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Asymptotic results for exponential functionals of Lévy processes

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1. DB-processes and rescaled limits

Let $\{\xi_{n,i} : n, i \geq 1\}$ be i.i.d. random variables in $\mathbb{N} := \{0, 1, 2, \cdots\}$.

Fix $X_0 \in \mathbb{N}$. A Discrete-time/state branching process (DB-process) $\{X_n : n \geq 0\}$ is defined by (Bienaymé, 1845; Galton–Watson, 1874):

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \qquad n \geq 1. \tag{1}$$

- Consider a sequence of DB-processes

$$\{X_n^{(k)} : n \geq 0\}, \qquad k = 1, 2, \cdots . \tag{2}$$

- A continuous-time/state branching process (CB-process) $\{x(t) : t \geq 0\}$ arises as the rescaled limit (Feller '51; Jiřina '58; Lamperti '67):

$$k^{-1} X_{[kt]}^{(k)} \to x(t), \qquad k \to \infty. \tag{3}$$

Basic idea: Theory of branching processes based on stochastic analysis.

2. CB-processes with stable branching

Recall that a DB-process $\{X_n : n \geq 0\}$ is defined by:

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \qquad n \geq 1.$$

Suppose that $\mu := \boldsymbol{P}(\xi_{1,1}) < \infty$. Then ($\mu - 1 =: \beta$, $1 < \alpha \leq 2$)

$$\begin{aligned} X_n &= X_{n-1} + \sum_{i=1}^{X_{n-1}} (\xi_{n,i} - \mu) + X_{n-1}(\mu - 1), \\ X_n &= X_{n-1} + \frac{\sqrt[\alpha]{X_{n-1}}}{\sqrt[\alpha]{X_{n-1}}} \sum_{i=1}^{X_{n-1}} (\xi_{n,i} - \mu) + \beta X_{n-1}. \end{aligned}$$

- A stable branching CB-process can be defined as the strong solution to (Fu–Li '10):

$$dx(t) = \sqrt[\alpha]{\alpha c x(t-)}\, dZ_\alpha(t) + \beta x(t-)dt, \tag{4}$$

where $\{Z_\alpha(t)\}$ is a spectrally positive α-stable Lévy process.

- When $\alpha = 2$, it reduces to a Feller branching diffusion (Feller '51).

- When $\beta = 0$, it has critical branching ($\Leftrightarrow \mu = \boldsymbol{P}(\xi_{1,1}) = 1$).

3. The branching property

Let $\boldsymbol{P}_v$ be the distribution of a CB-process $(x(t))_{t\geq 0}$ with $x(0) = v$ on the space

$$D_+[0,\infty) := \{\text{positive càdlàg paths}\}.$$

Theorem 1 (Branching property) *The family $(\boldsymbol{P}_v)_{v\geq 0}$ is a convolution semigroup, i.e.,*

$$\boldsymbol{P}_{v_1+v_2} = \boldsymbol{P}_{v_1} * \boldsymbol{P}_{v_2}, \qquad v_1, v_2 \geq 0. \tag{5}$$

Proof (for critical Feller branching) Suppose that $(x_1(t))_{t\geq 0}$ and $(x_2(t))_{t\geq 0}$ are independent with $x_i(0) = v_i$ and

$$dx_i(t) = \sqrt{2cx_i(t)}dB_i(t).$$

Let $\xi(t) = x_1(t) + x_2(t)$. Then

$$d\xi(t) = \sqrt{2c}\big[\sqrt{x_1(t)}dB_1(t) + \sqrt{x_2(t)}dB_2(t)\big] = \sqrt{2c\xi(t)}dW(t),$$

where $dW(t)$ is a Brownian motion defined by

$$dW(t) = \big(\sqrt{\xi(t)}\big)^{-1}\big[\sqrt{x_1(t)}dB_1(t) + \sqrt{x_2(t)}dB_2(t)\big]. \qquad \square$$

Problem Population models without branching property: (1) nonlinear branching; (2) random environment.

