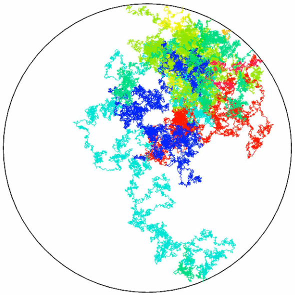
Critical behaviour of branching diffusions in bounded domains

The Fourth Bath-Paris Branching Structures Meeting

Ellen Powell

28th June 2016

Branching Diffusions in a Domain



- $D \subset \mathbb{R}^d$ bounded, C^1 domain.
- Start BBM at $x \in D$. Particles killed upon hitting the boundary.

Simulation by Henry Jackson.

Notations

β = branching rate.

$(X_1^t, \cdots, X_{N_t}^t)$ = system at time t.

Extinction (BBM with binary branching)

Phase Transition (Sevast'yanov 1958, Watanabe 1965)

There exists a critical value λ of the branching parameter β s.t.

- $\beta \leq \lambda \Rightarrow$ a.s. extinction
- $\beta > \lambda \Rightarrow$ survival with positive probability.

λ is the first eigenvalue of $-\frac{1}{2}\Delta$ on D.

Survival Probabilities

- Survival probability $= \mathbb{P}_x(N_t > 0)$.
- Decays exponentially in subcritical case.
- Critical case?

Survival Probabilities

- Survival probability $= \mathbb{P}_x(N_t > 0)$.
- Decays exponentially in subcritical case.
- Critical case?

Galton-Watson Case (Kolmogorov 1938, Miermont 2008)

Critical GW and multitype GW processes *with finite variance* have

$$\mathbb{P}(N_n > 0) \sim c/n$$

as $n \to \infty$.

Main Results (BBM with binary branching)

Theorem (P. 2015)

In the critical case $\beta = \lambda$, for all $x \in D$ we have

$$\mathbb{P}_x(N_t > 0) \sim \frac{1}{t} \times \frac{\varphi(x)}{\lambda \int_D \varphi(y)^3 \, dy}$$

as $t \to \infty$.

φ is the first eigenfunction of $-\frac{1}{2}\Delta$ on D, normalised to have unit L^2 norm.

Main Results (BBM with binary branching)

Theorem (P. 2015)

For any measurable $E \subset D$, if N_t^E is the number of particles in E at time t, we have

$$\left(\frac{N_t^E}{t} \middle| N_t > 0 \right) \to Z$$

in distribution as $t \to \infty$, where Z is an exponential random variable with mean

$$\lambda \langle \varphi, \mathbf{1}_E \rangle_{L^2(D)} \int_D \varphi^3.$$

Main Results (BBM with binary branching)

Corollary

Let

$$\mu_t := \frac{1}{N_t} \sum_{i=1}^{N_t} \delta_{X_t^i}$$

be the uniform distribution on all particles alive at time t, given survival. Then, for each measurable $E \subset D$, we have that

$$\mu_t(E) \to \mu(E)$$

in distribution, and hence in probability, as $t \to \infty$, where

$$\mu(E) = \frac{\int_E \varphi(x)\,dx}{\int_D \varphi(x)\,dx}.$$

Notations

$A =$ offspring distribution - mean m.

$L =$ generator.

Assume: D is C^1, $\mathbb{E}[A^3] < \infty$ and L is uniformly elliptic and self-adjoint with smooth coefficients.

Notations

$A =$ offspring distribution - mean m.

$L =$ generator.

Assume: D is C^1, $\mathbb{E}[A^3] < \infty$ and L is uniformly elliptic and self-adjoint with smooth coefficients.

BBM with binary branching: $A \equiv 2,\ L = \frac{1}{2}\Delta.$

Extinction (general case)

Phase Transition (Sevast'yanov 1958, Watanabe 1965)

There exists a critical value $\frac{\lambda}{m-1}$ of the branching parameter β s.t.

- $\beta \leq \frac{\lambda}{m-1} \Rightarrow$ a.s. extinction
- $\beta > \frac{\lambda}{m-1} \Rightarrow$ survival with positive probability.

