Some properties of the true self-repelling motion

L. Dumaz

Statistical laboratory, University of Cambridge

Bath-Paris branching structures meeting, June the 9th, 2014

What is a **(real-valued)** self-repelling process?

If $(X_t)_{t\geq 0}$ is a one-dimensional continuous process, define its occupation time measure:
For all $t \geq 0$, for all borel set A,

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$\longrightarrow L_t : \mathbb{R} \to \mathbb{R}_+$ is called "local time".

What is a **(real-valued)** self-repelling process?

Heuristical (vague) definition: $(X_t)_t$ is called self-repulsive when:

- $(X_t, L_t(\cdot))_t$ is a Markov process.
- $(X_t)_t$ prefers to go to the less visited places
  $\longrightarrow$ pushed away from $x \in \mathbb{R}$ where $L_t(x)$ is large.

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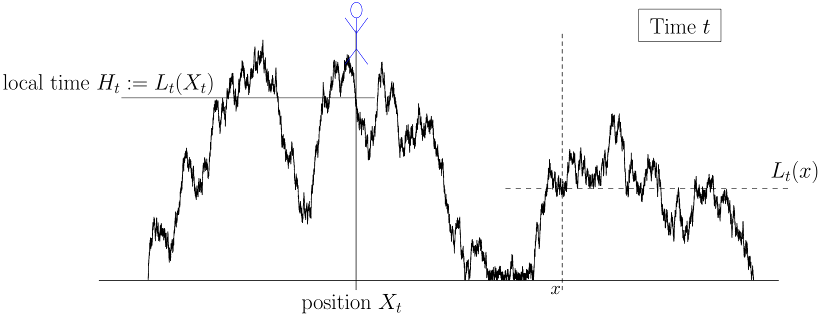

Figure : Picture at time t: X_t and its local time $L_t(\cdot)$

A first example: Brownian polymers

Brownian polymers were introduced by Durrett and Rogers in 1992.

$$dX_t = -\nabla(b * L_t(X_t + \cdot))|_0 \, dt + dB_t$$

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What happens when $\varepsilon \to 0$? Should converge to the true self-repelling motion (introduced after).

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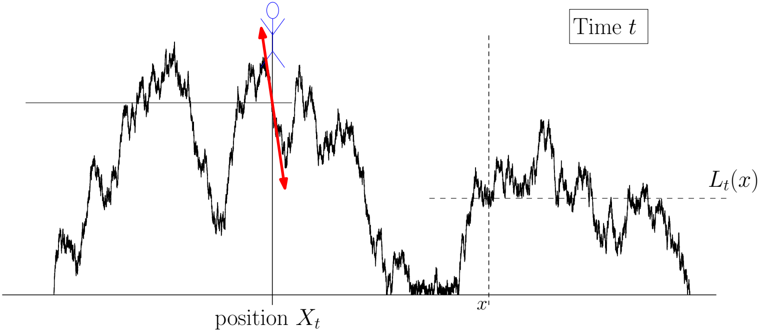

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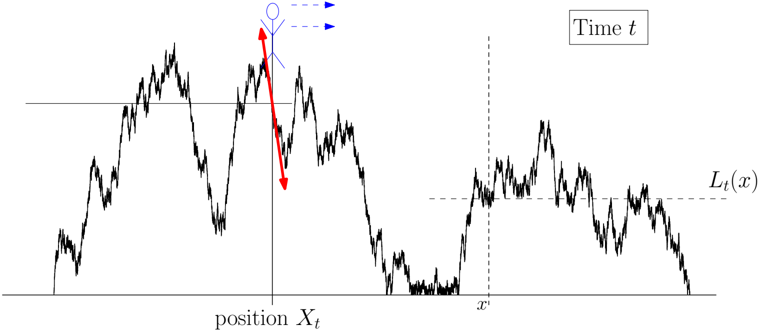

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TSRM definition and first properties

Results of the papers of B. Toth - W. Werner 1998 and F. Soucaliuc - B. Toth - W. Werner 2000

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For the construction, they used a family of coalescing reflected Brownian motions in the upper half plane now called **Brownian Web**. The TSRM is defined as the trace of the contour of the tree of these coalescing Brownian motions.

