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Propagation and localization of elastic waves in highly anisotropic periodic composites via two-scale homogenization

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ABSTRACT

Wave propagation in periodic elastic composites whose phases may have not only highly contrasting but possibly also (in particular) highly anisotropic stiffnesses and moderately contrasting densities is considered. A possibly inter-connected (i.e. not necessarily isolated) "inclusion" phase is assumed generally much softer than that in the connected matrix, although some components of its stiffness tensor may be of the same order as in the matrix. For a critical scaling, generalizing that of a "double porosity"-type for the highly anisotropic elastic case, we use the tools of "non-classical" (high contrast) homogenization to derive, in a generic setting, two-scale limiting elastodynamic equations. The partially high-contrast results in a constrained microscopic kinematics described by appropriate projectors in the limit equations. The effective equations are then uncoupled and explicitly analyzed for their band-gap structure. Their macroscopic component describes plane waves with a dispersion relation which is generally highly non-linear both in the frequency and in the wave vector. While it is possible in this way to construct band-gap materials without the high anisotropy, the number of propagating modes for a given frequency (including none, i.e. in the band-gap case) is independent of the direction of propagation. However the introduction of a high anisotropy does allow variation in the number of propagating modes with direction if the inclusion phase is inter-connected, including achieving propagation in some directions and no propagation in the others. This effect is explicitly illustrated for a particular example of a highly anisotropic fibrous composite.

1. Introduction

Composites or mixtures are known to often display unusual physical properties, of which a very impressive catalog of examples is found in the work of Milton (2002) with homogenization theory being a key for understanding those effects. In this spirit, recent interest in designing materials for e.g. 'cloaking' objects, photonic and phononic applications, etc., has intensified interest in propagation and localization of waves in highly heterogeneous composites. The "classical" homogenization, mathematically applicable to heterogeneous media with moderate contrasts in physical properties, replaces the heterogeneity by an equivalent effective medium with uniform physical characteristics and is therefore intrinsically incapable of accounting for a number of underlying microscopic effects for example due to "micro-resonances".

It has been known for a while that an introduction of high contrasts is capable of accounting for unusual effects such as the memory effect (e.g. Fenchenko and Khruslov, 1980; Sandrakov, 1999) and other non-local effects (e.g. Bellieud and Gruais, 2005; Briane, 2002; Camar-Eddine and Seppecher, 2003; Camar-Eddine and Milton, 2005; Cherednichenko et al., 2006; Cherednichenko, 2006). Problems of time-harmonic wave propagation in such media relate to homogenization of spectral problems with high contrast.

Models of particular interest in this context are related to the so-called double porosity models originally developed for single phase fluid flows in fractured porous media (Barenblatt and Zheltov, 1960). Their further

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generalizations (e.g. Arbogast, 1989) were realized to be rigorous two-scale homogenized limits of two-component mesoscopic Darcy flows with highly contrasting porosity and permeability, as long as the parameter of high-contrast δ is critically scaled against the periodicity size ε, $\delta \sim \varepsilon^2$ (e.g. Arbogast et al., 1990; Bourgeat et al., 2003). More general classes of high-contrast homogenization problems were also considered (e.g. Panasenko, 1991; Allaire, 1992; Sandrakov, 1999) all with the underlying critical scaling $\delta \sim \varepsilon^2$ which we will henceforth conventionally refer to as "double porosity"-type scaling.

Zhikov (2000, 2004), among other results, analyzed scalar double porosity-type spectral problems in bounded (Zhikov, 2000) and unbounded (Zhikov, 2004) periodic domains employing the techniques of two-scale convergence (Nguetseng, 1989; Allaire, 1992). He established the two-scale spectral convergence, associated two-scale compactness, and introduced and analyzed a two-scale limit operator. The resulting two-scale limit spectral problem couples in a particular way the limit macroscopic and microscopic behaviors. Upon uncoupling, the macroscopic component is described by a spectral problem with a highly non-linear dependence on the spectral parameter characterized by function $\beta(\lambda)$ introduced in Zhikov (2000, Section 8.1). This describes in an asymptotically explicit way both the structure of the band gaps (see also Hempel and Lienau, 2000) and of the Floquet–Bloch waves in the bands (Zhikov, 2004). The non-linear dependence on the spectral parameter appears in fact related, via Laplace or Fourier transform, to the time-non-locality (memory effect) in the macroscopic overall behavior for related parabolic (cf. Fenchenko and Khruslov, 1980) and hyperbolic (Sandrakov, 1999) problems. In a mathematically equivalent setting of *H*-polarized electromagnetic waves Bouchitté and Felbacq (2004) motivated by somewhat related effects in their earlier work (Felbacq and Bouchitté, 1997) on homogenization of highly conducting dilute fibers, among their other results essentially re-discovered some of the above observations of Zhikov. They additionally interpreted $\beta(\lambda)$ as "artificial magnetism" due to micro-resonances, implying that the limit magnetic permeability may be deemed negative, giving rise to a number of interesting physical effects associated with micro-resonances discussed, e.g. by O'Brien and Pendry (2002) among others. In Kamotski and Smyshlyaev (2006), the spectral convergence of eigenvalues in the gaps of the Floquet–Bloch spectrum due to defects in double porosity-type media were studied, and Cherdantsev (2009) supplemented this by the analysis of eigenfunction convergence based on an analysis of a two-scale uniform exponential decay; Babych et al. (2008) studied the case of high contrasts not only in (a scalar analog of) stiffness but also in the density. Recent work (Avila et al., 2004, 2008) specifically studied wave propagation is strongly heterogeneous linearly elastic media and, motivated by the analogy with Bouchitté and Felbacq (2004), argued that the effective density may be frequency dependent, anisotropic as well as non-positive definite, giving rise to "strong band gaps" (with no propagating waves for certain frequencies) as well as to "weak band gaps" with a reduced number of propagating modes in any direction. These effects do not however appear to display a "directional dependence" of the propagation and non-propagation: the number of propagating modes in the limit problem, although it may depend on the frequency, does not actually depend on the *direction*, see Section 4.2 below.

The underlying idea of an "unusual" effective density has been around for a while. It was suggested in Berryman (1980) that the effective density may be not a simple average and argued in Willis (1985) that in the ensemble-averaged sense it is generally non-local. A recent explosion of renewed interest in this topic was prompted by the ideas of the breakdown of conventional effective density laws near micro-resonances (e.g. Liu et al., 2005) with similar ideas advanced in the context of electromagnetism (e.g. Litchinitser et al., 2004; Kohn and Shipman, 2008) and applications to cloaking (e.g. Milton et al., 2006; Milton and Willis, 2007).

One of the aims of the present work is to show that an introduction of a "partially" high contrast, i.e., in particular, of some kind of "high anisotropy", does allow to vary the number of propagating modes with the direction, i.e. to achieve a "directional localization", if the "inclusion" phase is inter-connected (i.e. not isolated). In doing so, we build on some related ideas in Cherednichenko et al. (2006), where it was shown that in a scalar problem with highly anisotropic fibers the macroscopic component of the coupled two-scale limit system may be *spatially non-local* in the direction of the fibers. Guided by the analogy with the relation of a temporal non-locality to the non-linear dependence on frequency, one could expect in this setting a novel dependence on the wave vector, in particular of the direction of propagation which essentially explains the effect of the directional localization. We derive, in a generic setting, two-scale effective elastodynamic equations with the macro and micro components coupled in a particular way. Specifically, the partially high contrast leads to a constrained microscopic kinematics described by appropriate projectors in the limit equations. This appears a source of somewhat unusual effects physically, and mathematically allows in a sense treating from a unified perspective the "classical" homogenization, the high-contrast homogenization, and intermediate cases.

The two-scale limit equations are then analyzed for their band-gap structure. The high contrast implies the same frequency producing highly contrasting wavelengths in the materials of the matrix and the inclusions. Namely, for the adopted scaling, while it is still a long-wavelength (low frequency) regime in the matrix, it is a "resonance" scaling in the inclusions with the wavelength comparable to their size. The limit equations are then uncoupled. Their macroscopic component explicitly describes plane waves (47) with highly non-linear dispersion relation (63). This is such as if not only the effective density but possibly also the effective elastic stiffness (and the "coupling term", cf. Milton et al., 2006; Milton, 2007; Milton and Willis, 2007) were both frequency and wave vector dependent, anisotropic and possibly losing the positive definiteness. (A detailed study of the effective constitutive relations is postponed for future.) This allows achieving in particular, in the case of partially high-contrast and inter-connected inclusion phase, propagation in some directions and no propagation in the others, the effect potentially relevant to the cloaking. One can expect that, in a sense, this effect is due to some kind of directional sensitivity of the micro-resonances, being kinematically constrained by the "stiff" components. (Note in passing that the approach also gives a recipe for finding attenuating solutions, which correspond to complex solutions k of (63), in particular in the gaps, providing thereby an explicit asymptotic prediction for the attenuation rate which is essentially rigorous in contrast to various approximate models (e.g. Sabina et al., 1993).) If either the inclusions are isolated or the high contrast is not "partial" the dispersion relation simplifies to (49) which, although may still display the band-gap effect, does not display the direction localization any more.

Mathematically, the introduction of the "partially high contrast" poses, in the context of a two-scale asymptotic analysis, an interesting challenge of accounting for the microscopically constrained kinematics coupled to the macroscopic fields. This is done by appropriately modifying the two-scale asymptotic expansions approach and results in the two-scale effective equations explicitly involving "projectors" accounting for the kinematic constraints. A more systematic study of related general mathematical constructions and of their rigorous analysis, of possible microgeometries, as well as of the associated physical effects is beyond the scope of this paper, and will be addressed elsewhere. We give however two explicit illustrative examples demonstrating in particular the above effect of "directional localization" (Section 5.2).

2. Elastodynamics of high-contrast periodic media

2.1. Elastodynamic boundary-value problems

We consider the elastodynamic equations of motion in a heterogeneous medium, which can generally be expressed in the form

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} = \dot{\boldsymbol{p}}. \tag{1}$$

Here $\boldsymbol{\sigma}$ denotes stress, $\boldsymbol{p}$ is momentum density and $\boldsymbol{f}$ is body force, the dot denotes the differentiation with respect to time t. This is supplemented by constitutive equations relating stress to strain $\boldsymbol{e}$ and momentum density to velocity $\dot{\boldsymbol{u}}$

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{e}, \quad \boldsymbol{p} = \rho\dot{\boldsymbol{u}}. \tag{2}$$

The infinitesimal strain $\boldsymbol{e} = (e_{ij})_{i,j=1}^{d}$ is related to displacement $\boldsymbol{u} = (u_i)_{i=1}^{d}$ in d-dimensional space (physically, $d = 2$ or $d = 3$) through

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) =: \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,\ j = 1,\ldots,d$$

(henceforth, a comma in a subscript denotes spatial differentiation, and summation is applied to repeated indices).

Both the elasticity tensor $\boldsymbol{C}$ and the mass density ρ may depend on the position $\boldsymbol{x}$. The elasticity tensor $\boldsymbol{C} = (C_{ijpq}),\ i,\ j,\ p,\ q = 1,\ldots,d,$ is assumed to have the usual symmetries, and both $\boldsymbol{C}$ and ρ are strictly positive definite:

$$C_{ijpq} = C_{jipq} = C_{pqij}, \quad \alpha_{ij}C_{ijpq}(\boldsymbol{x})\alpha_{pq} \geqslant 0, \quad \rho(\boldsymbol{x}) > 0, \tag{3}$$

for all $\boldsymbol{x}$, any indices i, j, p, q and any symmetric tensor $\boldsymbol{\alpha}$ with the related equality held if and only if $\boldsymbol{\alpha} = \boldsymbol{0}$. The latter condition of positive definiteness will be denoted in the sequel as $\boldsymbol{C} > \boldsymbol{0}$.

Eqs. (1) and (2) have to be supplemented by appropriate initial and boundary conditions. One typical formulation of the associated initial boundary-value problem is to assume that for negative t the body force is identically zero and so is the solution,

$$\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0} \quad \text{for } t \leqslant 0. \tag{4}$$

This problem has an equivalent variational formulation in the form of Hamilton's principle of stationary action. In the case of $\Omega = \mathbb{R}^d$ for example, i.e. in the full space, it could be written as

$$\int_0^\infty \int_\Omega [\boldsymbol{\sigma} \cdot \boldsymbol{e}(\boldsymbol{\phi}) - \boldsymbol{p} \cdot \dot{\boldsymbol{\phi}}]\, d\boldsymbol{x}\, dt = \int_0^\infty \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt, \tag{5}$$

for any smooth "test function" $\boldsymbol{\phi}(\boldsymbol{x}, t)$ with "compact support", i.e. identically zero for large enough $\boldsymbol{x}$ or t. In (5) $e_{ij}(\boldsymbol{\phi}) := (\phi_{i,j} + \phi_{j,i})/2$ and the equivalence of (1) and (5) is verified by a straightforward integration by parts and the density property of the test functions $\boldsymbol{\phi}$.