λ is the first eigenvalue of $-L$ on D.

Main Results (general case)

Theorem (P. 2015)

In the critical case $\beta = \frac{\lambda}{m-1}$, for all $x \in D$ we have

$$\mathbb{P}_x\left(N_t > 0\right) \sim \frac{1}{t} \times \frac{2(m-1)\varphi(x)}{\lambda\left(\mathbb{E}[A^2] - \mathbb{E}[A]\right)\int_D \varphi(y)^3\, dy}$$

as $t \to \infty$.

φ is the first eigenfunction of $-L$ on D, normalised to have unit L^2 norm.

Main Results (general case)

Theorem (P. 2015)

For any measurable $E \subset D$, if N_t^E is the number of particles in E at time t, we have

$$\left(\frac{N_t^E}{t} \middle| N_t > 0 \right) \to Z$$

in distribution as $t \to \infty$, where Z is an exponential random variable with mean

$$\frac{\lambda \left(\mathbb{E}[A^2] - \mathbb{E}[A] \right) \langle \varphi, \mathbf{1}_E \rangle_{L^2(D)} \int_D \varphi^3}{2(m-1)}.$$

Main Results (general case)

Corollary

Let

$$\mu_t := \frac{1}{N_t} \sum_{i=1}^{N_t} \delta_{X_t^i}$$

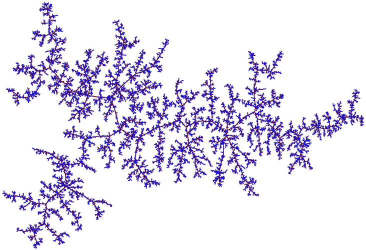
be the uniform distribution on all particles alive at time t, given survival. Then, for each measurable $E \subset D$, we have that

$$\mu_t(E) \to \mu(E)$$

in distribution, and hence in probability, as $t \to \infty$, where

$$\mu(E) = \frac{\int_E \varphi(x)\, dx}{\int_D \varphi(x)\, dx}.$$

Scaling Limit



Simulation by Igor Kortchemski.

- What does the conditioned tree look like?
- **Conjecture**: It converges (with appropriate rescaling) to the CRT.
- Work in progress!

FKPP equation

Key Tool: Let

$$u(x, t) = \mathbb{P}_x(N_t > 0).$$

Then u satisfies the FKPP equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda(u - u^2)$$

in D with correct boundary/initial conditions.

Heuristics

Reminder: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda(u - u^2)$.

Write

$$u(x,t) = \sum_i a_i(t)\varphi_i(x); \quad a_i(t) = \int_D u(x,t)\varphi_i(x)dx.$$

Heuristics

Reminder: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda(u - u^2)$.

Write

$$u(x,t) = \sum_i a_i(t)\varphi_i(x); \quad a_i(t) = \int_D u(x,t)\varphi_i(x)dx.$$

Then

$$\frac{da_i}{dt} = \int_D \left(\frac{1}{2}\Delta u + \lambda(u - u^2)\right)\varphi_i(x)\,dx.$$

Heuristics

Reminder: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda(u - u^2)$.

Write

$$u(x,t) = \sum_i a_i(t)\varphi_i(x); \quad a_i(t) = \int_D u(x,t)\varphi_i(x)dx.$$

Then

$$\frac{da_i}{dt} = \int_D \left(\frac{1}{2}\Delta u + \lambda(u - u^2)\right)\varphi_i(x)\,dx.$$

IBP $\Rightarrow$

$$\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right)\varphi_i(x)\,dx.$$

Heuristics

Reminder: $\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right) \varphi_i(x)\, dx.$

Heuristics

Reminder: $\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right) \varphi_i(x)\, dx.$

So

$$\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\, dx$$

Heuristics

Reminder: $\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right) \varphi_i(x)\, dx.$

So

$$\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\, dx$$

$$\frac{da_i}{dt} = (\lambda - \lambda_i)a_i(t) - \lambda \int_D u^2(x,t)\varphi_i(x)\, dx \text{ for } i \geq 2.$$

Heuristics

Reminder: $\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right) \varphi_i(x)\, dx.$

