Non-extensive condensation in reinforced branching processes

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June 28th, 2016

The preferential attachment tree

- At time 1, one node and one root-edge.
- At time n, add the n^{th} node in the tree: link it to a random node chosen with probability proportional to the degrees.

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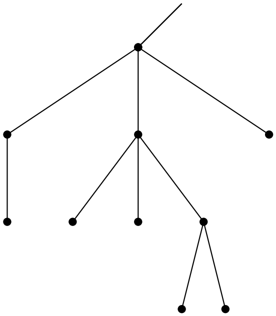

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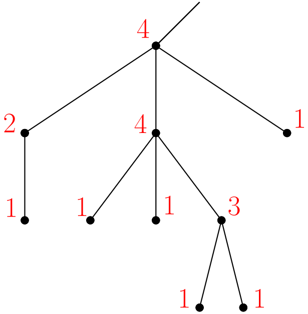

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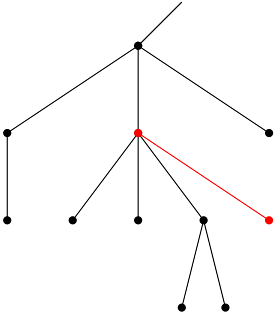

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Scale-free property

$$\frac{\#\{\text{nodes of degree } k \text{ at time } n\}}{n} \sim k^{-3} \text{ when } n \to \infty.$$

The preferential attachment tree with fitnesses

- Fix $(X_n)_{n\geq 1}$ i.i.d. fitnesses.
- At time 1, one node and one root-edge.
- At time n, add the n^{th} node in the tree: link it to a random node chosen with probability proportional to the degrees **times the fitnesses**.

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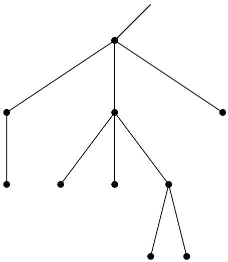

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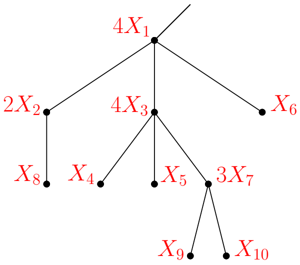

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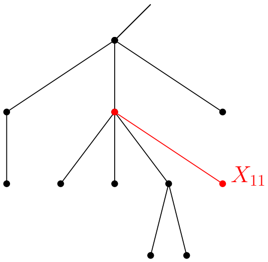

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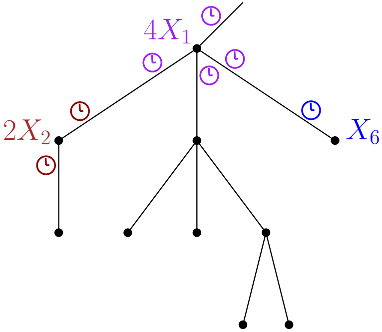
Two competing dynamics: rich-gets-richer and fit-gets-richer.

Conjecture – the winner takes it all [Bianconi & Barabási]

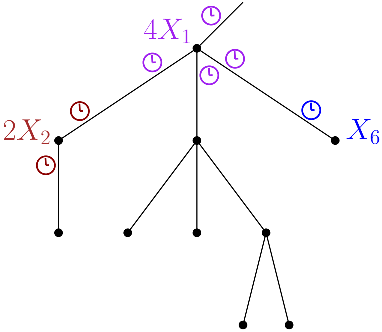
$$\liminf_{n\to\infty} \frac{\text{max degree at time } n}{n} > 0.$$

Embedding in continuous time



- Fix $(X_n)_{n\geq 1}$ i.i.d. fitnesses.
- At time 1, one node and one root-edge.
- Equip every half edge with a exponential clock of parameter the fitness of the node it's attached to.
- When a half-edge rings, add a new child to the node it's attached to.

Embedding in continuous time



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This is a branching process!