Other classes of solutions of (1) which are of particular interest are those with zero body force ($\boldsymbol{f} \equiv \boldsymbol{0}$) but harmonic time dependence, i.e.

$$\boldsymbol{u}(\boldsymbol{x}, t) = \exp(-i\omega t)\boldsymbol{U}(\boldsymbol{x}). \tag{6}$$

For bounded domains Ω, with appropriate boundary conditions on their boundary S, this poses a problem of vibration, which is a spectral problem on the "eigenfrequency" ω and "eigenmode" $\boldsymbol{U} \not\equiv \boldsymbol{0}$. For the whole space this corresponds to the problem of propagation or localization of waves of frequency ω. In particular, for infinite periodic media this relates to propagation of Floquet–Bloch waves which typically exist for most frequencies but may fail to exist for some other frequency ranges. The latter corresponds to the band-gap effect (the "phononic" band-gap effect in the context of elasticity).

We restrict attention to the case of periodic media having a very small spatial period ε, i.e.

$$\boldsymbol{C}(\boldsymbol{x}) = \boldsymbol{C}^\varepsilon(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x}/\varepsilon), \quad \rho(\boldsymbol{x}) = \rho_\varepsilon(\boldsymbol{x}) = \rho(\boldsymbol{x}/\varepsilon), \tag{7}$$

where $\boldsymbol{C}(\boldsymbol{y})$ and $\rho(\boldsymbol{y})$ are periodic in each variable y_i with period 1. Some analysis of the present work could, in principle, be generalized further, for example to include the so-called "locally periodic" media, i.e. with $\boldsymbol{C}^\varepsilon(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ and $\rho_\varepsilon(\boldsymbol{x}) = \rho(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$, where $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y})$ and $\rho(\boldsymbol{x}, \boldsymbol{y})$ are $\boldsymbol{y}$-periodic, as well as to include some random media. We remark that the periodic case (7) can itself be viewed as a realization of a simple random ensemble of "translated" periodic media (cf. e.g. Jikov et al., 1994):

$$\boldsymbol{C}^\varepsilon(\boldsymbol{x}, \boldsymbol{\omega}) = \boldsymbol{C}(\boldsymbol{x}/\varepsilon + \boldsymbol{\omega}), \quad \rho_\varepsilon(\boldsymbol{x}, \boldsymbol{\omega}) = \rho(\boldsymbol{x}/\varepsilon + \boldsymbol{\omega}). \tag{8}$$

Here $\boldsymbol{\omega}$ is an element of the reference periodicity cell $Q = [0, 1)^d$ describing the "phase shift" of the periodic medium. (Q is then identified with the probability space, and the usual Lebesgue measure on Q with the probability measure.)

The body force $\boldsymbol{f}(\boldsymbol{x}, t)$ is often regarded as ε-independent, although "configuration-dependent" body forces could also be considered (cf. Luciano et al., 2000).

As is well-known, for small but finite ε, the ensemble-averaged constitutive relations are generally non-local, and may display unusual coupling between the stress–strain and momentum density–velocity constitutive relations (see e.g. Willis, 1981a,b, 1997; Milton, 2007; Milton and Willis, 2007). From this perspective, the essence of the "classical homogenization" is that, for moderate contrasts in $\boldsymbol{C}$ and ρ, those "localize" in the limit of $\varepsilon \to 0$, and are described in terms of the homogenized elasticity tensor $\boldsymbol{C}^{\text{hom}}$ and the mean density $\langle\rho\rangle$, with the effective constitutive relations having the same local and uncoupled structure (2). As one possible interpretation of the present work, those in a sense remain non-local but become asymptotically more explicit at a *high-contrast homogenized limit*, where the parameters of the high contrasts in $\boldsymbol{C}$ are appropriately scaled against ε (cf. Cherednichenko, 2001, Section 4) for a prototype scalar case.

2.2. High-contrast scalings

We consider a general "two-phase" (in particular, matrix-inclusion) periodic medium occupying a domain Ω of wave propagation (in particular, $\Omega = \mathbb{R}^d,\ d = 3$, for the full physical space). The elastic matrix is assumed connected and characterized by the elasticity tensor $\boldsymbol{C}^1$ and mass density ρ_1. It is filled with an ε-periodic array of perfectly bonded "inclusions" with associated (possibly microscopically varying) elastic tensor $\boldsymbol{C}^0$ and density ρ_0, see Fig. 1. The "inclusions" are possibly but not necessarily isolated, so may be inter-connected which also includes the case of fibers (Fig. 2), but are also allowed to be "multicomponent", to include, e.g. the cases of several inclusions and/or several arrays of (differently oriented) fibers or other inter-connected components, of varying shapes and elastic properties. The reference periodicity cell Q is therefore generally divided into two parts: a reference inclusion Q_0 having a smooth boundary Γ inside Q, and its complement Q_1 corresponding to the matrix phase. The inclusion phase Ω_0^ε is therefore the intersection of Ω with the ε-contracted and ε-periodically replicated Q_0, and the matrix phase Ω_1^ε is the complement, Fig. 1. We denote Γ^ε the collection of all the inclusion–matrix interfaces and S the external boundary of Ω (if any). The limit value on Γ^ε of a function f in Ω_j^ε is denoted by $f|_j,\ j = 0, 1$. Let $\boldsymbol{n}^y$ be the outer unit normal to Q_0 on its boundary Γ and let $\boldsymbol{n}$ denote the similar normal on Γ^ε.

We assume some components of the elastic stiffness $\boldsymbol{C}$ to have *a high contrast*, namely to be substantially smaller in the inclusions than in the matrix. The density ρ is allowed to be moderately contrasting. Namely, introducing the small parameter δ of the contrast, most generally,

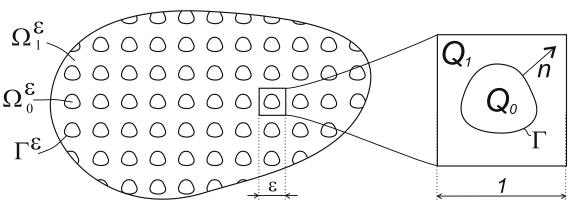

Fig. 1. The periodic geometry and a periodicity cell Q with isolated inclusions.

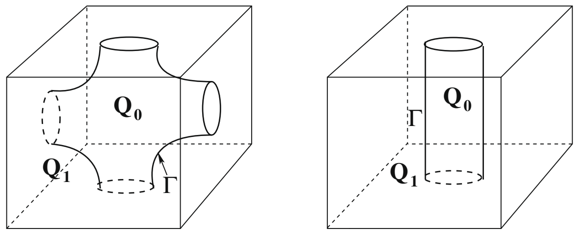

Fig. 2. Examples of inter-connected "inclusions", including fibres (right).

$$\boldsymbol{C}^\varepsilon(\boldsymbol{x}) = \begin{cases} \boldsymbol{C}^1(\boldsymbol{x}/\varepsilon), & x \in \Omega_1^\varepsilon \\ \delta\ \boldsymbol{C}^0(\boldsymbol{x}/\varepsilon) + \boldsymbol{C}^2(\boldsymbol{x}/\varepsilon), & x \in \Omega_0^\varepsilon \end{cases} \quad \text{and}$$

$$\rho_\varepsilon(x) = \begin{cases} \rho_1(\boldsymbol{x}/\varepsilon), & x \in \Omega_1^\varepsilon \\ \rho_0(\boldsymbol{x}/\varepsilon), & x \in \Omega_0^\varepsilon \end{cases}.$$

Here $\boldsymbol{C}^0(\boldsymbol{y}),\ \boldsymbol{C}^1(\boldsymbol{y}),\ \boldsymbol{C}^2(\boldsymbol{y})$ and $\rho_0(\boldsymbol{y}),\ \rho_1(\boldsymbol{y})$, are arbitrary non-negative (in particular possibly constant) elastic tensors and densities, respectively, all satisfying the symmetry conditions in (3), and $\rho_0,\ \rho_1,\ \boldsymbol{C}^0$ and $\boldsymbol{C}^1$ satisfying the uniform strict positivity conditions:

$$\alpha_{ij} C^r_{ijpq}(\boldsymbol{y})\alpha_{pq} \geqslant \nu\alpha_{ij}\alpha_{ij}, \quad \rho_r(\boldsymbol{y}) > \nu, \quad r = 0, 1, \tag{9}$$

for some $\nu > 0$ and any symmetric $\boldsymbol{\alpha}$.

The elasticity tensor $\boldsymbol{C}^2(\boldsymbol{y})$ does not have to be strictly positive definite. More precisely, $\boldsymbol{C}^2$ will typically vanish at a linear subspace $T(\boldsymbol{y})$ of the six-dimensional space of strains, being strictly positive at its orthogonal complement $(T(\boldsymbol{y}))^\perp$. [In the present work $\boldsymbol{C}^2(\boldsymbol{y})$ is also required to satisfy some additional technical restrictions, typically valid for particular examples, see next Section.]

Regarding ε and δ small, the asymptotic behavior will crucially depend on how $\delta = \delta(\varepsilon)$ is "scaled" against ε (cf. e.g. Poulton et al., 2001 and further references therein). It could be seen from dimensional analysis (and indeed retrospectively follows from the asymptotic analysis below) that there is only one "critical" scaling for δ, namely $\delta \sim \varepsilon^2$, which we call a double porosity-type scaling, where the phenomena at the macro and micro scales are coupled in a non-trivial way[1] (cf. e.g. Arbogast et al., 1990; Sandrakov, 1999; Zhikov, 2000, 2004; Bouchitté and Felbacq, 2004; Avila et al., 2008). We therefore set henceforth $\delta = \varepsilon^2$. We emphasize that a novelty of the present formulation, and indeed a source of key effects reported in this paper, is in allowing only a "part" of the elasticity tensor in the inclusion, corresponding in a sense to $\delta\boldsymbol{C}^0$, to asymptotically degenerate as δ becomes small, with the part corresponding to $\boldsymbol{C}^2$ remaining stiff. This allows for the materials in the inclusions to be in particular highly anisotropic, with possibly significant microscopic deformation but with the kinematics constrained by $\boldsymbol{C}^2$ (cf. Cherednichenko et al., 2006). Remark in passing that although $\boldsymbol{C}^2$

[1] This corresponds to the case when, in contrast to the "classical" homogenization, the two-scale asymptotic expansions display the dependence on the fast variable in the main-order term, although only in the "soft" inclusions, see (14) and (15) below.

can be taken constant within the inclusions (in particular in fibers, cf. Section 5), allowing $\boldsymbol{C}^2$ to vary within the inclusion results in additional kinematic degrees of freedom.

Hence, adopting the above scaling, we assume

$$\boldsymbol{C}^\varepsilon(\boldsymbol{x}) = \begin{cases} \boldsymbol{C}^1(\boldsymbol{x}/\varepsilon), & x \in \Omega_1^\varepsilon \\ \varepsilon^2\boldsymbol{C}^0(\boldsymbol{x}/\varepsilon) + \boldsymbol{C}^2(\boldsymbol{x}/\varepsilon), & x \in \Omega_0^\varepsilon \end{cases} \quad \text{and}$$
$$\rho_\varepsilon(x) = \begin{cases} \rho_1(\boldsymbol{x}/\varepsilon), & x \in \Omega_1^\varepsilon \\ \rho_0(\boldsymbol{x}/\varepsilon), & x \in \Omega_0^\varepsilon \end{cases}. \tag{10}$$

We therefore deal with asymptotic sequences of solutions of the elastodynamic boundary-value problems parametrized by ε. We will denote the related entities by $\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)$, $\boldsymbol{\sigma}^\varepsilon(\boldsymbol{x},t)$, etc. The boundary-value problems have to incorporate the conditions of continuity of the displacements and of the tractions across the interfaces Γ^ε:

$$\boldsymbol{u}^\varepsilon|_1 = \boldsymbol{u}^\varepsilon|_0, \quad \sigma_{ij}^\varepsilon n_j|_1 = \sigma_{ij}^\varepsilon n_j|_0. \tag{11}$$

Consider the initial-value problem in the unbounded domain corresponding to e.g. (4). Its variational formulation (5) specializes via (10) to the following. Find piecewise smooth, continuous $\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)$, identically zero for $t \leqslant 0$, such that

$$\int_0^\infty \int_\Omega [\boldsymbol{C}^\varepsilon(\boldsymbol{x})\boldsymbol{e}^\varepsilon \cdot \boldsymbol{e}(\boldsymbol{\phi}) - \rho^\varepsilon(\boldsymbol{x})\dot{\boldsymbol{u}}^\varepsilon \cdot \dot{\boldsymbol{\phi}}]d\boldsymbol{x}dt = \int_0^\infty \int_\Omega \boldsymbol{f}^{\,\varepsilon} \cdot \boldsymbol{\phi}\, d\boldsymbol{x}dt, \tag{12}$$

for all smooth compactly supported $\boldsymbol{\phi}(\boldsymbol{x},t)$. Here $\boldsymbol{f}^{\,\varepsilon}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}/\varepsilon,t)$, with a given $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},t)$ ($\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},t) \equiv \boldsymbol{0}$ for $t<0$) periodic in $\boldsymbol{y}$, to include the possibility of a configuration dependent body force. This of course includes the case of an ε-independent $\boldsymbol{f}$ provided it is constant in $\boldsymbol{y}$ for any $\boldsymbol{x}$ and t. Notice that the interface conditions (11) are automatically incorporated into the "weak" (variational) formulation (12).