TSRM construction

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$\longrightarrow$ space-filling curve.

Parametrization: by the area it has swept. For every (x,h) in the upper half plane, the process (X_t, H_t) visits the point (x,h) at the random time $t = T_{x,h} := \int_{\mathbb{R}} \Lambda_{x,h(y)} dy$

TSRM = first coordinate (X_t).

Some first properties

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- Continuity and recurrence.
- Scaling and local variation: For all $a > 0$, $(X_{at}, t \geq 0)$ and $(a^{2/3} X_t, t \geq 0)$ have the same distribution and the TSRM is of finite variation of order $3/2$.

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- Local time: The TSRM admits a local time $L_t(\cdot)$ and a.s., for every $(x, h) \in \mathbb{R} \times \mathbb{R}^+$, the Brownian Web curves corresponds to the local time at times $T_{x,h}$ (strong Ray Knight theorem). It implies $H_t = L_t(X_t)$.

Some first properties

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- Localization: Interaction is local: the law of X just after t depends only on L_t around the point X_t.
  Moreover, we have a dynamical equation:

  $$\text{“}dX_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left(L_t(X_t + \varepsilon) - L_t(X_t - \varepsilon) \right) dt.\text{”}$$

  Limit holds in the probability sense.

My contributions

Large deviations of the TSRM

TSRM is an unusual process and it gives motivation to study some of its finest properties to discover the features it shares/does not share with the other processes.

Proposition (L.D.)

- *When* $x \to \infty$, $P(X_1 > x) = \exp(-\kappa x^3 + O(\ln(x)))$ *for some explicit* κ *(in terms of zeros of Airy function).*
- *When* $h \to \infty$, $P(H_1 > h) = \exp(-8h^3/9 + O(\ln(h)))$.

Law of the iterated logarithm

Pushing forward those results permits to derive a LIL for the TSRM when both t is large and t is small:

Proposition (L.D.)

a.s., $\limsup_{t\to 0} t^{-2/3}(ln(ln(1/t)))^{-1/3}X_t = 1/\kappa^{1/3}$.

Marginal distributions

In a joint work with Bálint Tóth, we computed the marginal distributions of this process.

Proposition (L.D., B. Tóth)

- *The density of X_1 denoted by $\nu_1(x)$ is equal to:*

$$\nu_1(x) = \sum_{k=1}^{\infty} \frac{3^{2/3}}{2^{7/3}} \left(\frac{\Gamma(2/3)}{\Gamma(1/3)} \right)^2 |a'_k|^{-3} f_{2/3}(2^{1/3}|a'_k||x|)$$

where the scaling factors a'_k are the zeros of the derivative of the Airy function and $f_{2/3}$ is the Mittag-Leffler's function.

- *The density of H_1 denoted by $\nu_2(h)$ is equal to:*

$$\nu_2(h) = \frac{2 \cdot 6^{1/3}\sqrt{\pi}}{\Gamma(1/3)^2} \exp(-(8h^3)/9) U(1/6, 2/3; (8h^3)/9)$$

where U is the hypergeometric function.