4. Generalized CB-processes

A nonlinear DB-process $\{X_k : k \geq 0\}$ is defined by (controlled, Sevast'yanov–Zubkov '74; $\rho, r =$ positive functions):

$$X_k = \sum_{i=1}^{\rho(X_{k-1})} Y_{k,i} + \sum_{i=1}^{r(X_{k-1})} Z_{k,i}. \tag{6}$$

Suppose that $\mu + \nu := \boldsymbol{P}(Y_{1,1}) + \boldsymbol{P}(Z_{1,1}) < \infty$. Then [$f(x) := \mu\rho(x) + \nu r(x) - x$]:

$$X_k = X_{k-1} + \sum_{i=1}^{\rho(X_{k-1})} (Y_{k,i} - \mu) + \sum_{i=1}^{r(X_{k-1})} (Z_{k,i} - \nu) + \underline{\mu\rho(X_{k-1}) + \nu r(X_{k-1}) - X_{k-1}},$$

$$X_n = X_0 + \sum_{k=1}^{n} \sum_{i=1}^{\rho(X_{k-1})} (Y_{k,i} - \mu) + \sum_{k=1}^{n} \sum_{i=1}^{r(X_{k-1})} (Z_{k,i} - \nu) + \sum_{k=1}^{n} \underline{f(X_{k-1})}.$$

- A nonlinear CB-process can be defined as the strong solution to (Li '16+):

$$x(t) = x(0) + \int_0^t \int_0^{\rho(x(s))} \sqrt{2c} W(ds, du) + \int_0^t \int_0^{r(x(s-))} \int_0^\infty z \tilde{N}(ds, du, dz) - b \int_0^t f(x(s)) ds,$$

where $W(ds, du)$ is a Gaussian noise based on $dsdu$ and $\tilde{N}(ds, dz, du)$ a compensated Poisson noise based on $dsm(dz)du$ ($m =$ Lévy measure).

Typical special cases: $\rho(x) = r(x) = f(x) = x^\theta$ (linear: $\theta = 1$; sublinear: $\theta < 1$; superlinear: $\theta > 1$).

Li ('16+): Discrete-time/state approximation, Lamperti transformation, generator, martingale problem, entrance from or hitting to 0 and ∞, expressions for $\boldsymbol{P}[\zeta]$, $\boldsymbol{P}[\sigma]$, $\boldsymbol{P}[\zeta \wedge \sigma]$, where (explosion and extinction times):

$$\zeta = \inf\{t \geq 0 : x(t) = \infty\}, \quad \sigma = \inf\{t \geq 0 : x(t) = 0\}$$

Let ϕ be the branching mechanism and $q = \inf\{z > 0 : \phi(z) > 0\}$.

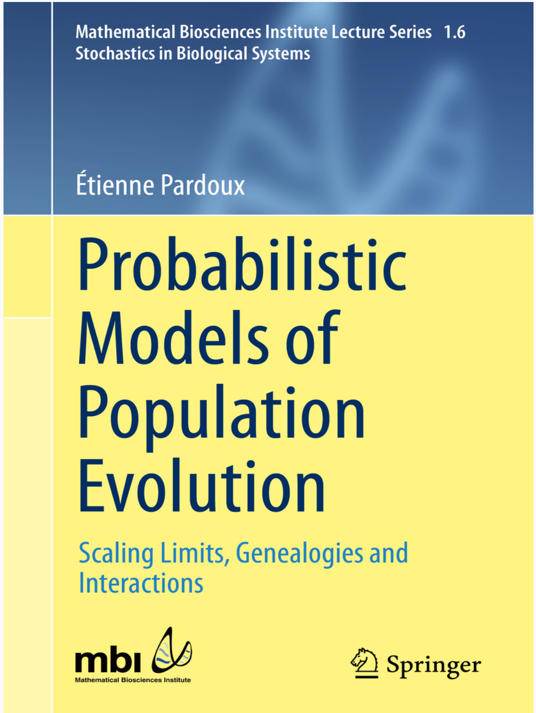
Laplace transform of the transition semigroup:

$$\rho_x(\eta, \lambda) = \int_0^\infty e^{-\eta t} dt \int_{[0,\infty)} e^{-\lambda y} P_t(x, dy), \qquad x, \eta, \lambda > 0. \tag{7}$$