The equivalent "classical" formulation is to find $\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)$, identically zero for $t \leqslant 0$, such that

$$\rho_\varepsilon(\boldsymbol{x})\ddot{\boldsymbol{u}}^\varepsilon - \mathrm{div}(\boldsymbol{C}^\varepsilon(\boldsymbol{x})\boldsymbol{e}^\varepsilon) = \boldsymbol{f}^{\,\varepsilon}(\boldsymbol{x},t), \tag{13}$$

in Ω_1^ε and Ω_0^ε, and the interfacial conditions (11) are satisfied at Γ^ε.

We study the asymptotic behavior of the solution $\boldsymbol{u}^\varepsilon$ of the above problem as $\varepsilon \to 0$. Notice that the above includes as limit cases both the "classical" homogenization ($\boldsymbol{C}^0 = \boldsymbol{0}$) and a double porosity-type high-contrast homogenization ($\boldsymbol{C}^2 = \boldsymbol{0}$). The following is a variant of a high-contrast version (cf. e.g. Kamotski and Smyshlyaev, 2006) of the method of two-scale asymptotic expansions (see e.g. Bensoussan et al., 1978; Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1989). Remark that an alternative/supplementary approach is to employ the tools of two-scale convergence[2] (e.g. Nguetseng, 1989; Allaire, 1992; Zhikov, 2000), which in the present context would require employing "two-scale" test functions in the weak formulation (12) and then passing to the limit.

[2] A further alternative is to use a conceptually somewhat similar to the two-scale convergence "periodic unfolding" method (see e.g. Cioranescu et al., 2002) applied in related context in Avila et al. (2008).

3. Homogenization and a class of two-scale limit problems

We describe in this section the formal asymptotic procedure for solving (13) as $\varepsilon \to 0$. One seeks a formal solution to the problem in the form of a standard two-scale asymptotic ansatz:

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x},t) = \boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{x}/\varepsilon,t) + \varepsilon\boldsymbol{u}^{(1)}(\boldsymbol{x},\boldsymbol{x}/\varepsilon,t) + \varepsilon^2\boldsymbol{u}^{(2)}(\boldsymbol{x},\boldsymbol{x}/\varepsilon,t) + \boldsymbol{r}^\varepsilon(\boldsymbol{x},t), \tag{14}$$

where $\boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{y},t)$, $\boldsymbol{u}^{(1)}(\boldsymbol{x},\boldsymbol{y},t)$ and $\boldsymbol{u}^{(2)}(\boldsymbol{x},\boldsymbol{y},t)$ are functions to be determined which are Q-periodic in $\boldsymbol{y}$, the remainder $\boldsymbol{r}^\varepsilon$ is expected to be small when $\varepsilon \to 0$, with $\boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{y},t)$ subsequently having the meaning of the solution of a "limit problem".

The substitution of (14) into (13) results upon straightforward calculation in the following structure of the main-order term $\boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{y},t)$, see Appendix A:

$$\boldsymbol{u}^{(0)}(\boldsymbol{x},\boldsymbol{y},t) = \begin{cases} \boldsymbol{u}^0(\boldsymbol{x},t), & \boldsymbol{x} \in \Omega,\ \boldsymbol{y} \in Q_1, \\ \boldsymbol{u}^0(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y},t), & \boldsymbol{x} \in \Omega,\ \boldsymbol{y} \in Q_0. \end{cases} \tag{15}$$

This highlights the fact that $\boldsymbol{u}^{(0)}$ varies only at the "slow" scale $\boldsymbol{x}$ (as is the case in the "classical", i.e. not high contrast, homogenization) everywhere in the matrix Ω_1^ε, i.e. outside the domain of "soft" inclusions Ω_0^ε where it may depend also on the fast variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ (in contrast to the classical homogenization). The pair of functions $(\boldsymbol{u}^0(\boldsymbol{x},t), v(\boldsymbol{x},\boldsymbol{y},t))$ solves a coupled "limit" initial-value problem derived in Appendix A and stated below. First, the function v, regarded for any $\boldsymbol{x}$ and t as a function of $\boldsymbol{y}$, $\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{w}(\boldsymbol{y})$, is required to belong to a linear subspace V of all possible solutions of the following homogeneous boundary-value problem

$$-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{w})) = \boldsymbol{0}, \quad \boldsymbol{y} \in Q_0, \tag{16}$$

$$\boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{w}) \cdot \boldsymbol{n}(\boldsymbol{y}) = \boldsymbol{0}, \quad \boldsymbol{y} \in \Gamma, \tag{17}$$

$$\boldsymbol{w}(\boldsymbol{y}) = \boldsymbol{0}, \quad \boldsymbol{y} \in \Gamma. \tag{18}$$

We hence define[3]

$$V := \{\boldsymbol{w}(\boldsymbol{y}),\ \ \boldsymbol{y} \in Q_0,\ \text{such that (16)–(18) holds}\}. \tag{19}$$

Depending on $\boldsymbol{C}^2$ and the geometry of Q_0 there may or may not be non-trivial solutions of (16)–(18). For example, if $\boldsymbol{C}^2$ is strictly positive definite obviously $\boldsymbol{w} \equiv \boldsymbol{0}$ and we are in the context of the classical homogenization, in which case V is trivial, $V = \boldsymbol{0}$. If $\boldsymbol{C}^2 \equiv \boldsymbol{0}$ we are in the context of a double porosity-type model, with V unconstrained apart from boundary condition (18). We are particularly interested in "intermediate" cases where on one hand $\boldsymbol{C}^2$ is not identically zero but on the other hand (16)–(18) still has non-trivial solutions.

One can further easily see (Appendix A) that (16)–(18) imply

$$C^2_{ijpq}\frac{\partial w_p}{\partial y_q} \equiv 0 \ \text{ for any } \ \boldsymbol{w} \in V. \tag{20}$$

[3] The rigorous definition would require specifying a functional space, namely V being a closed subspace of the Sobolev space $(H_0^1(Q_0))^d$ such that (16) is held in the whole Q in the sense of distributions, where v is regarded as extended by zero into Q_1.

To simplify some technical details, we additionally assume in this paper that $\boldsymbol{C}^2(\boldsymbol{y})$ is such that the following two conditions are satisfied. Generalizations of the present theory removing these restrictions (e.g. Cooper, 2008[4]), together with further generalizations, will be reported elsewhere.

First, any $\boldsymbol{w}$ satisfying (16)–(18) is also required to satisfy

$$C^2_{ijpq}\frac{\partial w_p}{\partial y_j} \equiv 0. \quad (21)$$

This assumption is trivially valid for both classical homogenization ($\boldsymbol{C}^2 > \boldsymbol{0}$), and for the double porosity-type models ($\boldsymbol{C}^2 = \boldsymbol{0}$). One further example is $\boldsymbol{C}^2$ being constant and having only one non-zero component, e.g. $C^2_{3333} = \alpha > 0$. Then V consists of all $\boldsymbol{w}$ with w_3 independent of y_3 and vanishing on the boundary of Q_0 and is obviously non-trivial. In particular, if Q_0 is a cylinder with its axis parallel to y_3 (cf. Section 5.2) and arbitrary cross-section, then any $\boldsymbol{w}$ with w_3 independent of y_3 and vanishing on the cylinder's boundary Γ obviously satisfies (16)–(20), as well as (21). Another example is constant $\boldsymbol{C}^2$ being strictly positive for any "in-plane", with respect to e.g. (y_1, y_2), deformation but degenerating otherwise, i.e. with $C^2_{ijpq} = 0$ if at least one of the indices i, j, p, q equals 3 (cf. Section 5.1). (The set V then consists of all $\boldsymbol{w}$ with $w_1 \equiv w_2 \equiv 0$ and arbitrary w_3.) The above examples are generalized for the multicomponent cases of several inclusions and/or fibers of varying shapes, directions and elastic stiffnesses, with $\boldsymbol{C}^2$ constant within any component.

Allowing for $\boldsymbol{C}^2(\boldsymbol{y})$ to vary point-wise, we introduce additional restriction:

$$\frac{\partial}{\partial y_i}C^2_{ijpq}(\boldsymbol{y}) \equiv 0 \quad (22)$$

(which is hence automatically satisfied for piecewise constant $\boldsymbol{C}^2$).[5]

The resulting coupled "limit" initial-value problem derived in Appendix A is the following. Find $\boldsymbol{u}^0(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}, t)$, such that $v \in V$ for any $\boldsymbol{x}$ and t; $\boldsymbol{u}^0 \equiv v \equiv \boldsymbol{0}$ for $t \leqslant 0$, and

$$\langle\rho\rangle\ddot{\boldsymbol{u}}^0 + \langle\rho_0\ddot{v}\rangle - \mathrm{div}(\boldsymbol{C}^{\mathrm{hom}}\boldsymbol{e}^0(\boldsymbol{x},t)) - \mathrm{div}(\langle\boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{x}}(v)\rangle) = \langle\boldsymbol{f}\rangle, \ \boldsymbol{x}\in\Omega; \quad (23)$$

$$\boldsymbol{P}\Big[\rho_0(\ddot{\boldsymbol{u}}^0 + \ddot{v}) - \mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^0\boldsymbol{e}^{\boldsymbol{y}}(v)) - \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^0(\boldsymbol{x},t)) - \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{x}}(v))\Big] = \boldsymbol{P}\boldsymbol{f}, \quad \boldsymbol{y} \in Q_0 \ (x \in \Omega). \quad (24)$$

Here

$$\langle g\rangle(\boldsymbol{x}, t) := |Q|^{-1}\int_Q g(\boldsymbol{x}, \boldsymbol{y}, t)d\boldsymbol{y} \quad (25)$$

denotes the averaging of relevant function g with respect to $\boldsymbol{y}$ over the periodicity cell Q (extending v by zero outside Q_0); $\rho(\boldsymbol{y}) := \rho_0\chi_0(\boldsymbol{y}) + \rho_1(1 - \chi_0(\boldsymbol{y}))$ with $\chi_0(\boldsymbol{y})$ denoting the characteristic function of Q_0; $e^{\boldsymbol{y}}_{ij}(v) := (\partial v_i/\partial y_j + \partial v_j/\partial y_i)/2$. In (23) $\boldsymbol{C}^{\mathrm{hom}} = (C^{\mathrm{hom}}_{ijpq})$ is the analog of the standard homogenized elasticity tensor for periodic medium $\boldsymbol{C}(\boldsymbol{y}) = \chi_0(\boldsymbol{y})\boldsymbol{C}^1 + (1 - \chi_0(\boldsymbol{y}))\boldsymbol{C}^2$ (cf. e.g. Jikov et al., 1994, Sections 3 and 12):

$$C^{\mathrm{hom}}_{ijpq}\xi_{ij}\xi_{pq} = \inf_{\boldsymbol{w}\in C^\infty_{\mathrm{per}}(Q)}\int_Q C_{ijpq}(\boldsymbol{y})(\xi_{ij} + e^{\boldsymbol{y}}_{ij}(\boldsymbol{w}))(\xi_{pq} + e^{\boldsymbol{y}}_{pq}(\boldsymbol{w}))d\boldsymbol{y} \quad (26)$$

for any symmetric $\xi = (\xi_{ij}) \in \mathbb{R}^{d\times d}$. It is well-known that, for connected matrix phases, $\boldsymbol{C}^{\mathrm{hom}}$ inherits all the symmetry properties (3) as well as the strict positive definiteness (this includes the porous case $\boldsymbol{C}^2 = \boldsymbol{0}$).

Finally, in (24) $\boldsymbol{P}$ can be viewed as the orthogonal projection on the (closure of the) space of all vector functions $\boldsymbol{w}(\boldsymbol{y})$ satisfying (16)–(18), i.e. on V, namely $\boldsymbol{Pg} = \boldsymbol{Pf}$ means, by definition[6]

$$\int_{Q_0} g_i(\boldsymbol{y})w_i(\boldsymbol{y})d\boldsymbol{y} = \int_{Q_0} f_i(\boldsymbol{y})w_i(\boldsymbol{y})d\boldsymbol{y}, \quad \text{for all } \boldsymbol{w} \in V, \quad (27)$$

cf. (A.18). We henceforth adopt the notation involving $\boldsymbol{P}$ as a shorthand for the associated integral identity (27).

Therefore $\boldsymbol{P}$ can be interpreted as describing the constrained microscopic kinematics due to the "stiff" part $\boldsymbol{C}^2$. Notice that in particular $\boldsymbol{P} = \boldsymbol{I}$ (the identity operator) for $\boldsymbol{C}^2 = \boldsymbol{0}$ (the double porosity-type model), and $\boldsymbol{P} = \boldsymbol{0}$ for $\boldsymbol{C}^2 > \boldsymbol{0}$ (classical homogenization). For an "intermediate" case of e.g. the only non-zero component of $\boldsymbol{C}^2$ being C^2_{3333} and Q_0 being a cylinder parallel to y_3, $\boldsymbol{P}$ is the averaging in y_3, see the example in Section 5.2.