Our model

Define a population process as follows: at time t,

- $N(t)$ particles (= half-edges)
- $M(t)$ families (= set of particles sharing the same fitness = nodes)
- n^{th} family born at time τ_n, has fitness X_n and size $Z_n(t)$ (= degree)

At time t, each family reproduces at rate $X_n Z_n(t)$. When a birth event happens in family n:

- with probability γ a new particle is added to family n;
- with probability β a mutant having fitness $X_{M(t)+1}$ is born.

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Three parameters:

- $0 \leq \beta, \gamma \leq 1$ mutation and selection probability
- μ the fitness distribution on $(0, 1)$

BB model

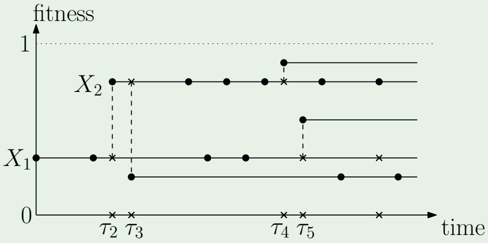
$\beta = \gamma = 1$

Kingman

$\gamma = 1 - \beta$

Empirical fitness distribution

Picture in the $\gamma = 1 - \beta$ case:

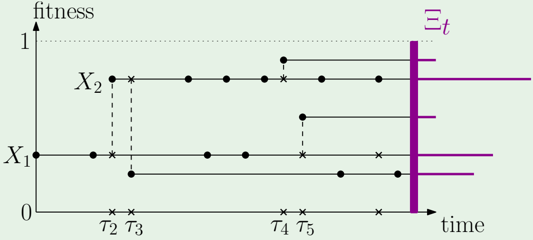


Remark: Given it's birth time τ_n, each family is a Yule process of parameter γX_n, independent of the rest of the system, implying that, almost surely when $t \to \infty$

$$e^{-\gamma X_n(t-\tau_n)} Z_n(t) \to \xi_n \quad \text{(indep. of } \tau_n\text{)}.$$

Empirical fitness distribution

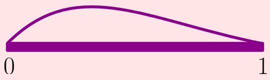
Picture in the $\gamma = 1 - \beta$ case:



Empirical fitness distribution:

$$\Xi_t := \frac{1}{N(t)} \sum_{n=1}^{M(t)} Z_n(t)\delta_{X_n}.$$

If (cond) fails:

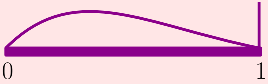


at $t = \infty$

(cond)

$$\boxed{\frac{\beta}{\beta+\gamma}\int_0^1 \frac{\mathrm{d}\mu(x)}{1-x} < 1}$$

If (cond) holds:



at $t = \infty$

Condensation result

Theorem [folklore] –

If (cond) fails then there exists $\lambda^\star \in [\gamma, \beta+\gamma)$ such that

$$\frac{\beta}{\beta+\gamma}\int_0^1 \frac{\lambda^\star}{\lambda^\star-\gamma x}\,\mathrm{d}\mu(x) = 1,$$

otherwise, we let $\lambda^\star = \gamma$. In both cases:

- $\int_0^1 x\,\mathrm{d}\Xi_t(x) \to \lambda^\star/\beta+\gamma$ a.s. when $t\to\infty$;
- $\Xi_t \to \pi$ a.s. weakly when $t\to\infty$, where

(i) if (cond) fails then $\mathrm{d}\pi(x) = \frac{\beta}{\beta+\gamma}\frac{\lambda^\star}{\lambda^\star-\gamma x}\mathrm{d}\mu(x)$;

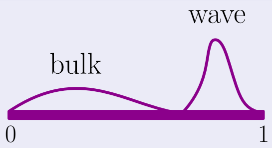
(ii) if (cond) holds then $\mathrm{d}\pi(x) = \frac{\beta}{\beta+\gamma}\frac{\mathrm{d}\mu(x)}{1-x} + \omega(\beta,\gamma)\delta_1$.

