$L \log L$ criterion for a class of multi-type superdiffusions with nonlocal branching mechanism

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Based on a working paper Zhen-Qing Chen and Renming Song

Outline

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The $L \log L$ criterion for G-W branching processes

$\{Z_n, n \geq 1\}$: a Galton-Watson branching process.
L: the number of children given by one particle.
$\{p_n, n \geq 1\}$: the distribution of L.

Set

$$m := \sum_{n=1}^{\infty} n p_n.$$

m is the mean number of children given by one particle.
Suppose $m > 1$ (supercritical).

It is known that $EZ_n = m^n$ and $\left\{\frac{Z_n}{m^n}; n \geq 1\right\}$ is a martingale and then

$$\lim_{n\to\infty} \frac{Z_n}{m^n} = W < \infty.$$

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Classical Question: When is W nondegenerate? or equivalently, when does m^n gives the right growth rate of Z_n?

In 1966, Kesten and Stigum proved that W is nondegenerate if and only if

$$(L \log L) \qquad E(L \log^+ L)) = \sum_{n=1}^{\infty} p_n (n \log n) < \infty. \qquad (1)$$

Moreover, if (1) is satisfied, $E(W) = 1$ and

$$P(W > 0) = P(Z_n = 0, \text{ for some } n > 0).$$

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In 1995, Lyon, Pemantle and Peres used a method of martingale change of measure to give a probabilistic proof of the $L \log L$ theorem of Kesten and Stigum.

Later this method were extended to branching processes in multiple and general multiple type cases (see Kurtz-Lyons-Pemantle-Peres(1997); Lyons(1997); Biggins-Kyprianou (2004)).

This technique was also used to study properties for branching random walk. See, for example, Hu-Shi(2009); Aidekon-Shi(2011, 2014); Faraud-Hu-Shi(2011, 2012); Gantert-Hu-Shi(2014).

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I would like to talk the $L \log L$ criterion for supercritical **multi-type superdiffusions** with **nonlocal** branching mechanisms.

Outline

Multi-type Superdiffusion

Let $S := \{1, 2, \cdots, K\}$ be the set of types, $2 \leq K < +\infty$.

For each $k \in S$, L_k is a second order strictly elliptic differential operator:

$$L_k = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}^{(k)} \frac{\partial}{\partial x_j}\right) \qquad \text{on } \mathbb{R}^d, \qquad (2)$$

with $A^k(x) = (a_{ij}^k(x))_{1 \leq i,j \leq d}$ being a symmetric matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic and bounded:

$$\Lambda_1 |v|^2 \leq \sum_{i,j=1}^{d} a_{i,j}^k(x) v_i v_j \leq \Lambda_2 |v|^2 \qquad \text{for all } v \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d$$

for some positive constants $0 < \Lambda_1 \leq \Lambda_2 < \infty$, where $a_{ij}^k(x) \in C^{2,\gamma}(\mathbb{R}^d), 1 \leq i,j \leq d$ for some $\gamma \in (0, 1)$.

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For any $i \in S$, $\{\xi_t^i, t \geq 0; \Pi_x^i\}$ is a diffusion with generator L_i starting from x.

Suppose D is a bounded domain in $\mathbb{R}^d$. For $x \in D$, we will use $(\xi_t^{i,D}, t \geq 0; \Pi_x^i)$ to denote the process obtained by killing ξ^i upon exiting from D.

$\mathcal{M}_F(D \times S)$: the space of finite measures on $D \times S$.
$N(D \times S)(\subset \mathcal{M}_F(D \times S))$: the space of integer-valued measures on $D \times S$.

Let $b \in \mathcal{B}^+(D \times i)$ and let $F(x, i, d\nu)$ be a Markov kernel from $D \times S$ to $N(D \times S)$ such that

$$\sup_{(x,i) \in D \times S} \int_{N(D)} \nu(1) F(x, i, d\nu) < \infty.$$

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Nonlocal branching particle system

A branching particle system with parameters $(\xi^{i,D}, b, F)$ is described by the following properties:

(a) The particles in D with type i move randomly according to the law of $\xi^{i,D}$.