Notice further that the above "limit problem" (23)–(24) can generally be interpreted as a two-scale variational problem which may itself be viewed as a limit of (12), as follows:

$$\int_0^\infty\int_\Omega \boldsymbol{C}^{\mathrm{hom}}\boldsymbol{e}(\boldsymbol{u}^0)\cdot\boldsymbol{e}(\boldsymbol{w}^0)d\boldsymbol{x}\,dt + \int_0^\infty\int_\Omega\int_{Q_0}[\boldsymbol{C}^0\boldsymbol{e}^{\boldsymbol{y}}(v)\cdot\boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{z}) + \boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{x}}(\boldsymbol{u}^0 + v)\cdot\boldsymbol{e}^{\boldsymbol{x}}(\boldsymbol{w}^0 + \boldsymbol{z}) - \boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{x}}(\boldsymbol{u}^0)\cdot\boldsymbol{e}^{\boldsymbol{x}}(\boldsymbol{w}^0)]d\boldsymbol{y}\,d\boldsymbol{x}\,dt - \int_0^\infty\int_\Omega\int_Q \rho(\boldsymbol{y})(\dot{\boldsymbol{u}}^0 + \dot{v})\cdot(\dot{\boldsymbol{w}}^0 + \dot{\boldsymbol{z}})d\boldsymbol{y}\,d\boldsymbol{x}\,dt = \int_0^\infty\int_\Omega\int_Q \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t)\cdot(\boldsymbol{w}^0 + \boldsymbol{z})d\boldsymbol{y}\,d\boldsymbol{x}\,dt, \quad (28)$$

for any $\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}, t)$ such that

[4] For the case of isolated inclusions the restrictions (21) and (22) are in fact not required, with all the results stated in the paper held (Cooper, 2008). However the results for non-isolated (inter-connected) inclusions, as stated, do require these restrictions.

[5] Many more examples of non-trivial V for which assumptions (21) and (22) are satisfied can be constructed by allowing $\boldsymbol{C}^2(\boldsymbol{y})$ to vary with $\boldsymbol{y}$. For example, consider any smooth $\boldsymbol{w} \in (H^1_0(Q_0))^d$ such that $\boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{w})$ is not (strictly) sign-definite on a set W of positive measure in Q_0. Then there exists $\boldsymbol{\eta}(\boldsymbol{y})$ a non-zero vector for any $\boldsymbol{y} \in W$ such that $\boldsymbol{\eta}(\boldsymbol{y})\cdot\boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{w})\boldsymbol{\eta}(\boldsymbol{y}) = 0$, smooth in $\boldsymbol{y}$. One can therefore see that $\boldsymbol{w} \in V$ for $C^2_{ijpq}(\boldsymbol{y}) = \xi_i(\boldsymbol{y})\xi_j(\boldsymbol{y})\xi_p(\boldsymbol{y})\xi_q(\boldsymbol{y})$ where $\xi(\boldsymbol{y}) = v(\boldsymbol{y})\boldsymbol{\eta}(\boldsymbol{y})$, $v(\boldsymbol{y})$ is an arbitrary scalar function, and we set $\boldsymbol{C}^2(\boldsymbol{y}) = \boldsymbol{0}$ for all $\boldsymbol{y}$ with a strictly sign-definite $\boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{w})$ (if any). The assumption (21) is then clearly held, and (22) is satisfied as long as $\mathrm{div}\xi = 0$. The latter can be always achieved by an appropriate choice of $v(\boldsymbol{y})$ which solves an ordinary differential equation along the vector field $\boldsymbol{\eta}(\boldsymbol{y})$. On the other hand, $C^2_{ijpq} = \delta_{ij}\delta_{pq}$, corresponding physically to the case of a degenerate shear modulus in the inclusion, is an example when the assumption (21) is *not* satisfied: the (non-trivial) set V consists of all the divergence-free vector fields in Q_0 vanishing on the boundary, while (21) would imply identically zero gradients with an obviously trivial V.

[6] More rigorously, considering V as a closed subspace of the Sobolev space $H^1_0(Q_0)$, and $\boldsymbol{g}$ and $\boldsymbol{f}$ as elements of the dual space $H^{-1}(Q_0)$ of linear continuous functionals on $H^1_0(Q_0)$, $(\boldsymbol{g}, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w})$, $\forall\, \boldsymbol{w} \in V$. We postpone rigorous analysis for future publications.

$$\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y},t)=\begin{cases}\boldsymbol{w}^0(\boldsymbol{x},t), & \boldsymbol{x}\in\Omega,\ \boldsymbol{y}\in Q_1,\\ \boldsymbol{w}^0(\boldsymbol{x},t)+\boldsymbol{z}(\boldsymbol{x},\boldsymbol{y},t), & \boldsymbol{x}\in\Omega,\ \boldsymbol{y}\in Q_0.\end{cases} \quad (29)$$

Here the microscopic parts of both the sought solution and the trial fields are required to be constrained to the space V, i.e.

$$v\in V,\quad \boldsymbol{z}\in V. \quad (30)$$

Functions $\boldsymbol{w}^0$ and $\boldsymbol{z}$ are assumed smooth in their arguments, compactly supported in $\boldsymbol{x}$ and t, and $\boldsymbol{z}$ periodic in $\boldsymbol{y}$. Any function defined on a smaller domain is regarded extended by zero to a larger domain whenever appropriate. The equivalence of (28)–(30) and (23)–(24) is directly verified: setting in (28) $\boldsymbol{z}\equiv\boldsymbol{0}$ and keeping $\boldsymbol{w}^0$ arbitrary yields (23), while choosing $\boldsymbol{w}^0\equiv\boldsymbol{0}$ and arbitrary $\boldsymbol{z}$ leads to (A.18) and hence (24).

Notice finally that the limiting equations (23)–(24) simplify further if all the inclusions are "isolated" (i.e. have bounded components), with the terms containing $\boldsymbol{C}^2\boldsymbol{e}^x(v)$ in both (23) and (24) and the term $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^0(\boldsymbol{x},t))$ in (24) vanishing identically (cf. Cooper, 2008) see (A.20) and (A.21). The corresponding terms vanish also in the variational formulation. (In particular, the vanishing in (24) of the term $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^0(\boldsymbol{x},t))$ corresponds to vanishing of the integral containing $\boldsymbol{C}^2\boldsymbol{e}^x(\boldsymbol{u}^0)\cdot\boldsymbol{e}^x(\boldsymbol{z})$ in (28), with the latter seen, e.g. using (A.21).)

4. Band-gap structure of the limit problem

The limit system (23)–(24) may be of interest from various points of view. For example, as was shown in Sandrakov (1999), in a general geometrical setting but in the particular case of $\boldsymbol{C}^2=\boldsymbol{0}$ the elimination of v from (23)–(24) results in a *time-non-local* equation for $\boldsymbol{u}^0$ i.e. one displaying a "memory effect". On the other hand, for $\boldsymbol{C}^2\neq\boldsymbol{0}$ the equation for $\boldsymbol{u}^0$ may be *spatially non-local* as was shown for a prototype scalar example in Cherednichenko et al. (2006). A related issue is the band-gap structure of the spectrum of the two-scale limit operator as we discuss below (cf. Zhikov, 2000, 2004; Avila et al., 2008) for the case of $\boldsymbol{C}^2=\boldsymbol{0}$.

We formally set in (23) and (24) $\boldsymbol{f}\equiv\boldsymbol{0}$ and seek time-harmonic solutions $\boldsymbol{u}^0(\boldsymbol{x},t)=\exp(-i\omega t)\boldsymbol{u}^0(\boldsymbol{x})$, $v(\boldsymbol{x},\boldsymbol{y},t)=\exp(-i\omega t)v(\boldsymbol{x},\boldsymbol{y})$ (ω is the frequency). Then we arrive at the following limit spectral problem[7] for $(\boldsymbol{u}^0,v)$ with spectral parameter $\Lambda:=\omega^2$:

$$-\mathrm{div}(\boldsymbol{C}^{\mathrm{hom}}\boldsymbol{e}^0(\boldsymbol{x}))-\mathrm{div}(\langle\boldsymbol{C}^2\boldsymbol{e}^x(v)\rangle)=\Lambda(\langle\rho\rangle\boldsymbol{u}^0+\langle\rho_0 v\rangle),\quad \boldsymbol{x}\in\Omega; \quad (31)$$

$$\boldsymbol{P}[-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^0\boldsymbol{e}^y(v))-\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^{\boldsymbol{0}}(\boldsymbol{x}))-\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^x(v))]=\Lambda\rho_0\boldsymbol{P}(\boldsymbol{u}^0+v),\quad \boldsymbol{y}\in Q_0\ (\boldsymbol{x}\in\Omega), \quad (32)$$

$$v=\boldsymbol{0},\quad \boldsymbol{y}\in\Gamma\ (x\in\Omega), \quad (33)$$

with the additional constraint

$$v\in V. \quad (34)$$

We will show that the above system decouples if either (or both) of the following holds: (i) the inclusions Q_0 are isolated; (ii) the domain is the whole space ($\Omega=\mathbb{R}^d$).

4.1. Case of isolated inclusions

In this case, the inclusions have bounded isolated periodic components (although could possibly have a multiple number of those). The terms in (31) and (32) containing $\boldsymbol{C}^2$ then identically vanish, cf. (A.20) and (A.21), and the above limit spectral problem therefore reads:

$$-\mathrm{div}(\boldsymbol{C}^{\mathrm{hom}}\boldsymbol{e}^0(\boldsymbol{x}))=\Lambda(\langle\rho\rangle\boldsymbol{u}^0+\langle\rho_0 v\rangle),\quad \boldsymbol{x}\in\Omega; \quad (35)$$

$$\boldsymbol{P}\left[-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^0\boldsymbol{e}^{\boldsymbol{y}}(v))\right]=\Lambda\rho_0\boldsymbol{P}(\boldsymbol{u}^0+v),\quad \boldsymbol{y}\in Q_0\ (\boldsymbol{x}\in\Omega), \quad (36)$$

$$v\in V. \quad (37)$$

To decouple it, seek the solution v of (36)–(37) in terms of $\boldsymbol{u}^0$, i.e. in components,

$$v_i(\boldsymbol{x},\boldsymbol{y})=\Lambda\eta_i^r(\boldsymbol{y})u_r^0(\boldsymbol{x}) \quad (38)$$

where $\boldsymbol{\eta}^r=(\eta_i^r(\boldsymbol{y}))\in V$ solves

$$-\boldsymbol{P}[(C^0_{ijpq}\eta^r_{p,q})_{,j}]-\Lambda\rho_0\eta_i^r=\boldsymbol{P}[\rho_0\delta_{ri}],\quad \boldsymbol{y}\in Q_0, \quad (39)$$

$$\eta_i^r=0,\quad \boldsymbol{y}\in\Gamma. \quad (40)$$

The substitution of (38) into (35) results in

$$-\mathrm{div}(\boldsymbol{C}^{\mathrm{hom}}\boldsymbol{e}^0(\boldsymbol{x}))=\boldsymbol{\beta}(\Lambda)\boldsymbol{u}^0,\quad \boldsymbol{x}\in\Omega. \quad (41)$$

Here $\boldsymbol{\beta}(\Lambda)=(\beta_{ij}(\Lambda))_{i,j=1}^d$ is the matrix with components

$$\beta_{ij}(\Lambda)=\Lambda\langle\rho\rangle\delta_{ij}+\Lambda^2\left\langle\rho_0\eta_i^j\right\rangle(\Lambda). \quad (42)$$

Equivalently,

$$\boldsymbol{\beta}(\Lambda)=\Lambda\langle\rho\rangle\boldsymbol{I}+\Lambda^2\boldsymbol{B}(\Lambda), \quad (43)$$

with $\boldsymbol{I}$ denoting the unit matrix, and

$$\boldsymbol{B}(\Lambda):=(B_{ij})=\left(\left\langle\rho_0\eta_i^j\right\rangle(\Lambda)\right). \quad (44)$$

Matrix $\boldsymbol{\beta}$ is well-defined for any Λ except for the spectrum of the related operator which is typically discrete[8] with the eigenvalues Λ_m of the, kinematically constrained by $\boldsymbol{C}^2$ via $\boldsymbol{P}$, inclusion vibration problem for which there exists an eigenfunction $\boldsymbol{\phi}^m\in V$ with a non-zero ρ_0-weighted mean $\langle\rho_0\boldsymbol{\phi}^m\rangle\neq\boldsymbol{0}$:

[7] We do not perform here rigorous analysis, mentioning however that the methods of Zhikov (2000, 2004) allow to rigorously justify all the analogous steps in a simpler scalar case with (analog of) $\boldsymbol{C}^2=\boldsymbol{0}$, in particular to rigorously prove the existence of formally derived band gaps, spectral convergence, etc.