So

$$\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\, dx$$

$$\frac{da_i}{dt} = (\lambda - \lambda_i)a_i(t) - \lambda \int_D u^2(x,t)\varphi_i(x)\, dx \ \text{ for } \ i \geq 2.$$

Guess: $u(x,t) \sim a_1(t)\varphi(x).$

Heuristics

Reminder: $\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right) \varphi_i(x)\, dx.$

So

$$\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\, dx$$

$$\frac{da_i}{dt} = (\lambda - \lambda_i)a_i(t) - \lambda \int_D u^2(x,t)\varphi_i(x)\, dx \;\text{ for }\; i \geq 2.$$

Guess: $u(x,t) \sim a_1(t)\varphi(x)$.

Sadly, doesn't quite work.

Heuristics

Reminder: $\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\,dx$.

Suppose $u(x,t) \sim a_1(t)\varphi(x)$. Then

$$\frac{da_1}{dt} \sim -a_1^2 \times \lambda \int_D \varphi(x)^3\,dx.$$

Heuristics

Reminder: $\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\,dx.$

Suppose $u(x,t) \sim a_1(t)\varphi(x)$. Then

$$\frac{da_1}{dt} \sim -a_1^2 \times \lambda \int_D \varphi(x)^3\,dx.$$

Easy to prove:

$$a_1(t) \sim \frac{1}{t} \times \frac{1}{\lambda \int_D \varphi(y)^3}.$$

Martingales

Goal: $u(x,t) \sim a_1(t)\varphi(x)$.

Martingales

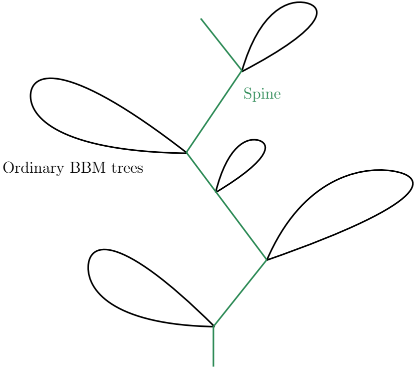
Goal: $u(x,t) \sim a_1(t)\varphi(x)$.

Key tool:

$$M_t = \sum_{i=1}^{N_t} \varphi(X_t^i)$$

is a martingale.

Change of Measure - Critical Case



No extinction under $\mathbb{Q}_x$ (new measure).

- Change measure by $\frac{M_t}{\mathbb{E}_x[M_t]} = \frac{\sum_i \varphi(X_i^t)}{\varphi(x)}$.
- Spine particle is BM conditioned to remain in D.
- Branches at rate 2λ.
- Offspring are ordinary BBM processes.

Heuristics

Reminder: want $\mathbb{P}_x(N_t > 0) \sim a_1(t)\varphi(x)$.

Heuristics

Reminder: want $\mathbb{P}_x\left(N_t > 0\right) \sim a_1(t)\varphi(x)$.

$$\frac{\mathbb{P}_x\left(N_t > 0\right)}{\varphi(x)} = \mathbb{E}_x\left[\frac{1}{M_t}\frac{M_t}{\varphi(x)}\mathbf{1}_{\{N_t>0\}}\right]$$

$$= \mathbb{Q}_x\left[\frac{1}{M_t}\right]$$

Heuristics

Reminder: want $\mathbb{P}_x(N_t > 0) \sim a_1(t)\varphi(x)$.

$$\begin{aligned}\frac{\mathbb{P}_x(N_t > 0)}{\varphi(x)} &= \mathbb{E}_x\left[\frac{1}{M_t}\frac{M_t}{\varphi(x)}\mathbf{1}_{\{N_t>0\}}\right] \\ &= \mathbb{Q}_x\left[\frac{1}{M_t}\right]\end{aligned}$$

This shouldn't depend on x for large t.

Heuristics

Reminder: want $\mathbb{P}_x(N_t > 0) \sim a_1(t)\varphi(x)$.

$$\frac{\mathbb{P}_x(N_t > 0)}{\varphi(x)} = \mathbb{E}_x\left[\frac{1}{M_t}\frac{M_t}{\varphi(x)}\mathbf{1}_{\{N_t>0\}}\right]$$
$$= \mathbb{Q}_x\left[\frac{1}{M_t}\right]$$

This shouldn't depend on x for large t.