Remark: we get some rough information about the growth rate:

$$\log N(t) = \lambda^\star t + o(t).$$

Motivations

If (cond) holds:

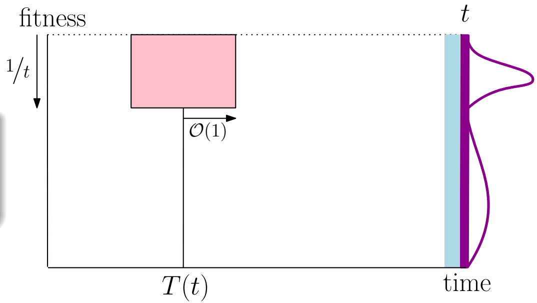


at large t

- how many families contribute to the condensation wave?
- how old are these families?
- what is the shape of the condensation wave?
- does the winner take it all? ...

One additional assumption

$$\mu(1-\varepsilon, 1) \sim \varepsilon^{\alpha}\ell(\varepsilon)$$
when $\varepsilon \to 0$.



Main results

Let $n(t) = \lfloor 1/\mu(1-1/t,1) \rfloor \approx t^{\alpha}$ and

$$T(t) = \inf\{s \geq 0 : M(s) \geq n(t)\}$$
$$\approx \text{ first time when there exists a fitness at least } 1 - 1/t$$
$$\approx \log t$$

Theorem [DMM++]

- Size $S(t)$ of the largest family: $e^{-\lambda^\star(t-T(t))}S(t) \Rightarrow \Gamma(\lambda^\star, \alpha)$.
- Fitness $V(t)$ of the largest family: $t(1-V(t)) \Rightarrow W$ (explicit).
- Time of birth $\Theta(t)$ of the largest family: $\Theta(t) - T(t) \Rightarrow Z$.

The winner does not take it all [DMM++]

In probability when $t \to \infty$, $\dfrac{S(t)}{N(t)} = \dfrac{\max_{1..M(t)} Z_n(t)}{N(t)} \to 0$.

Our approach

$$\Gamma_t := \sum_{n=1}^{M(t)} \delta\Big(\tau_n - T(t), t(1-X_n), \mathrm{e}^{-\gamma(t-T(t))} Z_n(t)\Big)$$

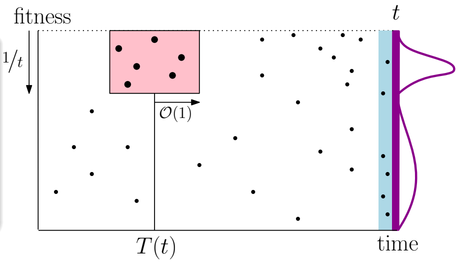
Theorem [DMM++]

When $t \to \infty$, Γ_t converges vaguely (compactly supported test functions) on $[-\infty, \infty] \times [0, \infty] \times (0, \infty]$ to the Poisson point process of intensity $d\zeta(s, x, z) = \lambda^\star \mathrm{e}^{\lambda^\star s} \alpha x^{\alpha-1} \mathrm{e}^{-z\mathrm{e}^{\gamma(s+x)}} \mathrm{e}^{\gamma(s+x)} \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z$.

Remark: When (cond) holds, $\lambda^\star = \gamma$.

Proof key points

1. vague convergence on $[-\infty, \infty) \times [0, \infty) \times [0, \infty]$;
2. young families are too small + non-fit families are too small.



Limit law of the size of the largest family

$$\Gamma_t := \sum_{n=1}^{M(t)} \delta\Big(\tau_n - T(t), t(1-X_n), \mathrm{e}^{-\gamma(t-T(t))} Z_n(t)\Big)$$

How to prove that $\Gamma_t \Rightarrow \mathtt{PPP}(\zeta)$ vaguely on $[-\infty,\infty]\times[0,\infty]\times(0,\infty]$ implies that $\mathrm{e}^{-\gamma(t-T(t))}\max_{1..M(t)} Z_n(t) \Rightarrow \Gamma(\lambda^\star,\alpha)$?