A Volterra-type equation (when $0 < \lambda < q$):

$$\Gamma(\theta)[\rho_x(\eta, \lambda) - \rho_x(\eta, q)] = \int_\lambda^q [\eta \rho_x(\eta, z) - e^{-zx}] \phi(z)^{-1} (z - \lambda)^{\theta - 1} dz. \tag{8}$$

A book on CB-processes with competition (nonlinear branching):



5. Random environments

Recall $\mathbb{N} = \{0, 1, 2, \cdots\}$. Suppose that:

(1) $\mathscr{G} := \{g_n : n \geq 1\}$ are i.i.d. random probability generating functions;

(2) $\{\xi_{n,i} : n, i \geq 1\}$ are $\mathbb{N}$-valued random variables;

(3) given g_n, $\{\xi_{n,i} : i \geq 1\}$ are i.i.d. and $\boldsymbol{P}(s^{\xi_{n,i}} | \mathscr{G}) = g_n(s)$;

(4) $\{\xi_{m,i} : i \geq 1\}$ and $\{\xi_{n,i} : i \geq 1\}$ independent for $m \neq n$.

Fix $X_0 \in \mathbb{N}$. A DB-process in random environment (DBRE-process) is defined by:

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \qquad n \geq 1. \tag{9}$$

See Smith–Wilkinson ('69) and Athreya–Karlin ('71), Vatutin–Dyakonova–Sagitov ('13).

● We are interested in a continuous-state counterpart of the DBRE-process, i.e., CB-process in random environment (CBRE-process).

6. Stable branching CBRE-processes

Recall that a GWRE-process is defined by [$\boldsymbol{P}(s^{\xi_{n,i}}|\mathscr{G}) = g_n(s)$]:

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \qquad n \geq 1.$$

Suppose that $0 < \mu_n := \boldsymbol{P}(\xi_{n,1}|\mathscr{G}) = g_n'(1-) < \infty$. Then ($0 < \alpha \leq 1$)

$$X_n = X_{n-1} + \sum_{i=1}^{X_{n-1}} (\xi_{n,i} - \mu_n) + X_{n-1}(\mu_n - 1),$$

$$X_n = X_{n-1} + \frac{\sqrt[1+\alpha]{X_{n-1}}}{\sqrt[1+\alpha]{X_{n-1}}} \sum_{i=1}^{X_{n-1}} (\xi_{n,i} - \mu_n) + X_{n-1}(\mu_n - 1).$$

- A CB-process in random environment (CBRE-process) can be defined by:

$$dX(t) = \sqrt[1+\alpha]{(1+\alpha)cX(t-)}\, dZ_\alpha(t) + X(t-)dL(t), \tag{10}$$

where $\{Z_\alpha(t)\}$ is a Brownian motion (if $\alpha = 1$) or a spectrally positive $(1+\alpha)$-stable process (if $0 < \alpha < 1$), and $\{L(t)\}_{t\geq 0}$ is a Lévy process with $L(t) - L(t-) > -1$.

Theorem 2 (Bansaye–Millan–Smadi '13; He–Li–Xu '16+) *For $\lambda \geq 0$ and $t \geq r \geq 0$ we have (quenched law):*

$$\boldsymbol{P}^L[e^{-\lambda X(t)}|X(s) : 0 \leq s \leq r] = \exp\{-X(r)v^L_{r,t}(\lambda)\},$$

where $r \mapsto v^L_{r,t}(\lambda)$ is defined by ($L(\overleftarrow{ds})$ = backward Itô integral, $\phi(\lambda) = c\lambda^{1+\alpha}$):

$$v^L_{r,t}(\lambda) = \lambda - \int_r^t \phi(v^L_{s,t}(\lambda))ds + \int_r^t v^L_{s,t}(\lambda)L(\overleftarrow{ds}). \tag{11}$$

- Suppose the environment $\{L(t)\}_{t\geq 0}$ has the following Lévy-Itô decomposition:

$$L(t) = \beta t + \sigma B(t) + \int_0^t \int_{[-1,1]} (e^z - 1)\tilde{N}(ds, dz) + \int_0^t \int_{[-1,1]^c} (e^z - 1)N(ds, dz).$$