(b) For a type i particle which is alive at time r and follows the path $(\xi_t^{i,D})_{t\geq r}$, the conditional probability of survival during the time interval $[r,t]$ is $\rho(r,t) := \exp\left(-\int_r^t b(\xi_s^{i,D}, i)ds\right)$.

(c) When a type i particle dies at a point $x \in D$, it gives birth to a random number of offspring in $D \times S$ according to the probability kernel $F(x, i, d\nu)$. The offspring then start to move from their locations.

It is assumed that the migrations, the lifetimes and the branchings of the particles are independent of each other.

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Let $\Gamma_t(B \times \{i\})$ denote the number of type i particles located in $B \in \mathcal{B}(D)$ that are alive at time $t \geq 0$ and assume $\Gamma_0(D \times S) < \infty$. Then $(\Gamma_t, t \geq 0)$ is a Markov process taking values in $\mathcal{M}_F(D \times S)$.

Let $(\Gamma_t^n : t \geq 0), n \geq 1$, be a sequence of branching particle systems with parameters $(\xi^{D,i}, b_n, F_n)$. Under some conditions, one has

$$\left(\frac{1}{n}\Gamma_t^n, \quad t \geq 0\right) \to (\chi_t : t \geq 0) \quad n \to \infty.$$

The measure-valued Markov process $(\chi_t : t \geq 0)$ is called a multi-type superdiffusion. For details see Dawson, Gorostiza and Li (2002).

branching mechanism $=$ local branching $+$ nonlocal branching.

We will focus on **nonlocal branching mechanism**.

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nonlocal branching mechanism

For $f \in B^+(D \times S)$, define

$$\zeta(x,i;f) = d(x,i)\pi(x,i;f) + \int_0^\infty \left(1 - e^{-u\pi(x,i;f)} - u\pi(x,i;f)\right) n(x,i;du),$$

where $d \in B^+(D \times S)$, $d(x,i) \geq \int_0^\infty u\, n(x,i;du) \in B^+(D \times S)$, and

$$\pi(x,i;f) = \sum_{j=1}^{K} p_j^{(i)}(x) f_j(x), \quad \text{where } p_j^{(i)}(x) \geq 0, \sum_{j=1}^{K} p_j^{(i)}(x) = 1.$$

Put

$$\psi(x,i;f) = b(x,i)\left(f_i(x) - \zeta(x,i;f)\right), \quad (x,i) \in \mathbb{R}^d \times S, f \in B^+(D \times S).$$

We suppose that $p_i^i(x) = 0$ for any $(x,i) \in D \times S$, which means that ψ is a purely nonlocal branching mechanism.

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We write $\chi_t = (\chi_t^1, \cdots, \chi_t^K)$.

We often use the convention

$$f(x) = (f(x,1), \cdots, f(x,K)) = (f_1(x), \cdots, f_K(x)), \quad x \in \mathbb{R}^d,$$

and $\langle f, \chi_t \rangle = \sum_{j=1}^K \langle f_j, \chi_t^j \rangle$.

The Laplace-functional of χ is given by

$$P_\mu \exp\langle -f, \chi_t \rangle = \exp\langle -u_t^f(\cdot), \mu \rangle, \tag{3}$$

where $u_t^f(x,i)$ is the unique bounded positive solution to the evolution equation

$$u_t^f(x,i) + \quad \Pi_x^i \left[\int_0^t \psi(\xi_s^{i,D}, i; u_{t-s}^f) ds \right] = \Pi_x^i f_i(\xi_t^{i,D}), \quad \text{for } t \geq 0, \tag{4}$$

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First moment

Let's first find $v(t,x,i) = P_{\delta_{(x,i)}}\langle f, \chi_t \rangle$ **(the mean of χ_t).**

