Branching random walk in random environment

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joint work with Piotr Miłoś

ENS, DMA

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Plan

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1. The branching random walk in random environment
2. The random walk in random environment
3. Brownian motion above a Brownian curve

Branching random walk in random environment

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Description of the process

Let $(\mathcal{L}_n, n \geq 0)$ be an i.i.d. sequence of point processes distributions on $\mathbb{R}$.

Description

- An individual alive at time 0.
- Gives birth to children around its current position.
- Each child then reproduces independently.
- Every new generation reproduces independently.
- We take interest in the maximal displacement.

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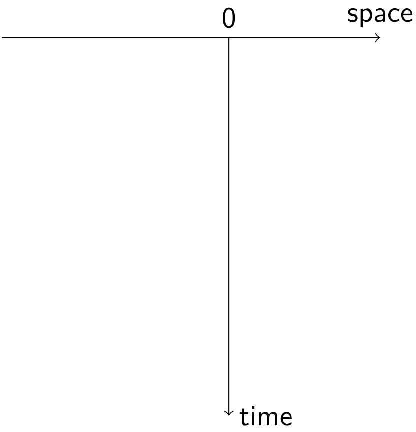

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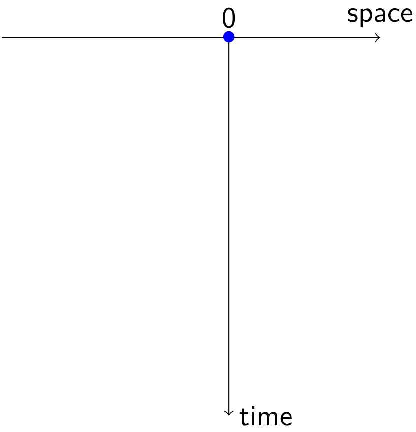

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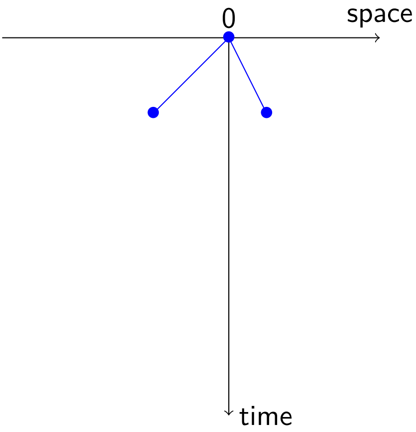

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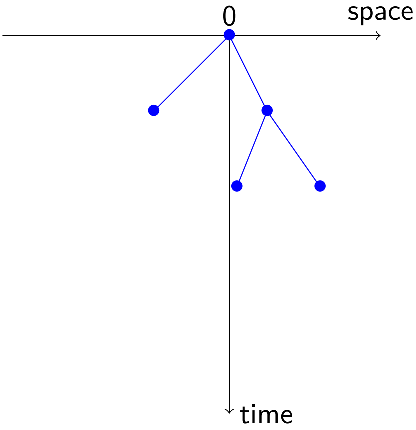

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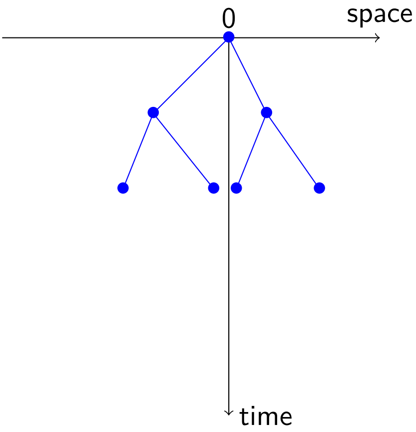

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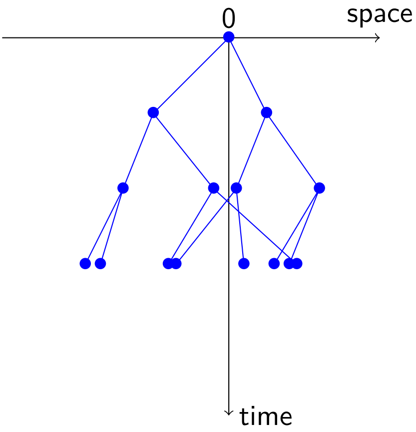