Take $K = [-\infty,\infty]\times[0,\infty]\times[x,\infty]$.

$$\sum_{n=1}^{M(t)} \mathbf{1}_K(\tau_n - T(t), t(1-X_n), \mathrm{e}^{-\gamma(t-T(t))} Z_n(t)) \Rightarrow \mathtt{Poi}\Big(\int_K \mathrm{d}\zeta(s,x,z)\Big),$$

implying that $\mathbb{P}\Big(\mathrm{e}^{-\gamma(t-T(t))}\max Z_n(t) \geq x\Big) = \mathbb{P}\Big(\int \mathbf{1}_K\,\mathrm{d}\Gamma_t(s,x,z) \geq 1\Big)$
$\to \mathbb{P}\Big(\mathtt{Poi}\Big(\int_K \mathrm{d}\zeta(s,x,z)\Big) \geq 1\Big) = 1 - \exp\Big(-\int_K \mathrm{d}\zeta(s,x,z)\Big)$
= the Gamma law we want.

Proof of the-winner-does-not-take-it-all

$$\Gamma_t := \sum_{n=1}^{M(t)} \delta\Big(\tau_n - T(t), t(1 - X_n), \mathrm{e}^{-\gamma(t-T(t))} Z_n(t)\Big)$$

How to prove that $\Gamma_t \Rightarrow \mathtt{PPP}(\zeta)$ vaguely on $[-\infty, \infty] \times [0, \infty] \times (0, \infty]$ implies that $\max_{1..M(t)} Z_n(t)/N(t) \to 0$ in probability?

We know that $\mathrm{e}^{-\gamma(t-T(t))} \max Z_n(t) \to \Gamma(\lambda^\star, \alpha)$.

$$\mathrm{e}^{-\gamma(t-T(t))} N(t) = \mathrm{e}^{-\gamma(t-T(t))} \sum_{n=1}^{M(t)} Z_n(t) = \int_0^\infty z \,\mathrm{d}\Gamma_t(s, x, z)$$

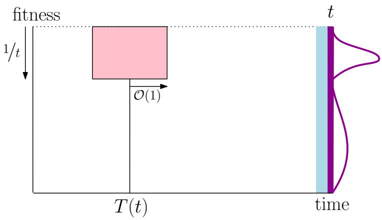
$$\geq \int_\varepsilon^\infty z \,\mathrm{d}\Gamma_t(s, x, z) \to \int_\varepsilon^\infty z \mathrm{d}\mathtt{PPP}_\zeta(s, x, z).$$

We then prove that, when $\varepsilon \to 0$,

$$\mathbb{E} \int_\varepsilon^\infty z \mathrm{d}\mathtt{PPP}_\zeta(s, x, z) = \int_\varepsilon^\infty z \mathrm{d}\zeta(s, x, z) \to +\infty,$$

and $\mathtt{Var} \int_\varepsilon^\infty z \,\mathrm{d}\mathtt{PPP}_\zeta(s, x, z) = O(1)$. (Apply Tchebychev to conclude.)

Conclusion and open problems



We have proved

- results about the largest family: we know its size, when it was born and its fitness.
- that the winner does not take it all (disproves BB's conjecture).

Still many open questions:

- Can we estimate better the growth rate? $N(t) = e^{\lambda^\star t + o(t)}$.
- What is the shape of the wave? Need to consider a wider box.
- What if the fitness distribution has a different behaviour near 1?
- What if the fitness distribution is unbounded? What is the growth rate? What kind of distribution can we tackle?

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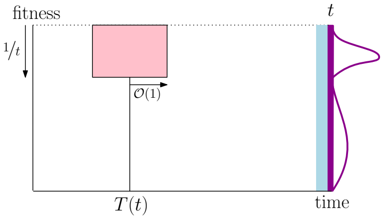

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Thanks!!