[8] This construction appears to typically define a self-adjoint operator in Hilbert space V of the constrained microscopic kinematics. The compactness of the resolvent operator is then a sufficient condition for the discreteness of the spectrum, however there is no a priori reason why it should generally hold (cf. Babych and Kamotski, in preparation), where the compactness is not held in a somewhat similar although different context. We remark that the compactness *is* held at least for the particular examples in Section 5, and postpone a systematic study of these issues for future publications.

$$-\boldsymbol{P}[(C^0_{ijpq}\phi^m_{p,q})_{,j}] = \Lambda_m \rho_0 \phi^m_i, \quad \boldsymbol{y} \in Q_0$$
$$\phi^m_i = 0, \quad \boldsymbol{y} \in \Gamma. \tag{45}$$

Here

$$0 < \Lambda_1 \leqslant \Lambda_2 \leqslant \Lambda_3 \leqslant \ldots \leqslant \Lambda_m \leqslant \ldots$$

are the eigenvalues, and the eigenfunctions are assumed normalized

$$\int_{Q_0} \rho_0 \boldsymbol{\phi}^m \cdot \boldsymbol{\phi}^n d\boldsymbol{y} = \delta_{mn}.$$

(Remark in passing that there is no a priori reason for the kinematically constrained system even to be infinite-dimensional: hence, in principle, the spectrum could also be finite.)

The matrix $\boldsymbol{\beta}(\Lambda)$ is *symmetric* as follows via (43) from the symmetry of $\boldsymbol{B}(\Lambda)$. To establish the latter, multiply (39) by η^s_i, sum over i and integrate over Q_0, using the self-adjointness of the projector $\boldsymbol{P}$ and $\boldsymbol{P}\boldsymbol{\eta}^s = \boldsymbol{\eta}^s$. The right-hand side then gives $\langle \rho_0 \eta^s_r \rangle = B_{rs}$ while the left-hand side, upon the integration by parts and using the symmetry properties of $\boldsymbol{C}^0$ (see (3)), yields

$$\int_{Q_0} (-(C^0_{ijpq}\eta^s_{i,j})_{,q} - \Lambda \rho_0 \eta^s_p)\eta^r_p d\boldsymbol{y} = \int_{Q_0} \delta_{sp}\rho_0 \eta^r_p d\boldsymbol{y} = \langle \rho_0 \eta^r_s \rangle = B_{sr},$$

implying $B_{rs} = B_{sr}$ for arbitrary r and s.

The homogenized equation (41) is the key for characterizing the spectrum of the limit problem, in both bounded (cf. Zhikov, 2000) and unbounded (cf. Zhikov, 2004) domains. In particular, it allows to determine the asymptotic behavior of the Floquet–Bloch waves and of the band-gap structure for small ε. Notice that, formally, it looks like an equation of wave propagation in a uniform medium with elasticity tensor $\boldsymbol{C}^{\text{hom}}$ and a "matrix" density $\boldsymbol{\beta}(\Lambda)$, with the latter depending on the frequency and might be both anisotropic as well as negative (cf. Berryman, 1980; Willis, 1985; Milton and Willis, 2007; Avila et al., 2008).

4.2. Limit Floquet–Bloch band-gap structure: isolated inclusions case

Assume now $\Omega = \mathbb{R}^d$ and re-write (41) in the component form:

$$C^{\text{hom}}_{ijpq} u^0_{p,jq} + \beta_{ip} u^0_p = 0. \tag{46}$$

Seek a plane wave solution to (46)

$$u^0_p(\boldsymbol{x}) = A_p \exp(ik\boldsymbol{n} \cdot \boldsymbol{x}), \tag{47}$$

where $\boldsymbol{n}$ is a unit vector describing the propagation direction, $k \geqslant 0$ is the wave number and $\boldsymbol{A} = (A_p) \in \mathbb{R}^d$ ($\boldsymbol{A} \neq \boldsymbol{0}$) is the polarization. Substituting (47) into (46) results in

$$[k^2 C^{\text{hom}}_{ijpq} n_j n_q - \beta_{ip}]A_p = 0. \tag{48}$$

This can be viewed as a 'dispersion relation' for $k = k(\boldsymbol{n}, \Lambda)$, with (48) implying

$$\det[k^2 \tilde{\boldsymbol{C}}^h(\boldsymbol{n}) - \boldsymbol{\beta}(\Lambda)] = 0. \tag{49}$$

Here

$$[\tilde{\boldsymbol{C}}^h(\boldsymbol{n})]_{ip} := C^{\text{hom}}_{ijpq} n_j n_q \tag{50}$$

is the 'acoustic tensor' associated with the homogenized elasticity tensor $\boldsymbol{C}^{\text{hom}}$. $\tilde{\boldsymbol{C}}^h(\boldsymbol{n})$ is a symmetric strictly positive definite matrix for any direction $\boldsymbol{n}$ as follows from the strict positivity (9) of $\boldsymbol{C}^{\text{hom}}$. Hence (49) may be interpreted as the requirement that $t = k^2 \geqslant 0$ is a non-negative solution of the (no more than) cubic (assuming dimension $d = 3$) equation.

We therefore conclude that, for varying frequency $\omega = \Lambda^{1/2}$, depending on the number of non-negative solutions of (49) one may have a varying number of propagating waves in a direction $\boldsymbol{n}$. Specializing to the three-dimensional case ($d = 3$), one may have three, two, one or no propagating waves. In particular, if $\boldsymbol{\beta}(\Lambda)$ is strictly negative definite no such waves exist in any direction $\boldsymbol{n}$, which corresponds to a band-gap frequency.

Following the pattern of Zhikov (2000) for the scalar and unconstrained case (see also Bouchitté and Felbacq, 2004; Avila et al., 2008) $\boldsymbol{\beta}(\Lambda)$ can generally be expressed in terms of Λ_m and $\boldsymbol{\phi}^m$ via the spectral decomposition as follows. For each r, denoting by $\boldsymbol{\eta}^r = (\eta^r_i)$ the solution of (39) and (40), it is decomposed, via (45), as follows

$$\boldsymbol{\eta}^r = \sum_{m=1}^{\infty} \frac{\langle \boldsymbol{e}^r \cdot \rho_0 \boldsymbol{\phi}^m \rangle}{\Lambda_m - \Lambda} \boldsymbol{\phi}^m(\boldsymbol{y}),$$

(where $\boldsymbol{e}^r = (\delta_{ri})$), implying

$$\boldsymbol{B}(\Lambda) := \langle \rho_0 \boldsymbol{\eta} \rangle(\Lambda) = \sum_{m=1}^{\infty} \frac{\langle \rho_0 \boldsymbol{\phi}^m \rangle \otimes \langle \rho_0 \boldsymbol{\phi}^m \rangle}{\Lambda_m - \Lambda}. \tag{51}$$

Substituting further into (42) yields

$$\boldsymbol{\beta}(\Lambda) = \Lambda \langle \rho \rangle \boldsymbol{I} + \Lambda^2 \sum_{m=1}^{\infty} \frac{\langle \rho_0 \boldsymbol{\phi}^m \rangle \otimes \langle \rho_0 \boldsymbol{\phi}^m \rangle}{\Lambda_m - \Lambda}, \tag{52}$$

or, in components,

$$\beta_{ij}(\Lambda) = \Lambda \langle \rho \rangle \delta_{ij} + \Lambda^2 \sum_{m=1}^{\infty} \frac{\langle \rho_0 \phi^m_i \rangle \left\langle \rho_0 \phi^m_j \right\rangle}{\Lambda_m - \Lambda}. \tag{53}$$

We notice that since $\tilde{\boldsymbol{C}}^h(\boldsymbol{n})$ is strictly positive definite and symmetric, the solutions $t = k^2$ to the dispersion relation (49) are non-negative eigenvalues of (symmetric matrix)

$$\boldsymbol{T}(\boldsymbol{n}, \Lambda) := \left(\tilde{\boldsymbol{C}}^h(\boldsymbol{n})\right)^{-1/2} \beta(\Lambda) \left(\tilde{\boldsymbol{C}}^h(\boldsymbol{n})\right)^{-1/2}. \tag{54}$$

If $\beta(\Lambda)$ is negative definite, clearly there are no such eigenvalues, so no waves propagate for the given frequency in any direction $\boldsymbol{n}$. It can in fact be shown by a direct adaptation of the analysis of Zhikov (2000, 2004) (see also Avila et al., 2008) that the gaps in the limit spectrum (and hence also for small enough ε in the initial problem) will correspond to precisely all such Λ for which $\beta(\Lambda)$ is negative definite and which are not eigenvalues of the inclusion spectral problem (45):

$$G = \{\Lambda : \boldsymbol{\beta}(\Lambda) \text{ negative definite}; \ \Lambda \neq \Lambda_m, \ m = 1, 2, \ldots\}. \tag{55}$$

In contrast to the scalar case (Zhikov, 2004), it is not immediately obvious from the representation (52) if such gaps will generally exist. However, it follows from (52) that the gaps will definitely exist at least to the right of such multiple eigenvalues Λ_m which have three eigenfunctions with the linearly independent means $\langle \boldsymbol{\phi} \rangle$ (cf. Avila et al., 2008). This condition does hold for sufficiently symmetric microgeometries, e.g. for balls, $\boldsymbol{C}^2 \equiv \boldsymbol{0}$ and isotropic $\boldsymbol{C}^0$. [Remark in passing that the limiting gaps can be evaluated in this case analytically in a fashion similar to that in Section 5 below.]

We also notice that, in the case of the isolated inclusions, outside the gaps i.e. in the "bands", the number of propagation modes in the limit problem *does not actually depend on the direction of propagation* $\boldsymbol{n}$ and simply coincides with the number of non-negative eigenvalues of $\boldsymbol{\beta}(\Lambda)$. This follows from noticing that the former is determined by the number of non-negative eigenvalues of $\boldsymbol{T}(\boldsymbol{n}, \Lambda)$ which, due to (54), coincides with that of $\boldsymbol{\beta}(\Lambda)$ for any $\boldsymbol{n}$. In particular, for the weak band gaps there is a reduced number of modes propagating in *any* direction, so one could say that, in a sense, waves of some *polarizations* fail to propagate, with these polarizations dependent on the direction.

The further refinement of the model to include non-isolated (i.e. inter-connected) inclusions with $\boldsymbol{C}^2 \neq \boldsymbol{0}$, e.g. highly anisotropic fibrous composites (cf. Cherednichenko et al., 2006) *does allow to display a "directional localization", i.e. supporting propagating waves in some directions while having none in the others*. This model will be developed further in the next subsection and then illustrated by an example in Section 5.2.

4.3. Limit Floquet–Bloch structure for inter-connected "inclusions"

In the case of inter-connected (non-isolated) inclusions Q_0, for example fibers, the more general spectral problem (31)–(32) has to be analyzed, since the terms containing $\boldsymbol{C}^2 \boldsymbol{e}^{\boldsymbol{x}}(v)$ and $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2 \boldsymbol{e}^0(\boldsymbol{x}))$ do not vanish any more in general. (This is in fact precisely the presence of these terms which gives rise to the spatial non-locality, cf. Cherednichenko et al., 2006.) Restricting ourselves in this case to the whole space case ($\Omega = \mathbb{R}^d$), it is convenient to immediately specialize (31) and (32) to the plane wave solutions (47),[9] yielding

$$k^2 \boldsymbol{C}^{\mathrm{hom}}_{ijpq} n_j n_q A_p e^{ik\boldsymbol{n}\cdot\boldsymbol{x}} - \frac{\partial^2}{\partial x_j \partial x_q} \langle C^2_{ijpq} v_p \rangle = \Lambda \langle \rho \rangle A_i e^{ik\boldsymbol{n}\cdot\boldsymbol{x}} + \Lambda \langle \rho_0 v_i \rangle, \quad \boldsymbol{x} \in \mathbb{R}^d; \tag{56}$$

$$\boldsymbol{P}\left[-\frac{\partial}{\partial y_j}\left(C^0_{ijpq} \frac{\partial v_p}{\partial y_q} \right) - C^2_{ijpq} \frac{\partial^2 v_p}{\partial x_j x_q} - \Lambda \rho_0 v_i \right] = \boldsymbol{P}[\Lambda \rho_0 A_i - k^2 C^2_{ijpq} n_j n_q A_p] e^{ik\boldsymbol{n}\cdot\boldsymbol{x}}, \tag{57}$$

$$v \in V. \tag{58}$$

The solution of (57)–(58) can then be sought in the form

$$v_i(\boldsymbol{x}, \boldsymbol{y}, k, \boldsymbol{n}) = e^{ik\boldsymbol{n}\cdot\boldsymbol{x}} \zeta^r_i A_r, \tag{59}$$

where $\zeta^r = (\zeta^r_i(\boldsymbol{y}, k, \boldsymbol{n})) \in V$ solves

$$\boldsymbol{P}[-(C^0_{ijpq} \zeta^r_{p,q})_{,j} + k^2 C^2_{ijpq} n_j n_q \zeta^r_p - \Lambda \rho_0 \zeta^r_i] = \boldsymbol{P}[\Lambda \rho_0 \delta_{ri} - k^2 C^2_{ijrq} n_j n_q], \quad \boldsymbol{y} \in Q_0, \tag{60}$$

$$\zeta^r_i = 0, \quad \boldsymbol{y} \in \Gamma. \tag{61}$$

Notice that (60) differs from (39) essentially for the extra middle term on the left-hand side. This makes ζ, in contrast to $\boldsymbol{\eta}$, depending not only on the spectral parameter Λ but also on the propagation direction $\boldsymbol{n}$ and the wavenumber k.

