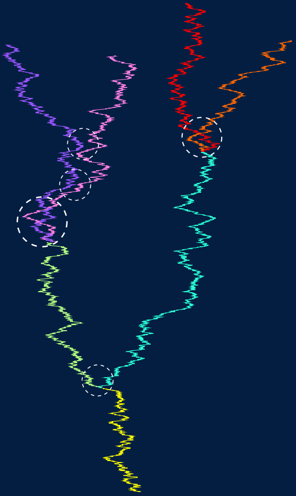

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Branching Brownian motion with decay of mass

Sarah Penington

Joint work with Louigi Addario-Berry

Department of Statistics
University of Oxford

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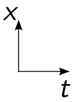
Branching Brownian motion

- Start with a single individual with an Exp(1) lifetime
- The individual moves according to (one-dimensional) Brownian motion
- When the individual dies, it produces two offspring individuals
- Each new individual has an independent Exp(1) lifetime and moves independently according to a Brownian motion until it dies and has offspring and so on.



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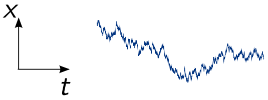

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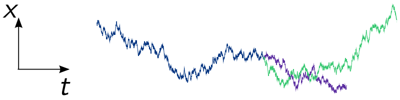

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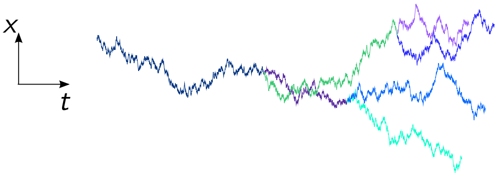


BBM with decay of mass

Individuals within distance one of each other have to share resources so their mass decays.

$n(t)$ is number of particles at time t.
Locations of particles given by $\mathrm{X}(t) = (X_i(t), 1 \leq i \leq n(t))$.
Masses of particles given by $\mathrm{M}(t) = (M_i(t), 1 \leq i \leq n(t))$.

Let $\zeta(t,x) = \sum_{\{i:|X_i(t)-x| \in (0,1)\}} M_i(t)$.

$M_i(t)$ decays at rate $M_i(t)\zeta(t, X_i(t))$ so

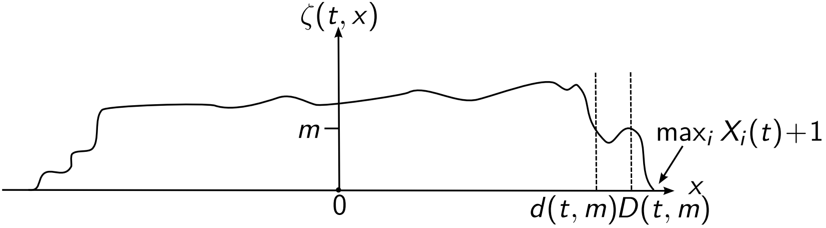
$$M_i(t) = \exp(-\int_0^t \zeta(s, X_{i,t}(s))\,ds)$$

where $X_{i,t}(s)$ is the location of the ancestor of $X_i(t)$ at time s.

Total mass increases through branching.

Front location

Let $d(t,m) = \min\{x > 0 : \zeta(t,x) < m\}$ and
$D(t,m) = \max\{x : \zeta(t,x) > m\}$.



If $x \geq \max_i X_i(t) + 1$ then $\zeta(t,x) = 0$.

Maximum particle in BBM

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Theorem (Bramson)

The rightmost particle location $\max_{i \geq 1} X_i(t)$ *has median* $med(t)$ *which satisfies*

$$med(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1).$$

Theorem (Hu and Shi)

Almost surely

$$\limsup_{t \to \infty} \frac{|\max_{i \geq 1} X_i(t) - med(t)|}{\log t} < \infty.$$

Main result

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Theorem (Addario-Berry, P.)

Write $c^* = 3^{4/3}\pi^{2/3}/2^{7/6}$. Then almost surely, for all $m < 1$,

$$\limsup_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} \geq c^* \quad \textit{and} \quad \liminf_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c^*.$$

There are

- large times t at which the first low-density region lags at least distance $c^* t^{1/3} + o(t^{1/3})$ behind the rightmost particle
- large times t at which there is some high-density region within distance $c^* t^{1/3} + o(t^{1/3})$ of the rightmost particle.

Simulation

Thanks to David Corlin-Marchand

Density self-correction

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Heuristically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t,x) \approx \zeta(t,x) - \sum_{\{i:|X_i(t)-x|\in(0,1)\}} M_i(t)\cdot\zeta(t,X_i(t))\,.$$

- If $\zeta(t,y) \ll 1$ for all y s.t. $|x-y| < 1$, get exponential growth.
- If $\zeta(t,y) \gg 1$ for all y s.t. $|x-y| < 1$, get exponential decay.

Upper bound

Fix $c \in (0, c^*)$ and let $g(s) = \sqrt{2}s - cs^{1/3}$ for $s \geq 0$.
Let $X_{i,t}(s)$ denote the location of the ancestor of $X_i(t)$ at time s.

Proposition (Jaffuel)

There exists $\delta = \delta(c) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq n(t) \text{ s.t. } X_{i,t}(s) \geq g(s)\, \forall s \leq t\right) \leq e^{-\delta t^{1/3}}.$$

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Fix $\beta > 0$ small and let $b(s) = \sqrt{2}s - c(s + \beta t)^{1/3}$ for $s \geq 0$.

Proposition (Jaffuel and Roberts)

There exists $\delta = \delta(c) > 0$ such that for K a constant and t sufficiently large,

$$\mathbb{P}\left(\exists i \leq n(t) \text{ s.t. } X_{i,t}(s) \geq b(s) - Kt^{1/6}\, \forall s \leq t\right) \leq e^{-\delta t^{1/3}}.$$

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Proposition

For any $C > 0$, there exists $\delta = \delta(c, C) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq n(t) \text{ s.t. } \mathrm{Leb}(\{s \leq t : X_{i,t}(s) \leq b(s)\}) \leq Ct^{1/3}\right) \leq e^{-\delta t^{1/3}}.$$

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The upper bound follows.

Proposition

For any $m > 0$, almost surely

$$\limsup_{t \to \infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} \geq c^*.$$

Lower bound

Proposition (Roberts)

There exists $C^* < \infty$ a.s. such that for all t,

$$\#\{i : \forall s \in [0,t], X_{i,t}(s) > \sqrt{2}s - c^* s^{1/3} + \frac{c^* s^{1/3}}{\log^2(s+e)} - C^*\} \geq 1$$

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Proposition

There exists a constant Z such that for all t sufficiently large,

$$\mathbb{P}\left(\sup\{\zeta(s,x) : 0 \leq s \leq t, x \in \mathbb{R}\} > Z \log t\right) \leq t^{-4}.$$

Lower bound

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Proposition

For any $m \in (0,1)$, almost surely

$$\liminf_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c^*.$$

Further work and open questions

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- Higher dimensions
- Bound on $\limsup_{t\to\infty} \sup_{x\in\mathbb{R}} \zeta(t,x)$.
- For $m \leq m^*$, there is some constant $A < \infty$ and random time $T < \infty$ a.s. such that for $t \geq T$,

$$d(t + A\log t, m) \geq D(t, m).$$

- Still open: almost surely, for all $m < 1$,

$$\lim_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} = c^* = \lim_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}}.$$

- More physical mechanism for mass growth?
- PDE approximation