathbb{P}_0(W_n = k \text{ for some } n \geq 0)$. State the values of y_{-1} and y_0. Show that for $k \geq 1$,

$$y_k = \frac{2}{7} y_{k+1} + \frac{2}{7} y_{k-1} + \frac{3}{7} y_{k-2}. \qquad (*)$$

Does this equation hold for $k = 0$?
Look for solutions to $(*)$ of the form $y_i \propto m^k$ and hence write down the auxillary equation that m must satisfy. Find the roots m_1, m_2, m_3 of the auxilliary equation. Since these roots are distinct, the general solution to the recurrence equation $(*)$ is $y_k = Am_1^k + Bm_2^k + Cm_3^k$ for $k \geq -1$, for some constants A, B, C. Use the boundary conditions and the fact that $0 \leq y_k \leq 1$ for every k to determine A, B, C and hence give an expression for $\mathbb{P}_0(\text{hit } k)$ for $k \geq 1$.
(c) Let $r := \mathbb{P}_0(\text{return to } 0)$. By considering the first move of the RW, find an expression for r in terms of y_1 and x_1. Deduce $r = 1/3$.

3. Let $W = (W_n)_{n\geq 0}$ be a simple Random Walk, $SRW(p)$.
   Let $y_k := \mathbb{E}_k(T_0) := E(T_0|W_0 = k)$ where $T_0 := \inf\{n \geq 0 : W_n = 0\}$ is the first hitting time of 0. By splitting according to the first step of the RW, show that $y_1 = (1-p) + p(1+y_2)$. Explain why $y_k = k\,y_1$ for $k \geq 1$. Deduce that $\mathbb{E}_1(T_0) = 1/(1-2p)$ when $p < 1/2$. What is $\mathbb{E}_1(T_0)$ when $p \geq 1/2$?
   Recall that $\mathbb{P}_1(\text{hit } 0) = \mathbb{P}_1(T_0 < \infty) = (1-p)/p$ when $p \geq 1/2$. Deduce $\mathbb{P}_1(T_0 = \infty)$ for $p \geq 1/2$. In particular, for the symmetric simple Random Walk $SRW(1/2)$, note that $\mathbb{P}_1(T_0 < \infty) = 1$ but $\mathbb{E}_1[T_0] = \infty$.

4. Consider any Random Walk $W = (W_n)_{n\geq 0}$ on $\mathbb{Z}$, that is, $W_n := a + X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are IID RVs.
   Let $r := \mathbb{P}_0(\text{return to } 0)$ be the probability that W returns to 0 given it starts at 0. Let N be the total number of visits to 0 including the visit at time 0. Since intuitively the Random Walk 'starts afresh' whenever it first returns to 0, show that for $r \in [0, 1)$,
   $$\mathbb{P}_0(N = k) = r^{k-1}(1-r) \qquad (k = 1, 2, \ldots),$$
   so that the total number of visits to 0 when started at 0 is a Geometrically distributed RV. Also note that $\mathbb{P}_0(N = +\infty) = 1$ when $r = 1$.
   Hence, calculate the expected total number of visits to 0 when starting at 0, $\mathbb{E}_0(N)$, in terms of r.
   On the other hand, with $\xi_n := I_{\{W_n=0\}}$, note $N = \xi_0 + \xi_1 + \xi_2 + \ldots$ where the sum counts 1 for every time 0 is visited. By taking expectations and comparing expressions, deduce that
   $$\frac{1}{1 - \mathbb{P}_0(W \text{ returns to } 0)} = \sum_{n=0}^{\infty} \mathbb{P}_0(W_n = 0). \qquad (+)$$

5. **Recurrence of symmetric simple Random Walks on $\mathbb{Z}$ and $\mathbb{Z}^2$**
(a) Let $W = (W_n)_{n\geq 0}$ be a simple Random Walk, $SRW(p)$. Show that

$$\mathbb{P}_0(W_n = 2k - n) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (k = 0, 1, 2, \ldots, n).$$

Deduce that for the symmetric Random Walk, $SRW(1/2)$,

$$\mathbb{P}_0(W_{2n} = 0) = \frac{(2n)!}{(n!)^2\, 2^{2n}}.$$

Stirling's formula says that $n! \sim n^n e^{-n}\sqrt{2\pi n}$, that is

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

Use this to show that $\mathbb{P}_0(W_{2n} = 0) \sim 1/\sqrt{\pi n}$. Using equation (+) above, deduce that $\mathbb{P}_0(W \text{ returns to } 0) = 1$, that is, the symmetric Simple Random Walk on $\mathbb{Z}$ is *recurrent* (it will visit 0 infinitely often).
(b) Consider the two dimensional Random Walk $\mathbf{W}$ where $\mathbf{W_n} = (U_n, V_n)$ with $U = (U_n)_{n\geq 0}$ and $V = (V_n)_{n\geq 0}$ two *independent* symmetric simple Random Walks, $SRW(1/2)$.
Draw a picture of how $\mathbf{W}$ moves on $\mathbb{Z}^2$ and notice that, after rotation by 45^o and a relabeling of vertices, $\mathbf{W}$ is equivalent to a symmetric simple Random walk on $\mathbb{Z}^2$ where a particle jumps from its current position to one of its four nearest neighbours with probability 1/4 each, independent of its past history. Show that

$$\mathbb{P}_{(0,0)}(\mathbf{W}_{2n} = (0,0)) = \mathbb{P}_0(U_{2n} = 0)\mathbb{P}_0(V_{2n} = 0) \sim \frac{1}{\pi n}$$

and hence deduce that *the return probability to* $(0,0)$ *for the symmetric simple Random Walk on* $\mathbb{Z}^2$ *is* 1, that is, the Random Walk is *recurrent.*