MA20034. Probability & random processes
Example Sheet Two

1. Suppose the failure time T for a component is exponentially distributed with rate $\lambda > 0$. Write down the probability density function, f, for RV T. Calculate $\mathbb{P}(T > t)$. Show that the exponential distribution has the *lack of memory property*, $\mathbb{P}(T > s + t | \, T > s) = \mathbb{P}(T > t)$.

2. The Smiths introduce you to one of their two children, Derek (a boy). (i) What is the probability that both children are boys? (Assume that the couple had an equal chance of having a boy or a girl at each birth.) (ii) Does your answer change if Derek tells you he is the younger of the two children? If so, how?

3. Suppose that you are at a birthday party at which there are n people where $n < 365$. Let p_n be the probability that at least two people share a common birthday. For simplicity, ignore leap years and assume that the birthdays are distributed uniformly over 365 days of the year.
(a) Show that $p_n = 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$.
(b) Recall that $1 - x \leq e^{-x}$ to show that $p_n \geq 1 - \exp\{-\frac{(n-1)n}{730}\}$.
(c) Show the probability at least two people share a common birthday is greater than $1/2$ whenever $n \geq 23$. [**Hint:** $730 \ln 2 \approx 505.997$].

4. Donated blood is screened for a certain disease. Suppose that one in thousand people have this disease. Suppose also that the test has 99% accuracy when a subject has the disease, and that it has a 2% false-positive rating. You have just donated blood.
(a) Let A be the event that you are infected, and B be the event that you test positive. What is the probability you test positive, $\mathbb{P}(B)$?
(b) Suppose that the test result is positive. What is the probability that you actually have the disease?

5. Suppose a bag contains red and black balls. A player successively picks a ball from the bag at random. If the chosen ball is Red he wins, otherwise he loses, and he returns the ball into the bag after each go.
(a) Suppose an additional black ball is added to the bag after each pick and initially there is only one red ball in the bag. What is the probability of winning on the n^{th} draw? Deduce that the player wins infinitely many games.

(b) Suppose initially there is one red and one black ball, and after each pick extra black balls are added to double the total number of balls in the bag. What is the probability of winning on the n^{th} draw? Does the player win infinitely many games here?

6. **The 'Car and Goats' problem.** (a) On the American TV 'Monty Hall game show', a contestant is shown three closed doors. Behind one of the doors is a car; behind each of the other two is a goat. The contestant chooses one of the three doors. The show's host, who knows which door conceals the car, opens one of the remaining two doors which he knows will definitely reveal a goat. He then asks the contestant whether of not she wishes to switch her choice to the remaining closed door. Should she switch or stick to her original choice?
   (b) Suppose there had been 1000 doors, 1 car and 999 goats. After the contestant's choice the host opens 998 doors each of which he knows will reveal a goat. By how much does switching now increase our contestant's chance of winning?

7. **Null and almost sure events.** (a) Suppose $\mathbb{P}(N_i) = 0$ for all $i \geq 1$. Show that $\mathbb{P}(\cup_i N_i) = 0$. Deduce that, if $\mathbb{P}(A_i) = 1$ for all $i \geq 1$, then $\mathbb{P}(\cap_i A_i) = 1$. [**Hint:** Use Boole's inequality $\mathbb{P}(\cup_{i=1}^n B_i) \leq \sum_{i=1}^n \mathbb{P}(B_i)$ and continuity of $\mathbb{P}$ (equivalently MON).]
   (b) Consider tossing a coin infinitely many times. Show that the probability that a given infinite sequence of coin tosses occurs is zero. Deduce that the probability of any one of a given *countable* number of infinite sequences occurring is zero.

8. **Second Borel-Cantelli Lemma: proof.** Suppose $J_1, J_2, \ldots$ is an infinite sequence of *independent* events where $\sum_i \mathbb{P}(J_i) = +\infty$. Show that $\mathbb{P}(\text{none of } J_1, \ldots, J_n \text{ occur}) \leq \exp -\sum_{i=1}^n \mathbb{P}(J_i)$. Deduce that $\mathbb{P}(\text{none of } J_1, \ldots \text{ occur}) = 0$.
   [**Hint:** Use $1 - x \leq e^{-x}$ and continuity of $\mathbb{P}$ (equivalently MON).]
   Let H_n be the event that at least one of the events $J_{n+1}, J_{n+2}, \ldots$ occurs. Deduce that $\mathbb{P}(H_n) = 1$ for all $n \geq 1$, hence $\mathbb{P}(\cap_n H_n) = 1$.
   Explain briefly why $\cap_n H_n = \cap_n \cup_{i=n+1}^{\infty} J_i$ is the event that infinitely many of the events $J_1, J_2, \ldots$ occurs.