The substitution of (59) into (56) results in

$$k^2 \tilde{C}^h_{ip}(\boldsymbol{n}) A_p + k^2 \left\langle \tilde{C}^2_{ip}(\boldsymbol{n}) \zeta^r_p \right\rangle A_r = \Lambda \langle \rho \rangle A_i + \Lambda \langle \rho_0 \zeta^r_i \rangle A_r.$$

Here $\tilde{\boldsymbol{C}}^h(\boldsymbol{n}) = (\tilde{C}^h_{ip}(\boldsymbol{n}))$ is defined by (50), and

$$[\tilde{\boldsymbol{C}}^2(\boldsymbol{n})]_{ip} := C^2_{ijpq} n_j n_q \tag{62}$$

is similar "acoustic tensor" corresponding to $\boldsymbol{C}^2$. This results in the following dispersion relation

$$\det[k^2(\tilde{\boldsymbol{C}}^h(\boldsymbol{n}) + \tilde{\gamma}(\boldsymbol{n}, \Lambda, k)) - \boldsymbol{\beta}(\Lambda, k, \boldsymbol{n})] = 0, \tag{63}$$

where

$$\boldsymbol{\beta}(\Lambda, k, \boldsymbol{n}) := \Lambda(\langle \rho \rangle \boldsymbol{I} + \langle \rho_0 \zeta \rangle), \tag{64}$$

$$\tilde{\gamma}(\Lambda, k, \boldsymbol{n}) := \langle \tilde{\boldsymbol{C}}^2(\boldsymbol{n}) \zeta \rangle, \tag{65}$$

where the matrix $\zeta = [\zeta]_{ij} := \zeta^j_i$. Notice that, although looking somewhat similar to the dispersion relation (49) in the case of isolated inclusions, (64) bears an essentially novel feature: the capability, in the simultaneous presence of high anisotropy (i.e. $\boldsymbol{C}^2 \neq \boldsymbol{0}$), to *alter the number of propagating modes with the direction* $\boldsymbol{n}$, via the dependence of $\tilde{\gamma}$ and $\boldsymbol{\beta}$ on $\boldsymbol{n}$. Section 5.2 gives an explicit example when the above dispersion relations lead to the directional localization effect in the case of highly anisotropic fibers, namely propagation in some directions combined with no propagation in others.

5. Examples: a directional localization

We give in this section two explicit examples of the limit Floquet–Bloch structure: one for isolated inclusions and one for fibers, with the latter demonstrating the effect of directional localization for highly anisotropic composites.

5.1. Isolated anisotropic inclusions

In the three-dimensional case ($d = 3$) let Q_0 be an isolated inclusion. Consider the case of a constant degenerate $\boldsymbol{C}^2$ which is strictly positive for all in-plane deformations for the (y_1, y_2)-plane and vanishes otherwise, i.e.

$$\alpha_{ij} C^2_{ijpq}(\boldsymbol{y}) \alpha_{pq} \geqslant \nu \alpha_{ij} \alpha_{ij}, \tag{66}$$

for some $\nu > 0$ and any symmetric $\boldsymbol{\alpha}$ such that $\alpha_{i3} = \alpha_{3i} = 0$, and

$$C^2_{ijpq} = 0 \quad \text{if } (3 - i)(3 - j)(3 - p)(3 - q) = 0 \tag{67}$$

(i.e. if at least one of the indices equals 3). Any a priori strictly positive elasticity tensor (for example isotropic), all of whose components with at least one of the indices

[9] What follows represents in effect application of the Fourier transform in $\boldsymbol{x}$ to (31)–(32), which is translationally invariant in $\boldsymbol{x}$.

equalling 3 are altered to vanish, will be within the described class.

The space V of the constrained microscopic kinematics then, according to e.g. (20), consists of all $\boldsymbol{w}(\boldsymbol{y})$ with $w_1 \equiv w_2 \equiv 0$[10] and with arbitrary w_3 vanishing on the inclusion's boundary Γ.

Then, according to (39) and (27), for $\boldsymbol{\eta}^r = (\eta_i^r(\boldsymbol{y})) \in V$ obviously $\eta_1^r \equiv \eta_2^r \equiv 0$ with $\eta_3^r(\boldsymbol{y})$ solving

$$-(A^0_{jq}\eta^r_{3,q})_{,j} - \rho_0 \Lambda \eta^r_3 = \rho_0 \delta_{r3} \quad \boldsymbol{y} \in Q_0, \tag{68}$$

$$\eta^r_3 = 0, \quad \boldsymbol{y} \in \Gamma, \tag{69}$$

where

$$A^0_{jq} := C^0_{3j3q}. \tag{70}$$

It immediately follows from (68) to (69) by uniqueness (away from the spectrum) that $\eta^r_3 \equiv 0$ for $r = 1, 2$, and $\tilde{\eta} := \eta^3_3$ solves

$$-(A^0_{jq}\tilde{\eta}_{,q})_{,j} - \rho_0 \Lambda \tilde{\eta} = \rho_0 \quad \boldsymbol{y} \in Q_0, \tag{71}$$

$$\tilde{\eta} = 0, \quad \boldsymbol{y} \in \Gamma. \tag{72}$$

Then, assuming ρ_0 to be also constant, by (42)

$$\beta_{ij}(\Lambda) = \Lambda \langle \rho \rangle \delta_{ij} + \Lambda^2 \rho_0 \delta_{i3}\delta_{j3} \langle \tilde{\eta} \rangle. \tag{73}$$

Suppose $\boldsymbol{C}^0$ is isotropic, with constant Lamé coefficients λ_0 and μ_0,

$$C^0_{ijpq} = \lambda_0 \delta_{ij}\delta_{pq} + \mu_0(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}). \tag{74}$$

Then via (70) matrix (A_{ij}) is diagonal and

$$A_{11} = A_{22} = \mu_0, \quad A_{33} = \lambda_0 + 2\mu_0.$$

The subsequent calculation is most explicit for Q_0 being a spheroid

$$\frac{y_1^2 + y_2^2}{a^2} + \frac{y_3^2}{b^2} = 1$$

with special aspect ratio

$$\frac{a}{b} = \left(\frac{\mu_0}{\lambda_0 + 2\mu_0}\right)^{1/2}.$$

Then by the change of variables

$$y_1' = y_1/a, \quad y_2' = y_2/a, \quad y_3' = y_3/b,$$
$$\eta' = \mu_0 \tilde{\eta}/(a^2\rho_0), \quad \Lambda' = a^2 \Lambda \rho_0/\mu_0, \tag{75}$$

(71) and (72) transforms to the following boundary-value problem in the unit ball $B = \{|\boldsymbol{y}'| \leqslant 1\}$:

$$-\Delta \eta' - \Lambda' \eta' = 1, \quad \boldsymbol{y}' \in B, \tag{76}$$

$$\eta' = 0, \quad |\boldsymbol{y}'| = 1. \tag{77}$$

This has an elementary radially symmetric solution (cf. Babych et al., 2008),

$$\eta'(\boldsymbol{y}') = \frac{\sin\left((\Lambda')^{1/2}|\boldsymbol{y}'|\right)}{\Lambda' \sin((\Lambda')^{1/2})|\boldsymbol{y}'|} - \frac{1}{\Lambda'}.$$

[10] (20) implies that w_1 and w_2 are constant in y_3, however the boundedness of the inclusion forces these constants to be zero.

Further, by change of variables (75),

$$\langle \tilde{\eta} \rangle := \int_{Q_0} \tilde{\eta}(\boldsymbol{y}) d\boldsymbol{y} = \frac{a^4 b \rho_0}{\mu_0} \int_B \eta'(\boldsymbol{y}') d\boldsymbol{y}'. \tag{78}$$

From (76), $\eta' = -(\Delta\eta' + 1)/\Lambda'$, which, upon plugging into (78) and applying the Green's formula, yields via a straightforward calculation

$$\langle \tilde{\eta} \rangle = 4\pi \frac{a^2 b}{\Lambda}\left[\frac{1}{\Lambda'} - \frac{1}{3} - \frac{\cot\mathrm{an}((\Lambda')^{1/2})}{(\Lambda')^{1/2}}\right].$$

In combination with (73) this finally gives

$$\beta_{ij}(\Lambda) = \Lambda \langle \rho \rangle \delta_{ij} + \delta_{i3}\delta_{j3} 4\pi \Lambda \rho_0 a^2 b \left[\frac{1}{\Lambda'} - \frac{1}{3} - \frac{\cot\mathrm{an}((\Lambda')^{1/2})}{(\Lambda')^{1/2}}\right], \tag{79}$$

with Λ' given in (75). It is readily seen that the matrix $\boldsymbol{\beta}(\Lambda)$ is diagonal, with β_{11} and β_{22} positive, whereas β_{33} changes the sign infinitely many times (in particular, near all the poles of $\cot\mathrm{an}((\Lambda')^{1/2})$).

Recall that the number of propagating modes is determined by the number of non-negative eigenvalues of $\boldsymbol{\beta}(\Lambda)$, see Section 4.2. We conclude that while for Λ with $\beta_{33}(\Lambda) > 0$ there exist three propagating modes in any direction, for Λ with $\beta_{33}(\Lambda) < 0$ the number of propagating modes reduces down to two.

5.2. Anisotropic fibers

In the three-dimensional case ($d = 3$) let Q_0 be a circular cylinder, cf. Fig. 2, with its axis parallel to y_3, i.e. (shifting the origin on the axis of the cylinder)

$$Q_0 = B_2(a) \times [0, 1), \quad B_2(a) = \{(y_1, y_2) : y_1^2 + y_2^2 \leqslant a\},$$
$$a < 1/2.$$

We consider the case when the only non-zero component of constant $\boldsymbol{C}^2$ is $C^2_{3333} \equiv \alpha > 0$.

Then, from (62), the only non-zero component of $\tilde{\boldsymbol{C}}^2(\boldsymbol{n})$ is $\tilde{C}^2_{33}$, i.e.

$$\tilde{C}^2_{ip}(\boldsymbol{n}) = \alpha \delta_{i3}\delta_{p3} n_3^2. \tag{80}$$

Let $\boldsymbol{C}^0$ be a constant isotropic elastic tensor given by (74), and let ρ_0 also be constant. Then (60) reads

$$\boldsymbol{P}[-\mu_0 \zeta^r_{i,pp} - (\lambda_0 + \mu_0)\zeta^r_{p,pi}] - (\Lambda \rho_0 \delta_{ip} - k^2 \alpha n_3^2 \delta_{i3}\delta_{p3})\zeta^r_p$$
$$= \boldsymbol{P}[\Lambda \rho_0 \delta_{ri} - k^2 \alpha n_3^2 \delta_{r3}\delta_{i3}] \quad \text{in } Q_0, \quad r = 1, 2, 3. \tag{81}$$

To find projector $\boldsymbol{P}$, notice that $\boldsymbol{w}(\boldsymbol{y}) = (w_i(\boldsymbol{y}))_{i=1}^3$ satisfies (16)–(18), i.e. $\boldsymbol{w} \in V$, if and only if w_3 is independent of y_3 and vanishes on the cylinder's boundary Γ. The operator $\boldsymbol{P}$ can then be seen, cf. (27), as simply the averaging of the third component in y_3:

$$(Pw)_i(\boldsymbol{y}) = \delta_{i1} w_1(\boldsymbol{y}) + \delta_{i2} w_2(\boldsymbol{y}) + \delta_{i3} \int_0^1 w_3(y_1, y_2, y_3) dy_3. \tag{82}$$

A plan for solving (81) is hence to first solve it without $\boldsymbol{P}$, i.e.

$$-\mu_0 \zeta^r_{i,pp} - (\lambda_0 + \mu_0)\zeta^r_{p,pi} - (\Lambda \rho_0 \delta_{ip} - k^2 \alpha n_3^2 \delta_{i3}\delta_{p3})\zeta^r_p$$
$$= \Lambda \rho_0 \delta_{ri} - k^2 \alpha n_3^2 \delta_{r3}\delta_{i3} \quad \text{in } Q_0, \quad r = 1, 2, 3, \tag{83}$$

and then to verify that the solutions will automatically solve also (81). (As will indeed be the case, since ζ_i^r will be independent of y_3 making $\boldsymbol{P}$ in (81) redundant.)

By the symmetries of the system (83), the matrix $\boldsymbol{D} := (\langle \rho_0 \zeta_i^j \rangle)$ entering (64) is diagonal in the chosen basis and $D_{11} = D_{22} =: \kappa$, which corresponds to $r = 1, 2$ in (83), describing the in-plane motion with $\zeta_1^r = \tilde{\zeta}_1(y_1, y_2)$, $\zeta_2^r = \tilde{\zeta}_2(y_1, y_2)$ and $\zeta_3^r \equiv 0$. We will set below $r = 1$ and evaluate $\kappa := \langle \rho_0 \tilde{\zeta}_1 \rangle$. Denoting also by $\theta := D_{33}$, we will select $r = 3$ in (83) which corresponds to an anti-plane shear motion, i.e. $\zeta_1^3 \equiv \zeta_2^3 \equiv 0$ and $\zeta_3^3 = \tilde{\zeta}(y_1, y_2)$. Then $\theta = \langle \rho_0 \tilde{\zeta} \rangle$ is also evaluated below.