Then $v(t,x,i)$ is the unique bounded solution to the following equation:

$$\frac{\partial}{\partial t}\mathbf{v}(t,x) = \mathcal{L}\mathbf{v}(t,x) + B(x)\cdot(R(x)-I)\mathbf{v}(t,x), \tag{5}$$

where

$$\mathcal{L} = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_K \end{pmatrix},$$

$$B(x) = \operatorname{diag}\left(B(x,1), \cdots B(x,K)\right), \quad x \in D,$$

$$R(x) = (r_{il}(x)), \quad r_{il}(x) = d(x,i)p_l^{(i)}(x) \quad x \in \mathbb{R}^d,\ i,l \in S.$$

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$$B(x)\cdot(R(x)-I) = \widehat{B}(x)\cdot(P(x)-I) + B(x)\left(D(x)-I\right), \tag{6}$$

where

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Put $Q(x) = \widehat{B}(x)\cdot(P(x)-I)$. We will assume that the matrix Q is symmetric and irreducible.

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$$P(x) = (p_{ij}(x))_{i,j\in S}\,, \quad p_{ij}(x) = p_j^{(i)}(x).$$

Put $Q(x) = \widehat{B}(x)\cdot(P(x)-I)$. We will assume that the matrix Q is symmetric and irreducible.

First moment

Let $\{(X_t, Y_t), t \geq 0\}$ be a switched diffusion with generator $\mathcal{A} := \mathcal{L} + Q(x)$ killed upon exiting from $D \times S$ and $\Pi_{(x,i)}$ be its law starting from (x, i).

Let $\{P_t^{\mathcal{A}+B\cdot(D-I)}, t \geq 0\}$ be the Feynman-Kac semigroup defined by

$$P_t^{\mathcal{A}+B\cdot(D-I)} f(x,i) = \Pi_{(x,i)}\left[f(X_t, Y_t) \exp\left(\int_0^t b(X_s, Y_s)(d(X_s, Y_s) - 1) ds \right) \right].$$

First moment $$P_{\delta_{(x,i)}} \langle f, \chi_t \rangle = P_t^{\mathcal{A}+B\cdot(D-I)} f(x,i). \quad (7)$$

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Recall that

$$\zeta(x,i;f) = d(x,i)\pi(x,i;f) + \int_0^\infty \left(1 - e^{-u\pi(x,i;f)} - u\pi(x,i;f)\right) n(x,i;du).$$

Define

$$\zeta_1(x,i;f) = d(x,i)\pi(x,i;f) = \sum_{l=1}^{K} r_{il}(x) f_l(x); \tag{8}$$

$$\zeta_2(x,i;f) = \int_0^\infty \left(1 - e^{-u\pi(x,i;f)} - u\pi(x,i;f)\right) n(x,i;du). \tag{9}$$

Then

$$\zeta(x,i;f) = \zeta_1(x,i;f) + \zeta_2(x,i;f). \tag{10}$$

Remark $\{\chi_t, t \geq 0\}$ can be regarded as a super-switched diffusion with (X_t, Y_t) as spatial motion on the space $D \times S$ and

$$\widehat{\psi}(x,i;f) = -b(x,i)d(x,i)f_i(x) + b(x,i)(f_i(x) - \zeta_2(x,i;f)), \quad f \in B^+(\mathbb{R}^d \times S),$$

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Outline

$\{(X_t, Y_t), t \geq 0\}$ has a transition density $p(t, (x, k), (y, l))$ which is positive and jointly continuous in $(x, y) \in D \times D$.

Let $\{P_t : t \geq 0\}$ be the semigroup of $\{(X_t, Y_t), t \geq 0\}$. For any $t > 0$, P_t is a compact self-adjoint operator.

Let $\{e^{\nu_k t} : k = 1, 2, \cdots\}$ be all the eigenvalues of P_t arranged in decreasing order, each repeated according to its multiplicity.