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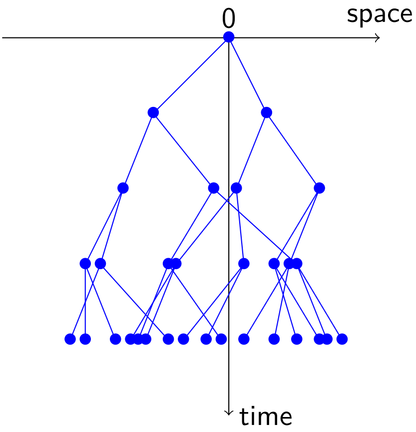

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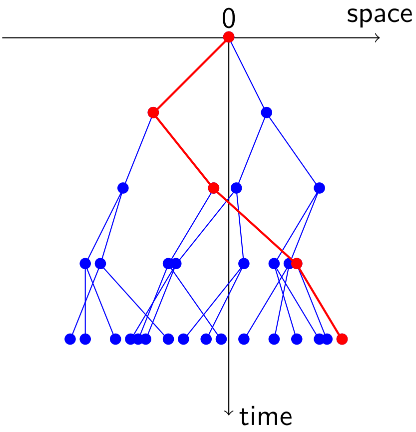

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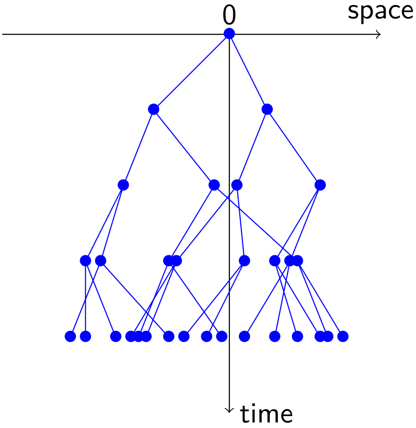

Notation

- Let u be an individual.
- $V(u)$: position of u.
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- u_k : ancestor of u at time k.
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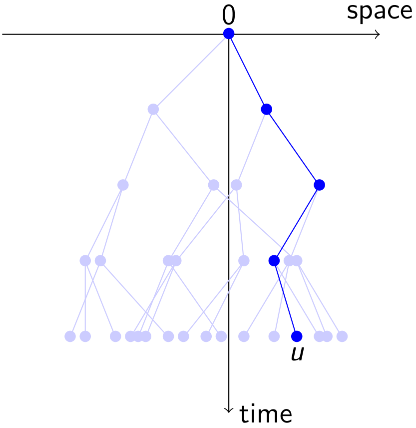

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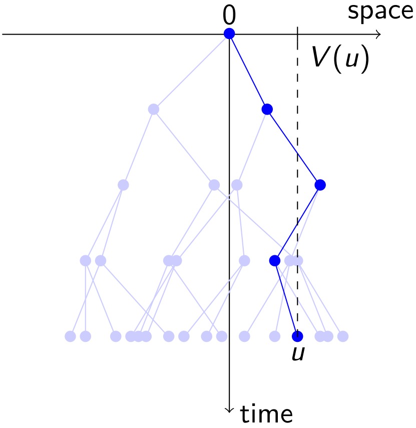

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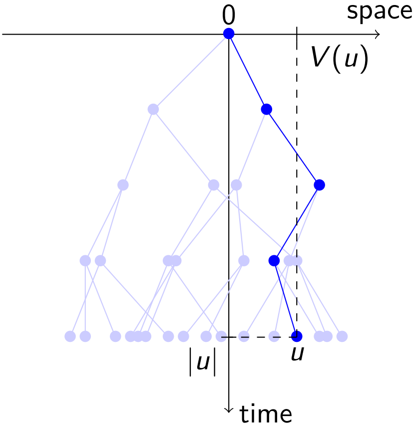

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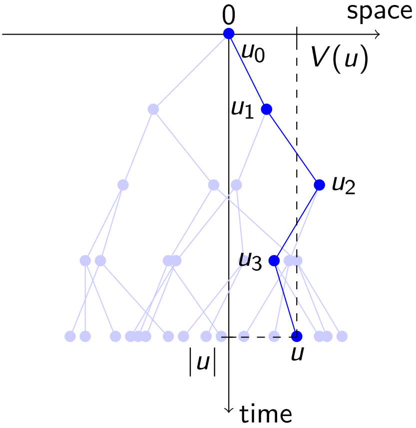