We aim at showing that there are frequencies Λ with the property that there exist propagating plane waves (47) in certain directions $\boldsymbol{n}$ while in some other directions no such waves would propagate. We argue that this can be realized by an appropriate choice of the involved parameters. Indeed, denoting $\boldsymbol{e}^3$ the unit vector in the y_3-direction, (80) implies that for any propagation direction $\boldsymbol{n}$ orthogonal to $\boldsymbol{e}^3$, i.e. such that $n_3 = 0$, $\tilde{\boldsymbol{C}}^2(\boldsymbol{n}) = \boldsymbol{0}$. Hence, via (63)–(65), $\tilde{\gamma} = \boldsymbol{0}$. Therefore, recalling that $\tilde{\boldsymbol{C}}^h$ is strictly positive definite for any $\boldsymbol{n}$, there are no real k satisfying the dispersion relation (63) as long as $\boldsymbol{\beta}$ is negative definite, i.e.

$$\kappa(\Lambda) < -\langle \rho \rangle, \quad \theta(\Lambda) < -\langle \rho \rangle, \tag{84}$$

where $\kappa(\Lambda)$ and $\theta(\Lambda)$ are found from (83) with $n_3 = 0$.

The plan is hence to first show that (84) is achievable for a range of values of Λ. Hence one will not have any propagation in any directions (close to those) orthogonal to $\boldsymbol{e}^3$ for these frequencies. On the other hand, for any so selected Λ and for example for $\boldsymbol{n} = \boldsymbol{e}^3$, we will argue that for appropriate α (63) *will* have a solution $k > 0$, i.e. a propagating mode. We start the more detailed analysis with the anti-plane component, i.e. with $r = 3$ in (83). Then $\theta = \rho_0 \langle \tilde{\zeta} \rangle$ where $\tilde{\zeta}$ is the solution to

$$-\mu_0 \Delta \tilde{\zeta} - \Lambda' \rho_0 \tilde{\zeta} = \Lambda' \rho_0, \quad (y_1, y_2) \in B_2(a), \quad \tilde{\zeta}|_\Gamma = 0, \tag{85}$$

where

$$\Lambda' := \Lambda - \frac{k^2 \alpha n_3^2}{\rho_0}. \tag{86}$$

For $0 < a < 1/2$ the exact solution of (85) is elementary and reads (cf. Babych et al., 2008):

$$\tilde{\zeta}(y_1, y_2) = J_0(\nu R)/J_0(\nu a) - 1, \quad R := (y_1^2 + y_2^2)^{1/2},$$
$$\nu := (\Lambda' \rho_0 / \mu_0)^{1/2}, \tag{87}$$

where J_0 is the Bessel function of the first kind (see e.g. Gradshtein and Ryzhik, 1994).

To evaluate $\theta(\Lambda') := \rho_0 \int_{B_2(a)} \tilde{\zeta}(\boldsymbol{y}) d\boldsymbol{y}$ we use (87) together with e.g. $J_0(\nu R) = -\Delta J_0(\nu R)/\nu^2$ and then the Green's formula. As a result,

$$\theta(\Lambda') = -\pi a^2 \rho_0 + \frac{2\pi a \rho_0}{\nu} \frac{J_1(\nu a)}{J_0(\nu a)}, \tag{88}$$

(having used $J_0'(z) = -J_1(z)$). Since J_0 is oscillatory function with an infinite number of zeros, clearly $\theta(\Lambda')$ can be both positive and large enough negative, in particular will satisfy the second inequality in (84), e.g. near its poles $J_0(\nu a) = 0$.

Consider next the requirement for the first inequality in (84). It follows from (83) that the functions $\tilde{\zeta}_i(y_1, y_2)$, $i = 1, 2$, are solutions to the system

$$-\mu_0 \Delta \tilde{\zeta}_1 - (\lambda_0 + \mu_0)(\tilde{\zeta}_{1,11} + \tilde{\zeta}_{2,21}) - \Lambda \rho_0 \tilde{\zeta}_1 = \Lambda \rho_0, \tag{89}$$

$$-\mu_0 \Delta \tilde{\zeta}_2 - (\lambda_0 + \mu_0)(\tilde{\zeta}_{1,12} + \tilde{\zeta}_{2,22}) - \Lambda \rho_0 \tilde{\zeta}_2 = 0 \ \text{ in } B_2(a), \tag{90}$$

vanishing additionally on the boundary Γ. Introducing

$$\tilde{\zeta}_1 = \hat{\zeta}_1 - 1, \quad \hat{\zeta}_2 = \tilde{\zeta}_2, \tag{91}$$

we equivalently have

$$\mu_0 \Delta \hat{\zeta}_1 + (\lambda_0 + \mu_0)(\hat{\zeta}_{1,11} + \hat{\zeta}_{2,21}) + \rho_0 \Lambda \hat{\zeta}_1 = 0, \tag{92}$$

$$\mu_0 \Delta \hat{\zeta}_2 + (\lambda_0 + \mu_0)(\hat{\zeta}_{1,12} + \hat{\zeta}_{2,22}) + \rho_0 \Lambda \hat{\zeta}_2 = 0 \quad \text{in } B_2(a), \tag{93}$$

$$\hat{\zeta}_1|_\Gamma = 1, \quad \hat{\zeta}_2|_\Gamma = 0. \tag{94}$$

Solutions $\hat{\zeta}_1$ and $\hat{\zeta}_2$ to (92)–(94) are sought in the form

$$\hat{\zeta}_1 = \phi_{,11} + \psi_{,22}, \quad \hat{\zeta}_2 = \phi_{,12} - \psi_{,21}, \tag{95}$$

with two radially symmetric potentials

$$\phi = C_1 J_0(m_1 R), \quad \psi = C_2 J_0(m_2 R)$$

with appropriate real constants C_1 and C_2 and

$$m_1 = \left(\frac{\rho_0 \Lambda}{\lambda_0 + 2\mu_0} \right)^{1/2} \quad m_2 = \left(\frac{\rho_0 \Lambda}{\mu_0} \right)^{1/2}. \tag{96}$$

Eqs. (92) and (93) are then automatically satisfied for any choice of C_1 and C_2.

Further, since in the polar coordinates (R, Θ)

$$\frac{\partial}{\partial x_1} = \cos\Theta \frac{\partial}{\partial R} - R^{-1} \sin\Theta \frac{\partial}{\partial \Theta},$$
$$\frac{\partial}{\partial x_2} = \sin\Theta \frac{\partial}{\partial R} + R^{-1} \cos\Theta \frac{\partial}{\partial \Theta}, \tag{97}$$

introducing

$$\tilde{\phi} := \phi_{,1} = -C_1 m_1 \cos\Theta J_1(m_1 R), \tag{98}$$

$$\tilde{\psi} := \psi_{,2} = -C_2 m_2 \sin\Theta J_1(m_2 R), \tag{99}$$

we re-write (95) as

$$\hat{\zeta}_1 = \tilde{\phi}_{,1} + \tilde{\psi}_{,2}, \quad \hat{\zeta}_2 = \tilde{\phi}_{,2} - \tilde{\psi}_{,1}. \tag{100}$$

We further obtain via direct manipulation, employing functional relations for Bessel functions (Gradshtein and Ryzhik, 1994),

$$\hat{\zeta}_1 = -\frac{C_1 m_1^2}{2} J_0(m_1 R) - \frac{C_2 m_2^2}{2} J_0(m_2 R) + X(R) \cos 2\Theta, \tag{101}$$

$$\hat{\zeta}_2 = X(R) \sin 2\Theta \tag{102}$$

where

$$X(R) := \frac{C_1 m_1^2}{2} J_2(m_1 R) - \frac{C_2 m_2^2}{2} J_2(m_2 R).$$

Hence to satisfy (94) we require

$$\frac{C_1 m_1^2}{2} J_2(m_1 a) - \frac{C_2 m_2^2}{2} J_2(m_2 a) = 0, \tag{103}$$

$$\frac{C_1 m_1^2}{2} J_0(m_1 a) + \frac{C_2 m_2^2}{2} J_0(m_2 a) = -1. \quad (104)$$

The latter has solution

$$C_1 = \frac{-2\, J_2(m_2 a)}{m_1^2 D(\Lambda, \lambda_0)}, \quad C_2 = \frac{-2\, J_2(m_1 a)}{m_2^2 D(\Lambda, \lambda_0)},$$

with

$$D(\Lambda, \lambda_0) := J_2(m_1 a) J_0(m_2 a) + J_0(m_1 a) J_2(m_2 a).$$

Note that constants $C_i,\ i = 1,\ 2$, depend on both μ_0 and λ_0 via (96).

We evaluate $\kappa(\Lambda) := \rho_0 \int_{B_2(a)} \tilde{\zeta}_1(\boldsymbol{y}) d\boldsymbol{y}$ similarly to that of $\theta(\Lambda)$, via (101) and (91). As a result,

$$\kappa(\Lambda) = -\pi a^2 \rho_0 - C_1 \pi m_1 a \rho_0\, J_1(m_1 a) - C_2 \pi m_2 a \rho_0\, J_1(m_2 a). \quad (105)$$

We can now see that varying the elastic modulus λ_0 one can shift the set of zeros for $D(\Lambda, \lambda_0)$ so that e.g. the corresponding pole of $\kappa(\Lambda)$ lies in the domain of $\theta(\Lambda)$ satisfying (84). Indeed, using for example the trigonometric asymptotics of the Bessel functions for large $\Lambda > 0$ (e.g. Gradshtein and Ryzhik, 1994),

$$D(\Lambda, \lambda_0) \sim \frac{-\,4}{\pi a (m_1 m_2)^{1/2}} \cos(m_1 a - \pi/4) \cos(m_2 a - \pi/4),$$

with the asymptotics of the zeros due to $\cos(m_1 a - \pi/4)$ controllable by choice of λ_0 via (96). This would allow to achieve simultaneously both inequalities in (84), as desired. Since for $n_3 = 0$ via (86) $\Lambda' = \Lambda$, this ensures the absence of any propagating modes in any direction orthogonal to the fibers for the chosen Λ.

On the other hand, for a chosen $\Lambda > 0$ satisfying (84) let $0 < \Lambda' < \Lambda$ be such that $\theta(\Lambda') > 0$. (This is always possible since for $\theta(\Lambda') := \Lambda' \langle \tilde{\eta} \rangle$, from e.g. (87), $\theta(0) = 0$ and $\theta'(0) > 0$.) Then, for $k^2 \alpha(k)$ fixed according to (86) (for example for $n_3 = 1$), by (65) and (80) the matrix $k^2 \tilde{\boldsymbol{\gamma}}$ with components

$$k^2 \tilde{\gamma}_{ip} = k^2 \alpha n_3^2 \rho_0^{-1} \delta_{i3} \delta_{p3} \theta(\Lambda') \quad (106)$$

is independent of k. Then the matrix $k^2 \tilde{\boldsymbol{\gamma}} - \boldsymbol{\beta}$ in (63) will cease to be positive definite (i.e. will have one negative eigenvalue), since via (106), (64) and (86)

$$k^2 \tilde{\gamma}_{33} - \beta_{33} = -\Lambda' \theta(\Lambda') - \Lambda \langle \rho \rangle < 0.$$

Since $k^2 \tilde{\boldsymbol{\gamma}} - \boldsymbol{\beta}$ is independent of k and has one negative and two positive eigenvalues (and hence has negative determinant), and $\tilde{\boldsymbol{C}}(\boldsymbol{n})$ is strictly positive, the dispersion relation (63) will have a solution $k > 0$, i.e. a propagating mode.

This qualitative argument ensures the simultaneous existence of propagating and "non-propagating" directions.

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Appendix A. Formal derivation of the limit problem (23) and (24)

This appendix derives the limit equations (23) and (24) via two-scale asymptotic expansions in the form (14) (cf. Kamotski and Smyshlyaev, 2006), for the case of scalar spectral problems in unbounded domains. The asymptotic solution to (13) is sought in the form of a two-scale ansatz (14), where $\boldsymbol{u}^{(0)}(\boldsymbol{x}, \boldsymbol{y}, t)$, $\boldsymbol{u}^{(1)}(\boldsymbol{x}, \boldsymbol{y}, t)$ and $\boldsymbol{u}^{(2)}(\boldsymbol{x}, \boldsymbol{y}, t)$ are required to depend periodically on the "fast" variable $\boldsymbol{y}$. The exact solution $\boldsymbol{u}^{\varepsilon}(\boldsymbol{x}, t)$ satisfies the continuity conditions (11) at the inclusions' boundary, specializing via (10) to

$$\boldsymbol{u}^{\varepsilon}(\boldsymbol{x}, t)|_1 = \boldsymbol{u}^{\varepsilon}(\boldsymbol{x}, t)|_0, \quad \boldsymbol{x} \in \Gamma^{\varepsilon}, \quad (A.1)$$

and

$$C^1_{ijpq} u^{\varepsilon}_{p,q} n_j|_1 = (\varepsilon^2 C^0_{ijpq} u^{\varepsilon}_{p,q} n_j + C^2_{ijpq} u^{\varepsilon}_{p,q} n_j)|_0, \quad \boldsymbol{x} \in \Gamma^{\varepsilon}, \quad (A.2)$$

(remind that $\boldsymbol{n}$ stands for unit normal to Γ^{ε} selected as outward for the inclusion phase Ω_0^{ε}, and hence inward for the matrix Ω_1^{ε}).