- The associated Lévy process $\{\xi(t)\}_{t\geq 0}$ is defined by:

$$\xi(t) = \beta_* t + \sigma B(t) + \int_0^t \int_{[-1,1]} z\tilde{N}(ds, dz) + \int_0^t \int_{[-1,1]^c} zN(ds, dz),$$

where (ν = Lévy measure)

$$\beta_* = \beta - \frac{\sigma^2}{2} - \int_{[-1,1]} (e^z - 1 - z)\nu(dz).$$

Note that $\sigma(\{L(t) : t \geq 0\}) = \sigma(\{\xi(t) : t \geq 0\})$.

Corollary *For $t > 0$ we have:*

$$\text{(quenched probability)} \qquad \boldsymbol{P}^L(X(t) = 0|X(0) = x) = e^{-x\bar{u}^{\xi}_{0,t}},$$

and so

$$\text{(annealed probability)} \qquad \boldsymbol{P}(X(t) = 0|X(0) = x) = \boldsymbol{P}(e^{-x\bar{u}^{\xi}_{0,t}}),$$

where $r \mapsto \bar{u}^{\xi}_{r,t}$ is defined by (a.e. $r \in [0,t]$, $\phi(\lambda) = c\lambda^{1+\alpha}$):

$$\frac{d}{dr}\bar{u}^{\xi}_{r,t} = e^{\xi(r)}\phi(e^{-\xi(r)}\bar{u}^{\xi}_{r,t}), \qquad \bar{u}^{\xi}_{t-,t} = \infty. \tag{12}$$

Corollary *For $t > 0$ we have (annealed probability):*

$$\boldsymbol{P}(X(t) > 0|X(0) = x) = \boldsymbol{P}\Big[F_x\Big(\int_0^t e^{-\alpha\xi(s)}ds\Big)\Big], \tag{13}$$

where $F_x(z) = 1 - \exp\{-x[c\alpha z]^{-1/\alpha}\}$.

Problem Asymptotics of $\boldsymbol{P}(X(t) > 0)$ as $t \to \infty$; Bansaye–Millan–Smadi ('13), Palau–Pardo ('15+), Li–Xu ('16), Palau–Pardo–Smadi ('16+).

7. Asymptotic results for the survival probability

Let $\Phi(\lambda) = \log \boldsymbol{P}_0[e^{\lambda\xi(1)}]$ and $\mathscr{D}(\Phi) = \{\lambda \in \mathbb{R} : \Phi(\lambda) < \infty\}$ with interior $\mathscr{D}^\circ(\Phi)$. Suppose that $\{0, 1\} \subset \mathscr{D}^\circ(\Phi)$. Then (Bansaye–Millan–Smadi '13; Palau–Pardo '15+; Li–Xu '16+; Palau–Pardo–Smadi '16+):

(1) [Supercritical case] *If* $0 < \Phi'(0) < \Phi'(1)$, *then* $\lim_{t\to\infty} \boldsymbol{P}_x(x(t) = 0) = D_1(\alpha, F_x)$.

(2) [Critical case] *If* $\Phi'(0) = 0 < \Phi'(1)$, *then* $\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_x(x(t) > 0) = D_2(\alpha, F_x)$.

(3) [Weakly subcritical case] *If* $\Phi'(0) < 0 < \Phi'(1)$, *then* $(0 < \theta < 1, \Phi'(\theta) = 0)$

$$\lim_{t\to\infty} t^{3/2} e^{-t\Phi(\theta)} \boldsymbol{P}_x(x(t) > 0) = D_3(\alpha, F_x).$$

(4) [Intermediately subcritical case] *If* $\Phi'(0) < 0 = \Phi'(1)$, *then*

$$\lim_{t\to\infty} t^{1/2} e^{-t\Phi(1)} \boldsymbol{P}_x(x(t) > 0) = x(c\alpha)^{-1/\alpha} D_4(\alpha).$$

(5) [Strongly subcritical case] *If* $\Phi'(0) < \Phi'(1) < 0$, *then*

$$\lim_{t\to\infty} e^{-t\Phi(1)} \boldsymbol{P}_x(x(t) > 0) = x(c\alpha)^{-1/\alpha} D_5(\alpha).$$