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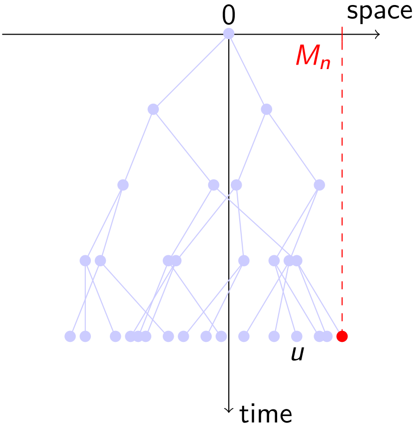

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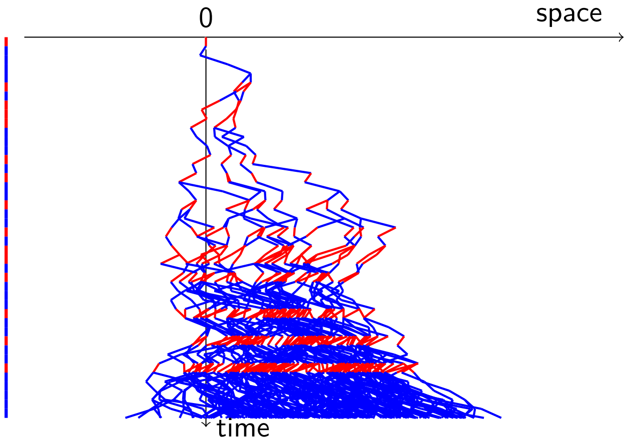

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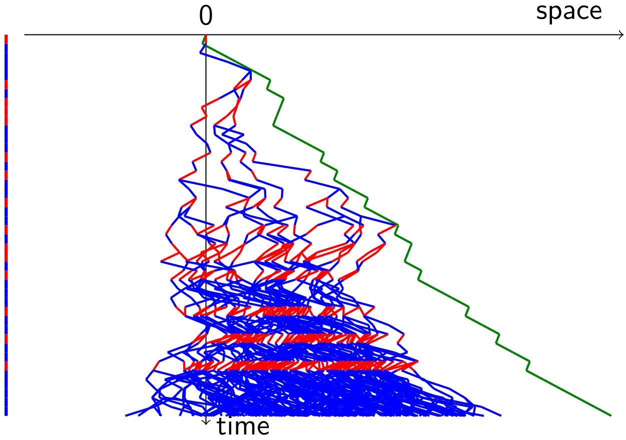

Additional notation

We write $\mathbb{E}_{\mathcal{L}}[.] = \mathbf{E}\left[. \,|\mathcal{L}_1, \mathcal{L}_2, \ldots\right]$ and L_n a point process with law $\mathcal{L}_n$.

Log-Laplace transform of $\mathcal{L}_n$

$$\forall \phi > 0, \kappa_n(\phi) = \log \mathbf{E}\left[\sum_{\ell \in L_n} e^{\phi \ell}\right].$$

Critical parameter

We set $K(\phi) = \mathbf{E}(\kappa_1(\phi))$. We assume there exists $\theta > 0$ such that

$$\theta K'(\theta) - K(\theta) = 0.$$

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Boundary of the branching random walk

From now on, we assume that θ is the critical parameter of the branching random walk.

Boundary of the branching random walk

For any $n \in \mathbb{N}$, we write $F_n = \sum_{j=1}^{n} \kappa_j(\theta)/\theta$.

Observations

The process $(F_n, n \geq 0)$ is a random walk that depend only on the environment $(\mathcal{L}_n, n \geq 1)$.

We have $\lim_{n\to+\infty} \frac{F_n}{n} = \frac{K(\theta)}{\theta} = K'(\theta)$.