Marginal distributions

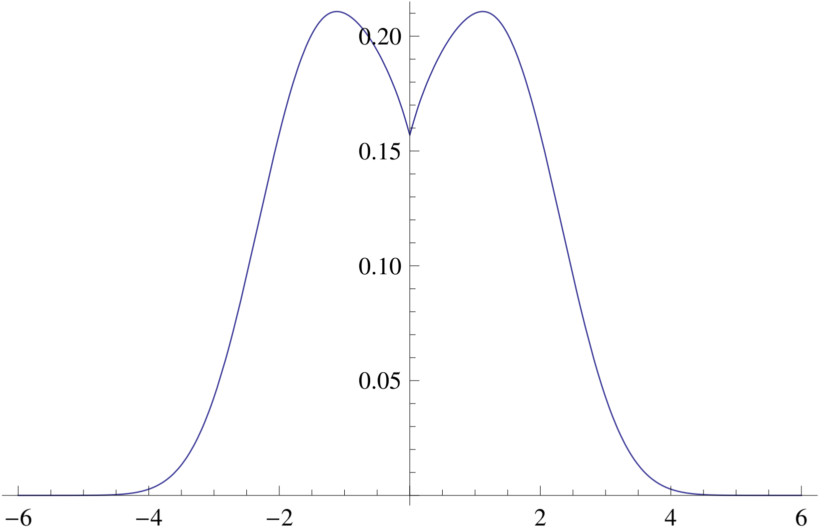
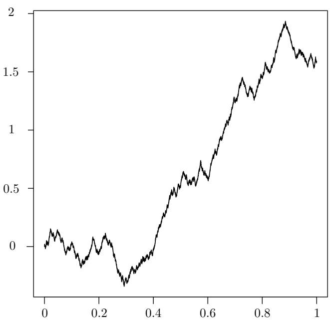


Figure : Density of X_1 (displacement at time 1)

A clever (self-repelling) burglar

What is the conditional law of the position X_1 knowing $L_1(\cdot)$?



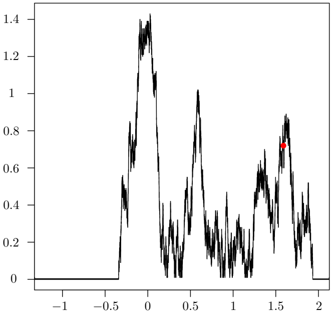


Figure : On the left $(X_t, t \in [0,1])$, and on the right the local time $L_1(\cdot)$

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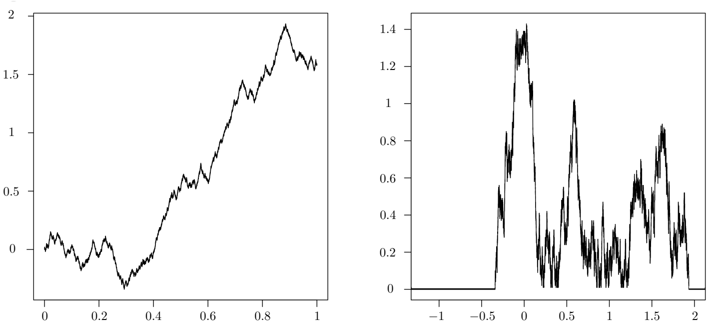

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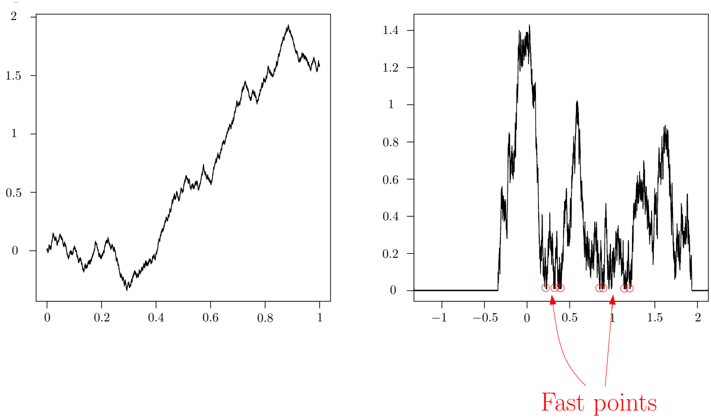

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Result

Proposition (L.D.)

The conditional law of X_1 knowing its local time at time 1, $L_1(\cdot)$, is uniform on the interval I defined by:

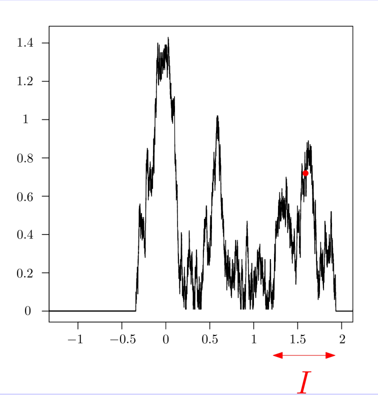

Figure : Definition of I

Thank you!