So $\mathbb{P}_x(N_t > 0) \sim c(t)\varphi(x)$ as $t \to \infty$,

Heuristics

Reminder: want $\mathbb{P}_x(N_t > 0) \sim a_1(t)\varphi(x)$.

$$\frac{\mathbb{P}_x(N_t > 0)}{\varphi(x)} = \mathbb{E}_x\left[\frac{1}{M_t}\frac{M_t}{\varphi(x)}\mathbf{1}_{\{N_t>0\}}\right] = \mathbb{Q}_x\left[\frac{1}{M_t}\right]$$

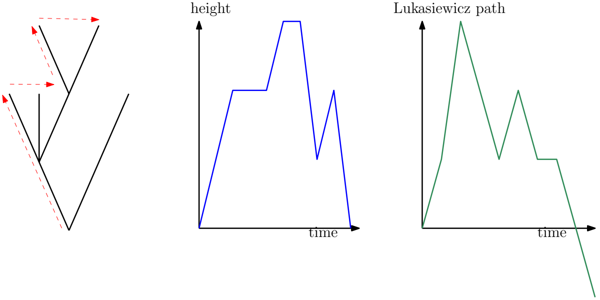
This shouldn't depend on x for large t.

So $\mathbb{P}_x(N_t > 0) \sim c(t)\varphi(x)$ as $t \to \infty$, and $c(t) \sim a_1(t)$.

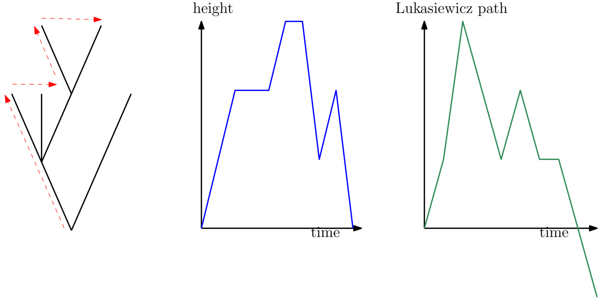
Conditioned results

- Asymptotic for survival probability.
- Many-to-few Lemma.
- Method of Moments.

Scaling Limit

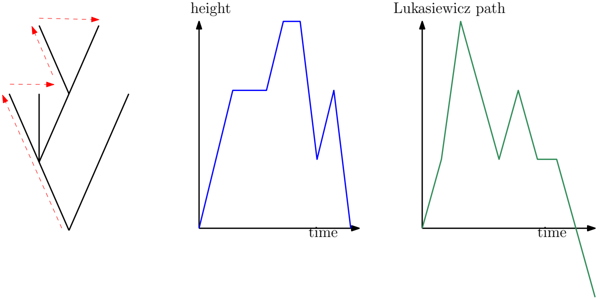


Scaling Limit



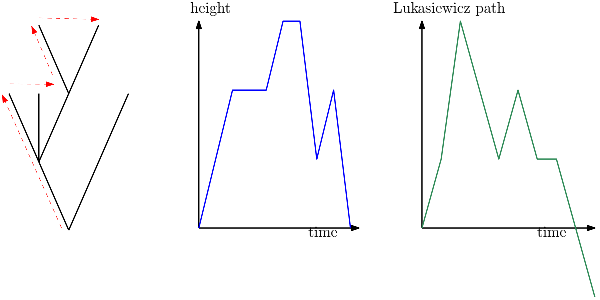
- Want concatenated sequence of iid height functions $\to$ reflecting Brownian motion.

Scaling Limit



- Want concatenated sequence of iid height functions $\rightarrow$ reflecting Brownian motion.
- **GW case:** connect height function with random walk.

Scaling Limit



- Want concatenated sequence of iid height functions $\to$ reflecting Brownian motion.
- **GW case:** connect height function with random walk.
- **BBM case:** connect height function with martingale.

Thanks for listening!