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The maximal displacement

Theorem

We assume there exists $\theta > 0$ *such that* $\theta K'(\theta) - K(\theta) = 0$*. We write*

$$F_n = \sum_{j=1}^{n} \frac{\kappa_j(\theta)}{\theta}, \ \sigma_Q^2 = \theta^2 \mathbf{E}\left[\kappa_1''(\theta)\right] \text{ and } \sigma_A^2 = \mathbf{V}\mathrm{ar}\left[\theta\kappa_1'(\theta) - \kappa_1(\theta)\right].$$

Let $m_n = \inf\{y \in \mathbb{R} : \mathbb{P}_{\mathcal{L}}(M_n \geq y) = 1/2\}$*, we have*

$$m_n = F_n - \phi \log n + o_{\mathbf{P}}(\log n) \quad \text{and} \quad M_n = F_n - \phi \log n + o_{\mathbf{P}}(\log n),$$

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$$\phi = \frac{1}{\theta}\left(\frac{1}{2} + 2\gamma(\sigma_A/\sigma_Q)\right).$$

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Many-to-one lemma

We write $\mu_n((-\infty, x]) = \mathbb{E}_{\mathcal{L}}\left[\sum_{\ell \in L_n} e^{\theta\ell - \kappa_n(\theta)} \mathbf{1}_{\{\ell \leq x\}}\right]$, and we set X_n a random variable with law μ_n.

Definition

The sequence $(\mu_n, n \in \mathbb{N})$ is a sequence of i.i.d. random probability distributions on $\mathbb{R}$.
The process $S_n = X_1 + X_2 + \cdots + X_n$ is a random walk with random environment $(\mu_n, n \in \mathbb{N})$.

Lemma (Many-to-one lemma)

Let f be a measurable positive function, we have

$$\mathbb{E}_{\mathcal{L}}\left[\sum_{|u|=n} f(V(u_j), j \leq n)\right] = \mathbb{E}_{\mathcal{L}}\left[e^{-\theta(S_n - F_n)} f(S_j, j \leq n)\right].$$

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Boundary of the process

We compute the number of individuals crossing for the first time $k \mapsto F_k + x$:

$$\mathbb{E}_{\mathcal{L}}\left[\sum_{u \in \mathbf{T}} \mathbf{1}_{\{V(u) > F_{|u|} + x\}} \mathbf{1}_{\{V(u_j) \leq F_j + x, j < |u|\}}\right]$$

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Upper bound for the maximal displacement

We compute the number of individuals who, while staying below the curve $k \mapsto F_k + x$, are at time n larger than $F_n - \lambda \log n$.

$$\mathbb{E}_{\mathcal{L}}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u) \geq F_n - \lambda \log n\}} \mathbf{1}_{\{V(u_j) \leq F_j + x, j \leq n\}}\right]$$
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$$\approx n^{\theta\lambda} \mathbb{P}_{\mathcal{L}}\left(S_n - F_n \geq -\lambda \log n, S_j - F_j \leq x, j \leq n\right).$$

To conclude, we need to compute the asymptotic behaviour of

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To conclude, we need to compute the asymptotic behaviour of

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Upper bound for the maximal displacement

We compute the number of individuals who, while staying below the curve $k \mapsto F_k + x$, are at time n larger than $F_n - \lambda \log n$.

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Persistence exponent for the random walk in random environment

Let S be a random walk in random environment.

Theorem

We assume that $\mathbf{E}(S_1) = 0$ *and there exists* $\lambda > 0$ *and* $C > 0$ *such that*

$$\mathbf{E}\left(e^{\lambda |X_1|}\right) < +\infty \quad et \quad \mathbb{E}_{\mathcal{L}}\left(e^{\lambda |X_1 - \mathbb{E}_{\mathcal{L}}(X_1)|}\right) \leq C \quad p.s.$$

There exists $\gamma \geq 1/2$ *such that*

$$-\frac{1}{\log n} \log \mathbb{P}_\mu \left(S_j \geq -\log n, j \leq n\right) = \gamma \quad p.s.$$

Thus $\mathbb{P}_\mu \left(S_j \geq -\log n, j \leq n\right) \approx n^{-\gamma}$ a.s.