Theorem 3 (Li–Xu '16) *The D_i's can be given explicitly.*

8. Exponential functionals of Lévy processes

Let $\xi = \{\xi(t) : t \geq 0\}$ be a one-dimensional Lévy process. Given $\alpha > 0$, we define the exponential functional:

$$A_t^\alpha(\xi) = \int_0^t e^{-\alpha\xi(s)} ds, \qquad 0 \leq t \leq \infty. \tag{14}$$

Yor ('92, Proposition 2): characterization of ($\xi =$ Brownian motion with drift)

$$\boldsymbol{P}(A_t^\alpha(\xi) \in du, \xi(t) \in dx).$$

Carmona–Petit–Yor ('94, '97): moments of $A_T^\alpha(\xi)$ for exponentially distributed T.

Bertoin–Yor ('05): $A_\infty^\alpha(\xi) < \infty$ if and only if $\lim_{t\to\infty} \xi(t) = \infty$.

Bertoin–Yor ('05): distribution of $A_\infty^\alpha(\xi)$ when it is finite.

Pardo–Patie–Savov ('12): Wiener-Hopf type factorization for $A_\infty^\alpha(\xi)$ when it is finite.

Problem For a positive decreasing function F satisfying $F(z) \to 0$ as $z \to \infty$, study the asymptotics (decay rate and limiting coefficient) of

$$\boldsymbol{P}[F(A_t^\alpha(\xi))] = \boldsymbol{P}\left[F\left(\int_0^t e^{-\alpha\xi(s)} ds\right)\right] \quad \text{as} \quad t \to \infty. \tag{15}$$

Kawazu–Tanaka ('93): $F(z) = a(a+z)^{-1}$ and $\xi =$ Br.m. with drift.

Carmona–Petit–Yor ('94, '97): $F(z) = z^{-1}$ or $= \log z$ and $\lambda^{-1/2}\xi(\lambda\cdot) \to$ Br.m.

Böeinghoff–Hutzenthaler ('12): $F(z) = 1 - \exp\{-cz\}$ and $\xi =$ Br.m. with drift.

Bansaye–Millan–Smadi ('13): $F(z) = 1 - \exp\{-cz^{-1/\alpha}\}$ $(0 < \alpha \leq 1)$ and ξ has bounded variation Lévy process.

Palau–Pardo ('15, '16): $F(z) = 1 - \exp\{-cz^{-1/\alpha}\}$ $(0 < \alpha \leq 1)$ and $\xi =$ Br.m. with drift.

For random walks: Afanasy'ev–Geiger–Kersting–Vatutin ('05), Dyakonova–Geiger–Vatutin ('04), Geiger–Kersting ('02), Geiger–Kersting–Vatutin ('03), Guivarc'h–Liu ('01), Kozlov ('76), Liu ('96), Vatutin–Dyakonova–Sagitov ('13) among others.

9. The key observation

Let $\Omega_0 = \{\inf_{s \geq 0} \xi(s) = -\infty\}$. Then:

(a) When $\Phi'(0) = \boldsymbol{P}[\xi(1)] > 0$, we have $\boldsymbol{P}(\Omega_0) = \boldsymbol{P}(\lim_{t \to \infty} \xi(t) = \infty) = 1$, and so $\boldsymbol{P}_0(A^\alpha_\infty(\xi) < \infty) = 1$ by the result of Bertoin–Yor ('05).

(b) When $\Phi'(0) = \boldsymbol{P}[\xi(1)] \leq 0$, we have $\boldsymbol{P}(\liminf_{t \to \infty} \xi(t) = -\infty) = 1$, so $\boldsymbol{P}(\Omega_0) = 0$.

