MA20034. Probability & random processes .
Example Sheet One

1. Which of the following three events is most likely?
   (a) You get at least 1 head when 2 coins are flipped.
   (b) You get at least 2 heads when 4 coins are flipped.
   (c) You get at least 3 heads when 6 coins are flipped.

2. Suppose you are at a birthday party with n other people. How large must n be to ensure that the probability that you share your birthday with at least one other person is greater than 1/2?
   [***Hint:*** *Assume 365 days per year and birthdays are uniformly distributed. Note,* $\ln 2 \approx 0.693$ *and* $\log(1+x) \approx x$ *for* x *small.*]

3. A tennis tournament for 2^n equally matched players is organised as a knock-out with n rounds, the last round being the final. Two players are chosen at random. Calculate the probabilities that they meet:
   (i) in the first round, (ii) in the final, (iii)* in any round.

4. **The hat matching problem.** N people go to a party, each leaving their coat at the door. When they leave, each takes a coat at random.
   (a) What's the probability that nobody gets the correct coat?
   (b) What happens to this probability as $N \to \infty$?
   (c) Show the expected number of people to get the correct coat is 1.
   [***Hints:*** *(a) Let* A_i *be the event the* i^{th} *person gets the correct coat, and use the inclusion-exclusion formula. (c) Let* $R_N := \sum_{i=1}^N I_{A_i}$.]

5. A committee of size r is chosen at random from a set of n people. Calculate the probability that m given people will all be on the committee (a) directly, and (b) using the inclusion-exclusion formula. Deduce that

$$\binom{n-m}{r-m} = \sum_{j=o}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r}.$$

6. *Each packet of Frosties contains one of five possible plastic figures from the Simpsons, the probability a packet contains any specific figure being 1/5, independently of all other packets. After purchasing six packets, show that the probability of having figures of Homer, Bart and Lisa is

$$1 - 3\left(\frac{4}{5}\right)^6 + 3\left(\frac{3}{5}\right)^6 - \left(\frac{2}{5}\right)^6.$$

7. **Continuity of $\mathbb{P}$.** If events $F_n \uparrow F$, show that $\mathbb{P}(F_n) \uparrow \mathbb{P}(F)$.
Deduce that $G_n \downarrow G \Rightarrow \mathbb{P}(G_n) \downarrow \mathbb{P}(G)$.
[***Hint:*** *Note* $F = \cup_n F_n = \cup_n \{F_n \setminus F_{n-1}\}$ *with* $F_0 := \emptyset$.]

8. **Properties of the Distribution Function (DF) F_X of RV X.**
(i) State what it means for X to be an $\mathbb{R}$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Define the *distribution function* $F_X : \mathbb{R} \to [0,1]$ by $F_X(x) := \mathbb{P}(X \le x)$ for all $x \in \mathbb{R}$. Show that $\mathbb{P}(a < X \le b) = F_X(b) - F_x(a)$, and
(a) F_X is non-decreasing,
(b) $\lim_{y \downarrow -\infty} F_X(y) = 0$ and $\lim_{y \uparrow +\infty} F_X(y) = 1$,
(c) F_X is right continuous: for $x \in \mathbb{R}$, $\lim_{y \downarrow x} F_X(y) = F_X(x)$.
[***Hint:*** *Use the continuity of* $\mathbb{P}$ *for (b) and (c).*]
(ii) If Y is a RV on $[-\infty, +\infty]$, what can you say about F_Y?

9. ‡ **Extending the fundamental model.** Suppose $F : \mathbb{R} \to [0,1]$ is a function satisfying properties 8(a)-(c) above.
(a) Use right-continuity to show that
$$G(u) := \inf\{y : F(y) \ge u\} = \min\{y : F(y) \ge u\},$$
that is, the infimum is attained. Illustrate G for a simple choice of F.
(b) Show that $G(u) \le x \iff u \le F(x)$.
(c) Let U be a uniformly distributed on $[0,1]$. Deduce that $X := G(U)$ is a RV on $\mathbb{R}$ with distribution function F.

10. ‡* **Extending the fundamental model, again.** Let U be a uniformly distributed RV on $[0,1]$. Write U as its binary expansion, that is, let $U = 0 \cdot B_1 B_2 B_3 B_4 \ldots$ where $(B_i)_{\ge 1}$ is a sequence with $B_i \in \{0,1\}$ for each i and $U = \sum_{i \ge 1} B_i 2^{-i}$.
(i) What can you say about $B_1, B_2, \ldots$?
(ii) Explain intuitively how this can be used to *model infinitely many tosses a fair coin.* Draw some diagrams to illustrate your answer.
(iii) Starting from the single uniform random variable U, explain intuitively you can create an *infinite sequence of independent RVs* $U_1, U_2, \ldots$ *each uniformly distributed on* $[0,1]$. Amazingly, this means the fundamental model contains infinitely many copies of itself!!! [***Hint:*** *Place numbered balls in order inside an infinite triangle from the top down!*]

Note: Questions marked with ‡ are *optional* and * are *harder*.