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From random walk to Brownian motion I

We set $E_j = S_j - \mathbb{E}_{\mathcal{L}}(S_j)$ and $M_j = -\mathbb{E}_{\mathcal{L}}(S_j)$. We have

$$\mathbb{P}_\mu\left(S_j \geq -\log n, j \leq n\right) = \mathbb{P}_\mu\left(E_j \geq M_j - \log n, j \leq n\right).$$

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Set $\sigma_A^2 = \mathbf{V}\mathrm{ar}(M_1) = \mathbf{V}\mathrm{ar}(\mathbb{E}_{\mathcal{L}}(X_1))$ and $\sigma_Q^2 = \mathbf{E}(E_1^2) = \mathbf{E}(\mathbb{V}\mathrm{ar}_\mu(X_1))$.

Sakhanenko inequalities

- As M is a random walk, there exists a Brownian motion W such that $|M_j - \sigma_A W_j| \lesssim \log n$.
- Conditionally on $\mathcal{L}$ and W, E is the sum of i.i.d. centred random variables, there exists a Brownian motion B such that $|E_j - \sigma_Q B_j| \lesssim \log n$.

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From random walk to Brownian motion II

Therefore, we have

$$\mathbb{P}_\mu\left(S_j \geq -\log n, j \leq n\right) \approx \mathbf{P}\left(\sigma_Q B_s \geq \sigma_A W_s - \log n, s \leq n \middle| W\right) \quad \text{a.s.}$$

Thus $\gamma = \gamma(\sigma_A/\sigma_Q)$, where we define

$$\gamma(\beta) = \lim_{t \to +\infty} -\frac{1}{\log t} \log \mathbf{P}\left(B_s \geq \beta W_s - 1, s \leq t \middle| W\right).$$

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Brownian motion above a Brownian curve

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Let B and W be two independent Brownian motions.

Theorem

There exists a convex pair function γ such that for all $\beta \in \mathbb{R}$,

$$\lim_{t \to +\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_s + 1 \geq \beta W_s, s \leq t \middle| W \right) = -\gamma(\beta) \quad p.s.$$

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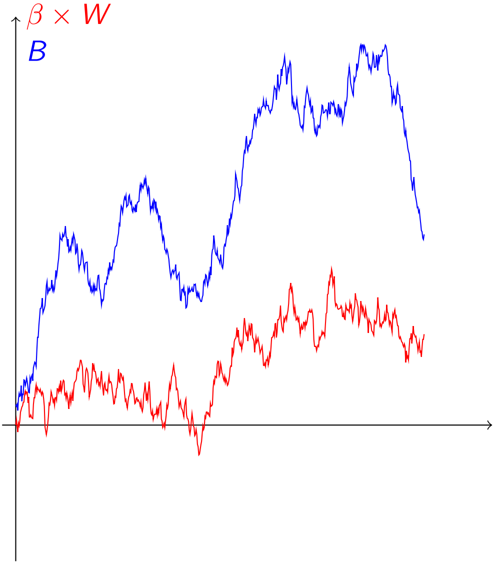

Properties of γ

Symmetry

As $W \overset{(d)}{=} -W$, we have

$$\left(\mathbf{P}\left(B_s + 1 \geq \beta W_s, s \leq t | W\right)\right)_{t \geq 0} \overset{(d)}{=} \left(\mathbf{P}(B_s + 1 \geq -\beta W_s, s \leq t | W)\right)_{t \geq 0},$$

thus $\gamma(\beta) = \gamma(-\beta)$.

Lower bound

By Jensen inequality,

$$\mathbf{E}\left[-\log \mathbf{P}\left(B_s + 1 \geq \beta W_s, s \leq t | W\right)\right] \geq -\log \mathbf{P}(B_s + 1 \geq \beta W_s, s \leq t),$$

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The Gaussian random variables being log-concave, we have

$$\mathbf{P}(B_s \geq f(s), s \leq t)\mathbf{P}(B_s \geq g(s), s \leq t) \leq \mathbf{P}\left(B_s \geq \frac{f(s)+g(s)}{2}, s \leq t\right)^2.$$

Therefore, we have

$$\mathbf{P}(B_s + 1 \geq \beta W_s, s \leq t | W)\mathbf{P}(B_s + 1 \geq \beta' W_s, s \leq t | W) \leq \mathbf{P}\left(B_s + 1 \geq \frac{\beta+\beta'}{2} W_s, s \leq t \middle| W\right)^2,$$

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By similar arguments, $\beta \mapsto \gamma(\sqrt{|\beta|})$ is convex.