Theorem 4 (i) *If* $0 < \Phi'(0) < \Phi'(\beta)$, *then*

$$\lim_{t \to \infty} \boldsymbol{P}[F(A^\alpha_t(\xi))] = \boldsymbol{P}[F(A^\alpha_\infty(\xi))] = \boldsymbol{P}\Big[F\Big(\int_0^\infty e^{-\alpha \xi(s)} ds\Big)\Big].$$

Observation When $\Phi'(0) \leq 0$, the asymptotics of $\boldsymbol{P}[F(A^\alpha_t(\xi))]$ is determined by the behavior of $\boldsymbol{P}$ near the "boundary" of Ω_0. Note that

$$\Omega_0 = \bigcup_{x \geq 0} \bigcap_{t \geq 0} \{\tau_{-x} > t\} = \uparrow\lim_{x \to \infty} \downarrow\lim_{t \to \infty} \{\tau_{-x} > t\}. \tag{16}$$

where $\tau_{-x} = \inf\{t > 0 : \xi_t \leq -x\}$.

10. Transformations of Lévy processes

Let Ω be the set of càdlàg paths from $[0, \infty)$ to $\mathbb{R}$.

Let $\xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \boldsymbol{P}_x)$ be the canonical realization of a one-dimensional Lévy process. See Bertoin ('96) and Kyprianou ('14).

Let $\{L(t) : t \geq 0\}$ be the local time at 0 of the reflected process $t \mapsto \sup_{s\in[0,t]} \xi(s) - \xi(t)$. Let

$$L^{-1}(t) = \begin{cases} \inf\{s > 0 : L(s) > t\}, & t < L(\infty); \\ \infty, & \text{otherwise.} \end{cases}$$

The ladder height process $\{H(t) : t \geq 0\}$ of ξ is a subordinator defined by

$$H(t) = \begin{cases} \xi(L^{-1}(t)), & t < L(\infty); \\ \infty, & \text{otherwise.} \end{cases} \tag{17}$$

The renewal function V is defined by

$$V(x) = \int_0^\infty \boldsymbol{P}_0(H(t) \leq x)dt, \qquad x \geq 0. \tag{18}$$

Let $\hat{\xi} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\boldsymbol{P}}_x)$ be the dual process of ξ, where $\hat{\boldsymbol{P}}_x$ is the law of $\{-\xi(t) : t \geq 0\}$ under $\boldsymbol{P}_{-x}$.

Let $\hat{}$ denote the quantities associated with $\hat{\xi}$.

Let $\tau_x = \inf\{t > 0 : \xi_t \leq x\}$ for $x \in \mathbb{R}$.

For any $x > 0$ the process $t \mapsto \hat{V}(\xi(t)-)1_{\{\tau_0 > t\}}$ is a $\boldsymbol{P}_x$-martingale and $t \mapsto V(\xi(t)-)1_{\{\tau_0 > t\}}$ is a $\hat{\boldsymbol{P}}_x$-martingales.

We can define the probability measures $\boldsymbol{Q}_x$ and $\hat{\boldsymbol{Q}}_x$ by ($A \in \mathscr{F}_t$ and $t \geq 0$):

$$\boldsymbol{Q}_x(A) = \hat{V}(x-)^{-1} \int_A \hat{V}(\xi(t)-)1_{\{\tau_0 > t\}} d\boldsymbol{P}_x \tag{19}$$

and

$$\hat{\boldsymbol{Q}}_x(A) = V(x-)^{-1} \int_A V(\xi(t)-)1_{\{\tau_0 > t\}} d\hat{\boldsymbol{P}}_x. \tag{20}$$

Call $\Xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \boldsymbol{Q}_x)$ and $\hat{\Xi} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\boldsymbol{Q}}_x)$ the h-transformations by the renewal functions.

Let $\Phi(\lambda) = \log \boldsymbol{P}_0[e^{\lambda\xi(1)}]$ be the Laplace exponent and let

$$\mathscr{D}(\Phi) = \{\lambda \in \mathbb{R} : \Phi(\lambda) < \infty\}.$$

For any $\theta \in \mathscr{D}(\Phi)$, the process $t \mapsto e^{\theta\xi(t) - \Phi(\theta)t}$ is a $\boldsymbol{P}_x$-martingale.