The ansatz (14) is substituted into Eq. (13), separately in the matrix and the inclusion phases, and the interface conditions (A.1) and (A.2).

By equating first the terms of order ε^{-2} in (13) and of order ε^{-1} in (A.2),

$$-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^1 \boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{u}^{(0)})(\boldsymbol{x}, \boldsymbol{y}, t)) = 0, \quad \boldsymbol{y} \in Q_1; \quad (A.3)$$

$$C^1_{ijpq} \frac{\partial u_p^{(0)}}{\partial y_q} n_j^{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, t) = C^2_{ijpq} \frac{\partial u_p^{(0)}}{\partial y_q} n_j^{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, t), \quad \boldsymbol{y} \in \Gamma, \quad (A.4)$$

$$-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^2 \boldsymbol{e}^{\boldsymbol{y}}(\boldsymbol{u}^{(0)})(\boldsymbol{x}, \boldsymbol{y}, t)) = 0, \quad \boldsymbol{y} \in Q_0 \quad (A.5)$$

(with $\boldsymbol{n}^{\boldsymbol{y}}$ being the inward unit normal for Q_1). This is a homogeneous problem in Q with periodic boundary conditions. Since Q_1 is connected and $\boldsymbol{C}^1$ strictly positive definite, its solution is an arbitrary constant in Q_1 (i.e. independent of $\boldsymbol{y}$)[11] which implies (15). The balance of the terms of order ε^0 in (A.1) implies (18).

The relations (15) together with (A.5), (A.4) and (18) imply then the following restrictions on v:

$$-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{C}^2 \boldsymbol{e}^{\boldsymbol{y}}(v)(\boldsymbol{x}, \boldsymbol{y}, t)) = 0, \quad \boldsymbol{y} \in Q_0; \quad (A.6)$$

$$C^2_{ijpq} \frac{\partial v_p}{\partial y_q} n_j^{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, t) = 0, \quad \boldsymbol{y} \in \Gamma, \quad (A.7)$$

$$v(\boldsymbol{x}, \boldsymbol{y}, t) = \boldsymbol{0}, \quad \boldsymbol{y} \in \Gamma, \quad (A.8)$$

which yields (16)–(18). Taking dot product of (A.6) with v and integrating over Q_0 in $\boldsymbol{y}$, upon further integration by parts and using non-negativeness and symmetry of $\boldsymbol{C}^2$ results in (20).

Equating next the terms of order ε^{-1} in (13) and of order ε^0 in (A.2), using also (A.8), and (20)–(22), we arrive at:

[11] This is seen in a standard way, e.g. by taking the dot product of (A.3) and (A.5) with $\boldsymbol{u}^{(0)}$, integrating over the respective domains and using (A.4) and the strict positivity of $\boldsymbol{C}^1$ together with the non-negativeness of $\boldsymbol{C}^2$.

$$-\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\right)=\frac{\partial}{\partial y_j}(C^1_{ijpq}(\boldsymbol{y}))\frac{\partial u_p^{(0)}}{\partial x_q},\quad \boldsymbol{y}\in Q_1 \tag{A.9}$$

$$-\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\right)=\frac{\partial}{\partial y_j}(C^2_{ijpq}(\boldsymbol{y}))\frac{\partial u_p^{(0)}}{\partial x_q},\quad \boldsymbol{y}\in Q_0 \tag{A.10}$$

$$C^1_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\bigg|_1 n_j^{\boldsymbol{y}}-C^2_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\bigg|_0 n_j^{\boldsymbol{y}}=-C^1_{ijpq}\frac{\partial u_p^0}{\partial x_q}n_j^{\boldsymbol{y}}+C^2_{ijpq}\frac{\partial u_p^0}{\partial x_q}n_j^{\boldsymbol{y}},\ \boldsymbol{y}\in\Gamma, \tag{A.11}$$

together with the periodicity conditions in $\boldsymbol{y}$. This is a version of a standard "corrector" problem, in particular, for $\boldsymbol{C}^2>\boldsymbol{0}$ for "classical" (two-phase) homogenization, and for $\boldsymbol{C}^2=\boldsymbol{0}$ for "porous" (or "perforated") periodic domains. As a result,

$$u_p^{(1)}(\boldsymbol{x},\boldsymbol{y},t)=N_s^{pr}(\boldsymbol{y})\frac{\partial u_r^0(\boldsymbol{x},t)}{\partial x_s}, \tag{A.12}$$

where N_s^{pr} are the solutions of the linear elastic "unit cell" problems with periodic boundary conditions (cf. e.g. Bakhvalov and Panasenko, 1989, Section 4.4; Jikov et al., 1994, Sections 3.1 and 13), which reads (in the weak form):

$$\int_Q C_{ijpq}(\boldsymbol{y})\left(\frac{\partial N_s^{pr}}{\partial y_q}+\delta_{pr}\delta_{qs}\right)\frac{\partial\psi}{\partial y_j}d\boldsymbol{y}=0,\quad \text{for any } \psi\in C^\infty_{per}(Q). \tag{A.13}$$

Here $\boldsymbol{C}(\boldsymbol{y}):=\chi_0(\boldsymbol{y})\boldsymbol{C}^1+(1-\chi_0(\boldsymbol{y}))\boldsymbol{C}^2$ with χ_0 denoting the characteristic function of Q_0. The above procedure determines N_s^{pr} up to a constant within Q_1, which is specified arbitrarily, e.g. by the condition that the average of N_s^{pr} over Q_1 is zero. In Q_0 it is specified up to an arbitrary non-trivial solution of (A.6)–(A.8), if any, and we assume that one of those is selected.[12]

Equate finally the terms of order ε^0 in (13). In the matrix, taking into account (15) and (A.12):

$$\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\right)=\rho_1\ddot{u}_i^0-C^1_{ijpq}u^0_{p,qj}-C^1_{ijpq}N^{pr}_{s,q}u^0_{r,sj}-(C^1_{ijpq}N_s^{pr})_{,j}u^0_{r,sq}-f_i(\boldsymbol{x},\boldsymbol{y},t),\ y\in Q_1. \tag{A.14}$$

Similarly, in the inclusion,

$$\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\right)=\rho_0(\ddot{u}_i^0+\ddot{v}_i)-\frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right)-C^2_{ijpq}u^0_{p,qj}-C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q}-C^2_{ijpq}N^{pr}_{s,q}u^0_{r,sj}-(C^2_{ijpq}N_s^{pr})_{,j}u^0_{r,sq}-f_i(\boldsymbol{x},\boldsymbol{y},t),\ y\in Q_0. \tag{A.15}$$

[12] Notice that a necessary condition for the solvability of (A.10) is for the right-hand sides to be "orthogonal" (in appropriate sense) to any solution of the homogeneous system, i.e. to any constant vector and any solution of (A.6)–(A.8) (equivalently, to any $\boldsymbol{w}\in V$, see (19)). Those are satisfied automatically, as seen by multiplying (A.9) and (A.10) by an arbitrary constant vector and $\boldsymbol{w}\in V$, integrating by parts and using (A.11). Establishing the sufficiency of the solvability condition requires a more advanced analysis though, developing for example appropriate versions of Poincaré and/or Korn's inequalities. We postpone this for future publications, announcing here that the solvability *can be* proved for general right-hand sides, not only in the limit cases of classical homogenization and perforated domains but also, at least, for the examples considered in this paper (Section 5) and for the case of $C^2_{ijpq}=\delta_{ij}\delta_{pq}$, i.e. of vanishing shear modulus in the inclusion.

Eqs. (A.14) and (A.15) have to be supplemented by boundary conditions which result from equating in (A.2) the terms of order ε^1, yielding:

$$C^1_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\bigg|_1 n_j^{\boldsymbol{y}}-C^2_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\bigg|_0 n_j^{\boldsymbol{y}}=-C^1_{ijpq}N_s^{pr}u^0_{r,sq}n_j^{\boldsymbol{y}}+C^2_{ijpq}N_s^{pr}u^0_{r,sq}n_j^{\boldsymbol{y}}+C^0_{ijpq}\frac{\partial v_p}{\partial y_q}n_j^{\boldsymbol{y}}. \tag{A.16}$$

Treating (A.14)–(A.16) as a boundary-value problem for $\boldsymbol{u}^{(2)}$ in $\boldsymbol{y}$ for any fixed $\boldsymbol{x}$ and t, the Green's formula together with the periodicity boundary conditions in $\boldsymbol{y}$ imply:

$$\int_Q\frac{\partial}{\partial y_j}\left(C_{ijpq}(\boldsymbol{y})\frac{\partial u_p^{(2)}}{\partial y_q}\right)d\boldsymbol{y}=-\int_\Gamma\left(C^1_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\bigg|_1 n_j^{\boldsymbol{y}}-C^2_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\bigg|_0 n_j^{\boldsymbol{y}}\right)d\boldsymbol{y}.$$

Substituting into these the right-hand sides of (A.14)–(A.16) yields:

$$\langle\rho\rangle\ddot{u}_i^0-\langle C_{ijpq}\rangle u^0_{p,qj}+\rho_0\langle\ddot{v}_i\rangle-u^0_{r,sj}\int_Q C_{ijpq}(\boldsymbol{y})N^{pr}_{s,q}(\boldsymbol{y})d\boldsymbol{y}-u^0_{r,sq}\int_Q(C_{ijpq}(\boldsymbol{y})N_s^{pr}(\boldsymbol{y}))_{,j}d\boldsymbol{y}-\int_{Q_0}\frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right)d\boldsymbol{y}-\int_{Q_0}C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q}d\boldsymbol{y}=\int_\Gamma\left((C^1_{ijpq}-C^2_{ijpq})N_s^{pr}(\boldsymbol{y})u^0_{r,sq}(\boldsymbol{x})n_j^{\boldsymbol{y}}-C^0_{ijpq}\frac{\partial v_p}{\partial y_q}n_j^{\boldsymbol{y}}\right)d\boldsymbol{y}+\int_Q f_i(\boldsymbol{x},\boldsymbol{y},t)d\boldsymbol{y}.$$

Applying the integration by parts to the right-hand side surface integrals we arrive at the limiting equation (23), where

$$C^{\text{hom}}_{ijpq}:=\left\langle C_{ijrs}(\boldsymbol{y})(\delta_{pr}\delta_{qs}+N^{rp}_{q,s}(y))\right\rangle_y. \tag{A.17}$$

This is a well-known representation of the entries of the homogenized linear elastic tensor C^{hom}, equivalent to (26) (see e.g. Bakhvalov and Panasenko, 1989, Section 4.4).

On the other hand, multiply (A.15) by an arbitrary function $\boldsymbol{w}(\boldsymbol{y})=(w_i)(\boldsymbol{y})$ satisfying (16)–(18), i.e. $\boldsymbol{w}\in V$, see (19), and integrate over Q_0. Upon integration by parts and employing again (20)–(22), the last two terms containing $\boldsymbol{C}^2$ will vanish, yielding the integral identity

$$\int_{Q_0}\left[\rho_0(\ddot{u}_i^0+\ddot{v}_i)w_i+C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\frac{\partial w_i}{\partial y_j}-C^2_{ijpq}u^0_{p,qj}w_i-C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q}w_i\right]d\boldsymbol{y}=\int_{Q_0}f_i(\boldsymbol{x},\boldsymbol{y},t)w_i d\boldsymbol{y},\quad\forall\,\boldsymbol{w}\in V. \tag{A.18}$$

This is equivalent to the second limit equation (24).

Finally, notice that (Cooper, 2008)

$$C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q}=\frac{\partial}{\partial y_q}\left(y_r C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_r}\right), \tag{A.19}$$

having used (20)–(22). Since the expression inside the brackets in (A.19) vanishes on Γ, it admits a periodic extension provided Q_0 is isolated. This yields, integrating by parts and again using (20)–(22),

$$\int_{Q_0} C^2_{ijpq} \frac{\partial^2 v_p}{\partial x_j \partial x_q} d\boldsymbol{y} = \int_{Q_0} C^2_{ijpq} \frac{\partial^2 v_p}{\partial x_j \partial x_q} w_i d\boldsymbol{y} = 0, \quad \forall\, \boldsymbol{w} \in V. \qquad (A.20)$$

This implies that in the limit system (23) and (24) the terms containing $\boldsymbol{C}^2\boldsymbol{e}^x(v)$ vanishing identically provided the inclusions are isolated. The vanishing in (24) of the term $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{C}^2\boldsymbol{e}^0(\boldsymbol{x},t))$ is, via (27), equivalent to

$$\int_{Q_0} C^2_{ijpq} \frac{\partial^2 u^0_p}{\partial x_j \partial x_q} w_i d\boldsymbol{y} = 0, \quad \forall\, \boldsymbol{w} \in V. \qquad (A.21)$$

The latter can be seen by e.g. multiplying the left-hand side of (A.21) by an arbitrary test function $\psi(\boldsymbol{x})$, integration by parts in $\boldsymbol{x}$ and then applying (A.20) with v replaced by $\psi\boldsymbol{w}$.

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