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Existence of γ I

Let $X_t = e^{-t/2}(1 + B_{e^t-1})$ et $Y_t = e^{-t/2} W_{e^t-1}$.

X and Y are two independent Ornstein-Uhlenbeck processes, moreover:

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We set $\tau_0 = 0$ and

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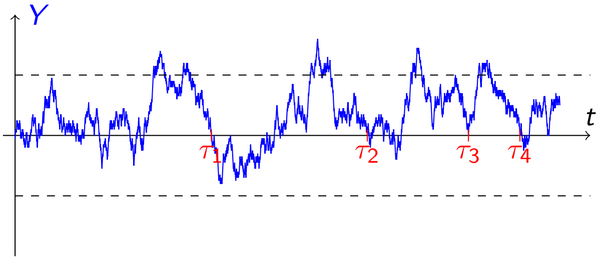

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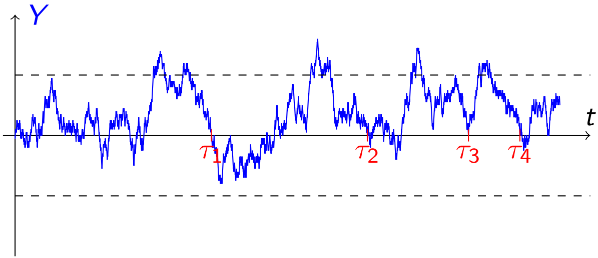

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By the Markov property, we have

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Kingman's subbaditive ergodic theorem

We have

$$\lim_{n \to +\infty} \frac{p_{0,n}}{n} = -\tilde{\gamma}(\beta) > -\infty \quad \text{p.s},$$

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We set $q_n = \log \mathbf{P}(X_s \geq \beta Y_s, s \leq \tau_n | Y)$. Observe that $q_n \geq p_n$.

FKG inequality

By FKG inequality for the Brownian motion, we have

$$q_n + \log \mathbf{P}(X_{\tau_n} \geq 1 | Y) \leq p_n.$$

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Existence of γ V

For all $\tau_n \leq t \leq \tau_{n+1}$,

$$\frac{q_{n+1}}{\tau_n} \leq \frac{1}{t} \log \mathbf{P}(X_s \geq Y_s, s \leq t | Y) \leq \frac{q_n}{\tau_{n+1}},$$

thus

$$\lim_{t \to +\infty} \frac{1}{t} \log \mathbf{P}(X_s \geq Y_s, s \leq t | Y) = -\frac{\tilde{\gamma}(\beta)}{\mathbf{E}(\tau_1)} = -\gamma(\beta) \quad \text{p.s.}$$

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The γ function

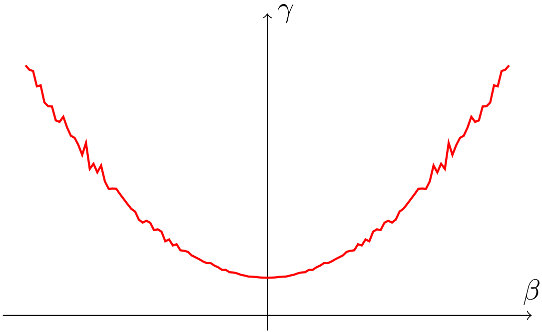

Figure : An approximation of $\mathbf{E}\left[-\log \mathbf{P}(B_s \geq \beta W_s - 1, s \leq 1000)\right]$.

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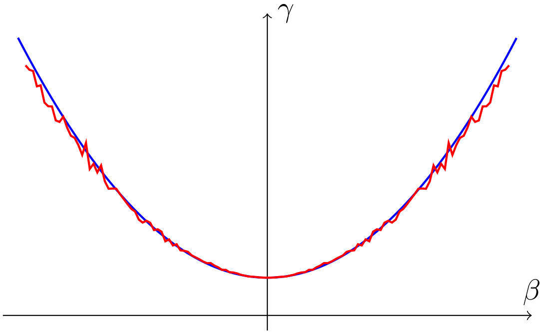

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