By Escheer's theorem, we can define the probability measure $\boldsymbol{P}_x^{(\theta)}$ on $(\Omega, \mathscr{F})$ by

$$\boldsymbol{P}_x^{(\theta)}(A) = \int_A e^{\theta\xi(t) - \Phi(\theta)t} d\boldsymbol{P}_x, \qquad A \in \mathscr{F}_t, t \geq 0. \tag{21}$$

The Escheer's transform $\xi^{(\theta)} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \boldsymbol{P}_x^{(\theta)})$ is a Lévy process with Laplace exponent $\lambda \to \Phi(\lambda + \theta) - \Phi(\theta)$.

Let $\hat{\xi}^{(\theta)} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\boldsymbol{P}}_x^{(\theta)})$ be the dual process of $\xi^{(\theta)}$.

Let ${}^{(\theta)}$ denote the quantities associated with $\xi^{(\theta)}$.

Let $\Xi^{(\theta)} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \boldsymbol{Q}_x^{(\theta)})$ and $\hat{\Xi}^{(\theta)} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\boldsymbol{Q}}_x^{(\theta)})$ be the h-transformations by the renewal functions of $\xi^{(\theta)}$ and $\hat{\xi}^{(\theta)}$, respectively.

11. The case $\Phi'(0) = \boldsymbol{P}_0[\xi(1)] = 0$

Recall that $\Phi(\lambda) = \log \boldsymbol{P}_0[e^{\lambda\xi(1)}]$ and $\mathscr{D}(\Phi) = \{\lambda \in \mathbb{R} : \Phi(\lambda) < \infty\}$.

Let $\alpha > 0$ and assume $\{0, \beta\} \subset$ interior of $\mathscr{D}(\Phi) \cap [0, \infty)$.

Condition A *There is a constant $K > 0$ so that $F(z) \leq K z^{-\beta/\alpha}$ for $z \geq 1$.*

Theorem 4 (ii) *If $\Phi'(0) = 0 < \Phi'(\beta)$ and Condition A holds, then*

$$\lim_{t\to\infty} t^{1/2} \boldsymbol{P}_0[F(A_t^\alpha(\xi))] = \sqrt{\frac{2}{\pi\Phi''(0)}} \hat{\boldsymbol{P}}_0[H(1)] D_2(\alpha, F),$$

where $D_2(\alpha, F) = \lim_{x\to\infty} \hat{V}(x-) \boldsymbol{Q}_x[F(e^{-\alpha x} A_\infty^\alpha(\xi))]$.

Lemma (Hirano '01) *For any $s \geq 0$ and $x > 0$ we have, as $t \to \infty$,*

$$t^{1/2} \boldsymbol{P}_0(\tau_{-x} > t) \to \sqrt{\frac{2}{\pi\Phi''(0)}} \hat{\boldsymbol{P}}_0[H(1)] \hat{V}(x-)$$

and

$$\boldsymbol{P}_0(\{\xi(r)\}_{r\in[0,s]} \in \cdot | \tau_{-x} > t) \to \boldsymbol{Q}_x(\{\xi(r) - x\}_{r\in[0,s]} \in \cdot).$$

Proof of Theorem 2 (ii):

Step 1. Write

$$\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_0[F(A_t^\alpha(\xi))] = \lim_{x\to\infty}\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_0[F(A_t^\alpha(\xi));\tau_{-x} > t] + \lim_{x\to\infty}\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_0[F(A_t^\alpha(\xi));\tau_{-x} \le t]. \quad (22)$$

Step 2. Prove the second term vanishes.

Step 3. Use Lemma 3 to see

$$\begin{aligned}
&\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_0[F(A_t^\alpha(\xi));\tau_{-x} > t] \\
&= \lim_{s\to\infty}\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_0[F(A_s^\alpha(\xi));\tau_{-x} > t] \\
&= \lim_{s\to\infty}\lim_{t\to\infty} t^{1/2}\boldsymbol{P}_0(\tau_{-x} > t)\cdot \boldsymbol{P}_0[F(A_s^\alpha(\xi))|\tau_{-x} > t] \\
&= \lim_{s\to\infty}\sqrt{\frac{2}{\pi\Phi''(0)}}\hat{\boldsymbol{P}}_0[H(1)]\hat{V}(x-)\cdot \boldsymbol{Q}_x[F(e^{-\alpha x}A_s^\alpha(\xi))] \\
&= \sqrt{\frac{2}{\pi\Phi''(0)}}\hat{\boldsymbol{P}}_0[H(1)]\hat{V}(x-)\cdot \boldsymbol{Q}_x[F(e^{-\alpha x}A_\infty^\alpha(\xi))]. \qquad \square
\end{aligned}$$