Then $\nu_k \downarrow -\infty$ and the corresponding eigenfunctions $\{\varphi_k\}$ can be chosen so that they form an orthonormal basis of $L^2(D \times S, dx \times di)$. All the eigenfunctions φ_k are continuous.

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Assumption 1 $\lambda_1 > 0$ (supercritical).

Assumption 2 The semigroup $\{P_t : t \geq 0\}$ is intrinsically ultracontractive, that is, for any $t > 0$, there exists $c_t > 0$ such that

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Lemma Define

$$W_t(\phi) := e^{-\lambda_1 t}\langle \phi, \chi_t \rangle. \quad (11)$$

Then $W_t(\phi), t \geq 0$ is a non-negative martingale and therefore there exists a limit $W_\infty(\phi) \in [0, \infty)$, P_μ-a.s.

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Outline

Main Results

We define a new kernel $n^{\pi(\phi)}(x,i;dr)$ from $D \times S$ to $(0,\infty)$ such that for any nonnegative measurable function f on $(0,\infty)$,

$$\int_0^\infty f(r)n^{\pi(\phi)}(x,i;dr) = \int_0^\infty f(\pi(x,i;\phi)r)n(x,i;dr), \quad (x,i) \in D \times S.$$

Define

$$I(x,i) := \int_0^\infty r\log^+(r)n^{\pi(\phi)}(x,i;dr). \tag{12}$$

Theorem 1 Suppose that $\{\chi_t; t \geq 0\}$ is a multi-type superdiffusion and that Assumptions 1 and 2 are satisfied. Then $W_\infty(\phi)$ is non-degenerate under P_μ for any nonzero measure $\mu \in M_F(D \times S)$ if and only if

$$(L\log L) \qquad \int_D \phi(x,i)b(x,i)I(x,i)dx < \infty, \quad \forall i \in S, \tag{13}$$

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The proof of this theorem is based on a “spine decomposition”. The new feature here is that we consider a nonlocal branching mechanism as opposed to the local branching mechanisms considered before.

The nonlocal branching mechanism results in *nonlocal immigration*, as opposed to the local immigration in Liu-R.-Song (2009).

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Spine decomposition

Let $\mathcal{F}_t = \sigma(\chi_s;\ s \le t)$. We define a probability measure $\widetilde{P}_\mu$ by:

$$\left.\frac{d\widetilde{P}_\mu}{dP_\mu}\right|_{\mathcal{F}_t} = \frac{1}{\langle \phi, \mu \rangle} W_t(\phi). \tag{14}$$

We aim to give a spine decomposition of $\{\chi_t, t \ge 0\}$ under $\widetilde{P}_\mu$.

Let $\mathcal{E}_t = \sigma(X_s, Y_s;\ s \le t)$. Define a measure $\Pi^\phi_{(x,i)}$ by

$$\left.\frac{d\Pi^\phi_{(x,i)}}{d\Pi_{(x,i)}}\right|_{\mathcal{E}_t} = e^{-\lambda_1 t} \frac{\phi(X_t, Y_t)}{\phi(x,i)} \exp\left(\int_0^t b(X_s, Y_s)(d(X_s, Y_s) - 1) ds\right). \tag{15}$$

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The generator of (X, Y) under $\Pi^{\phi}_{(x,i)}$ is given by

$$\mathcal{L}^{\phi} + \operatorname{diag}\left(\frac{bd\pi(\phi)}{\phi}(x,1),\cdots\frac{bd\pi(\phi)}{\phi}(x,K)\right)(\widetilde{P}(x) - I), \tag{16}$$

which is a generator of a new switched diffusion,

where

$$\mathcal{L}^{\phi} = \begin{pmatrix} L_1^{\phi(\cdot,1)} & 0 & \cdots & 0 \\ 0 & L_2^{\phi(\cdot,2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_K^{\phi(\cdot,K)} \end{pmatrix},$$