12. The case $\Phi'(0) = \boldsymbol{P}_0[\xi(1)] < 0$

Theorem 4 (iii) *Suppose that* $\Phi'(0) < 0 < \Phi'(\beta)$ *and Condition A holds. Let* $\varrho \in (0, \beta)$ *be the solution of* $\Phi'(\varrho) = 0$. *Then*

$$\lim_{t \to \infty} t^{3/2} e^{-t\Phi(\varrho)} \boldsymbol{P}_0[F(A_t^\alpha(\xi))] = \frac{c(\varrho)}{\sqrt{2\pi \Phi''(\varrho)}} D_3(\alpha, F),$$

where

$$c(\varrho) = \exp\left\{ \int_0^\infty (e^{-t} - 1) t^{-1} e^{-t\Phi(\varrho)} \boldsymbol{P}_0(\xi(t) = 0) dt \right\}, \tag{23}$$

$$D_3(\alpha, F) = \lim_{x \to \infty} e^{\varrho x} \hat{V}^{(\varrho)}(x-) \int_0^\infty e^{-\varrho y} V^{(\varrho)}(y) G(x, y) dy \tag{24}$$

and

$$G(x, y) = \boldsymbol{Q}^{(\varrho)}_{(x,y)}\{F(e^{-\alpha x}[A^\alpha_\infty(\xi) + A^\alpha_\infty(\hat{\xi})])\} \tag{25}$$

with $(W, \mathscr{G}, \mathscr{G}_t, (\xi(t), \hat{\xi}(t)), \boldsymbol{Q}^{(\varrho)}_{(x,y)})$ *being the independent coupling of* $\Xi^{(\varrho)}$ *and* $\hat{\Xi}^{(\varrho)}$.

Condition B *There is a constant* $K > 0$ *so that* $F(z) \sim Kz^{-\beta/\alpha}$ *as* $z \to \infty$.

Theorem 4 (iv) *If* $\Phi'(0) < \Phi'(\beta) = 0$ *and Condition B holds, then*

$$\lim_{t\to\infty} t^{1/2} e^{-t\Phi(\beta)} \boldsymbol{P}_0[F(A_t^\alpha(\xi))] = K\sqrt{\frac{2}{\pi\Phi''(\beta)}}\boldsymbol{P}_0^{(\beta)}[H(1)]D_4(\alpha,\beta),$$

where

$$D_4(\alpha,\beta) = \lim_{x\to\infty} V^{(\beta)}(x-)\boldsymbol{Q}_x^{(\beta)}[e^{-\beta x}A_\infty^\alpha(-\xi)^{-\beta/\alpha}].$$

Theorem 4 (v) *If* $\Phi'(0) < \Phi'(\beta) < 0$ *and Condition B holds, then*

$$\lim_{t\to\infty} e^{-t\Phi(\beta)} \boldsymbol{P}_0[F(A_t^\alpha(\xi))] = K\boldsymbol{P}_0^{(\beta)}[A_\infty^\alpha(-\xi)^{-\beta/\alpha}].$$

Summary

A stable branching CBRE-process can be constructed as the strong solution of a stochastic equation, where the environment is modeled by a Lévy process $\{L(t) : t \geq 0\}$ with $L(t) - L(t-) > -1$.

The survival probability of the population model is given by some expectation involving an exponential functional of another Lévy process $\{\xi(t) : t \geq 0\}$ determined by the environment.

Some results for the asymptotics, convergence rates and limiting coefficients, of exponential functionals of the Lévy process are presented.

The limiting coefficients are represented in terms of some transformations based on the renewal functions.

The key of the work is the observation that the asymptotics only depends on sample paths with local infimum decreasing slowly, i.e., those in “neighborhoods” of

$$\Omega_0 := \Big\{ \inf_{s \geq 0} \xi(s) > -\infty \Big\}.$$

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