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Using the non-local Feynman-Kac transform, we get

Proposition 1 Suppose $\mu \in M_F(D \times S)$ and $g \in B^+(D \times S)$. Let D_J be the set of jump times of (X, Y). Then

$$\begin{aligned} &\widetilde{P}_\mu\left(\exp\langle -g, \chi_t\rangle\right) \\ &= P_\mu\Big(\exp\langle -g, \chi_t\rangle\Big) \\ &\quad \cdot \Pi^{\phi}_{\phi\mu}\left[\prod_{s\in D_J, 0<s\le t} \int_0^\infty \exp(-u\pi(X_s, Y_s; u^g_{t-s}))\widetilde{n}(X_s, Y_s; du)\right], \quad (19) \end{aligned}$$

where u^g_{t-s} is the unique solution of (4) with f replaced by g and

$$\eta_2(x, i; \lambda) = \int_{[0,\infty)} e^{-u\lambda} u\, n(x, i; du), \quad \lambda \ge 0, \ (x, i) \in D \times S.$$

Spine decomposition

Theorem 2 (Spine decomposition):

$$\{\chi_t, t \geq 0; \widetilde{P}_\mu\} = \{\chi_t + \widehat{\chi}_t, t \geq 0; P_{\mu,\phi}\} \quad \text{in distribution}, \tag{20}$$

Here, under $P_{\mu,\phi}$, χ and $\widehat{\chi}$ are two independent processes,

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and $\widehat{\chi}$ is obtained by taking an "immortal particle" that moves according to the law of (X, Y) under $\Pi^{\phi}_{\phi\mu}$ and spins off pieces of mass that continue to evolve according to the dynamics of χ.

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a) Suppose that $(\widehat{X}, \widehat{Y})$ is defined on some probability space $(\Omega, P_{\mu,\phi})$, and $(\widehat{X}, \widehat{Y})$ has the same law as $((X, Y); \Pi^{\phi}_{\phi\mu})$. $(\widehat{X}, \widehat{Y})$ serves as the spine or the immortal particle, which is ergodic. Let D_J be the set of jump points of $(\widehat{X}, \widehat{Y})$. D_J is countable.

b) Conditioned on $s \in D_J$, a measure-valued process $\{\chi_t^s, t \geq s\}$ started at $m_s \delta_{(\widehat{X}_s, l)} (l \in S)$ is immigrated at space position $\widehat{X}_s$ and the new immigrated particles choose their types independently according to the distribution $\{p_l^{(i)}(x), l \in S\}$. We suppose $\{m_s; s \in D_J\}$ is also defined on $(\Omega, P_{\mu,\phi})$ such that, given $s \in D_J$ and $(\widehat{X}_s, \widehat{Y}_s)$, the distribution of m_s is $\widetilde{n}(\widehat{X}_s, \widehat{Y}_s; dr)$.

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c) Once the particles are in the system, they begin to move and branch according to the $\{(X, Y), \widehat{\psi}\}$-superprocess independently.

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Recall that $\{\chi_t^s,\ t \geq s\}$ denote the measure-valued process generated by the mass immigrated at time s and the position $(\widehat{X}_s, \widehat{Y}_s)$.

Conditional on $\{(\widehat{X}_t, \widehat{Y}_t)_{t\geq 0}; m_s, s \in D_J\}$, $\{\chi_t^s, t \geq s\}$ for different $s \in D_J$ are independent $((X, Y), \widehat{\psi})$-superprocesses. Set

$$\widehat{\chi}_t = \sum_{s\in(0,t]\cap D_J} \chi_t^s. \tag{21}$$

Remark The nonlocal immigration process appears to be a new feature not seen before in previous spine decompositions and is a consequence of non-local branching. Simultaneously to our work, we learnt that a similar phenomenon has been observed by Kyprianou and Palauy for super Markov chain (prepint 2016).

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Thank you!